Kruskal's Tree Theorem for Term Graphs*

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In this extended abstract we study termination of term graph rewriting, where we restrict our attention to *acyclic* term graphs. More precisely, we establish a variant of Kruskal's Tree Theorem formulated for term graphs. To this end we suitably adapt the original embedding relation on trees to a relation on directed, acyclic graphs. The proof then follows Nash-Williams' minimal bad sequence argument.

1 Introduction

In this extended abstract we study termination of term graph rewriting, where we restrict our attention to *acyclic* term graphs. More precisely we establish a variant of Kruskal's Tree Theorem formulated for term graphs.

It is well-known that term graph rewriting is *adequate* for term rewriting. However, this result requires suitable care in the treatment of *sharing*, cf. [13, 4]. In particular, if we only consider term graph rewrite steps then termination of a given graph rewrite system does not imply termination of the corresponding term rewrite system [12]. This follows as the representation of a term as a graph enables us to share equal subterms. However, if we do not provide the possibility to unshare equal subterms, we change the potential rewrite steps. Then not every term rewrite step can be simulated by a graph rewrite step. This motivates our interest in termination techniques *directly* for term graph rewriting. In our definition of term graph rewriting we essentially follow Barendsen [4], but also [3, 1], which are notationally closest to our presentation. Our motivation for term graphs stems from term graph rewriting as implementation of first-order term rewriting. Thus, we restrict our attention to term graphs, which represent such (finite) terms, that is in our context term graphs are directed, *rooted*, and *acyclic* graphs with node labels over a set of function symbols and variables.

In term rewriting, termination is typically established via compatibility with a reduction order. Well-foundedness of such an order is more often than not a consequence of Kruskal's Tree Theorem [9] (e.g. in [7]). In particular, Kruskal's Tree Theorem underlies the concept of simple termination (see e.g. [10]). Indeed, Plump [12] defines a simplification order for acyclic term graphs. This order relies on the notion of *tops*. The top of a term graph is its root *and* its direct successors—thus keeping information on how these successors are share.

We briefly recall: If for any infinite sequence, we can find two elements a_i, a_j with $i < j$ where $a_i \sqsubseteq_{\mathsf{emb}} a_j$, then $\sqsubseteq_{\mathsf{emb}}$ is a well-quasi order. Now, Kruskal's Tree Theorem states, that if we can find a well-quasi order $\sqsubseteq$ on the symbols in a term, we can find a well-quasi order $\sqsubseteq_{\mathsf{emb}}$ on terms. In our setting, we consider term graphs, not terms, and our symbols are tops.

Usually, the relation $\sqsubseteq_{\mathsf{emb}}$ is called *embedding relation*. Plump [12] defines $\sqsubseteq_{\mathsf{emb}}^{[12]}$, but as he notes, for the following two term graphs, his definition of $\sqsubseteq_{\mathsf{emb}}^{[12]}$ holds in both directions.

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$$\begin{tikzcd} \mathsf{f} \arrow[d] \arrow[dd, bend left] \\ \mathsf{g} \arrow[d] \\ \mathsf{a} \end{tikzcd} \quad \sqsubseteq^{[12]}_{\mathsf{emb}} \quad \begin{tikzcd} & \mathsf{f} \arrow[dl] \arrow[dr] & \\ \mathsf{g} \arrow[d] & & \mathsf{a} \\ \mathsf{a} & & \end{tikzcd} \quad \text{but also} \quad \begin{tikzcd} & \mathsf{f} \arrow[dl] \arrow[dr] & \\ \mathsf{g} \arrow[d] & & \mathsf{a} \\ \mathsf{a} & & \end{tikzcd} \quad \sqsubseteq^{[12]}_{\mathsf{emb}} \quad \begin{tikzcd} \mathsf{f} \arrow[d] \arrow[dd, bend left] \\ \mathsf{g} \arrow[d] \\ \mathsf{a} \end{tikzcd}$$

In particular, [12] does not take sharing into account—except for direct successors through tops. This is a consequence of identifying each sub-graph independently.

This is the inspiration and starting point for our work: We want to define an embedding relation, which also takes sharing into account. With this new embedding relation we re-prove Kruskal's Tree Theorem. Also here we take a slightly different approach to [12], which relies on an encoding of tops to function symbols with different arities. It is stated that there is a direct proof based on [11], which will be our direction.

As already mentioned, the context of this paper is the quest for termination techniques for term graph rewriting. Here *termination* refers to the well-foundedness of the graph rewrite relation $\to_{\mathcal{G}}$, induced by a graph rewrite system $\mathcal{G}$, cf. [4]. In particular we seek a technique based on orders. This is in contrast to related work in the literature. There termination is typically obtained through interpretations or weights, cf. Bonfante et al. [5]. Also Bruggink et al. [6] use an interpretation method, where they use type graphs to assign weights to graphs to prove termination.

2 Preliminaries: Term Graphs, Tops & Embedding

First, we introduce our flavour of term graphs based on term dags and give the sharing relation. Then we investigate tops with respect to a function symbol but also with respect to a node in a term graph. Based on this, we will consider a precedence on tops and—finally—give an embedding relation on term graphs.

Definition 1. Let $\mathcal{N}$ be a set of nodes, $\mathcal{F}$ a set of function symbols, and $\mathcal{V}$ a set of variables. A *graph* is $G = (N, \mathsf{succ}, \mathsf{label})$, where $N \subseteq \mathcal{N}$, $\mathsf{succ} : N \to N^*$, and $\mathsf{label} : N \to \mathcal{F} \cup \mathcal{V}$. Here, succ maps a node n to an ordered list of *successors* $[n_1 \ldots n_k]$. Further, label assigns labels, where *(i)* for every node $n \in G$ with $\mathsf{label}(n) = f \in \mathcal{F}$ we have $\mathsf{succ}(n) = [n_1, \ldots, n_{\mathsf{arity}(f)}]$, and *(ii)* for every $n \in G$ with $\mathsf{label}(n) \in \mathcal{V}$, we have $\mathsf{succ}(n) = [\,]$. If G is acyclic, then G is a *term dag*.

The *size* of a graph $|G|$ is the number of its nodes N. We write $n \in G$ and mean $n \in N$, and call G *ground*, if $\mathsf{label} : N \to \mathcal{F}$. If $\mathsf{succ}(n) = [\ldots, n_i, \ldots]$, we write $n \xrightarrow{i} n_i$, or simply $n \rightharpoonup n_i$ for any i. Further, $\rightharpoonup^+$ is the transitive, and $\rightharpoonup^*$ the reflexive, transitive closure. If $n \rightharpoonup^* n'$, then n' is *reachable* from n. In the sub-graph $G{\restriction}[n_1, \ldots, n_k]$ all nodes reachable from $n_1, \ldots, n_k$ are collected, i.e. $N = \{n \mid n_i \rightharpoonup^* n, 1 \leqslant i \leqslant k\}$, and the domains of succ and label are restricted accordingly.

Definition 2. Let T be a term dag. If all nodes are reachable from one node called $\mathsf{root}(T)$, that is, T is *rooted*, then T is a *term graph*. The *argument graph* of T is the term dag $T{\restriction}\mathsf{inlets}$ where $\mathsf{inlets} = \mathsf{succ}(\mathsf{root}(T))$.

Example 3. On the right we show the term graph $T = (\{①, ②\}, \mathsf{succ}, \mathsf{label})$, with $\mathsf{succ} : ① \mapsto [②, ②], ② \mapsto [\,]$, and $\mathsf{label} : ① \mapsto \mathsf{f}, ② \mapsto \mathsf{a}$. The term representation of T is $\mathsf{f}(\mathsf{a}, \mathsf{a})$, $|T| = 2$, and T is ground. The argument graph of T is $\mathsf{a} : ②$ with $\mathsf{inlets} = [②, ②]$.

$$\begin{tikzcd} \mathsf{f} : ① \arrow[d, bend right] \arrow[d, bend left] \\ \mathsf{a} : ② \end{tikzcd}$$

In the following S and T denote term graphs. We may ask: Is S a "more shared" version of T? Are S and T "equal"? To answer this, we look for a *morphism* from the nodes in S to the nodes in T.

Definition 4. A function $m : S \to T$ is *morphic* if for a node $n \in S$, we have *(i)* $\mathsf{label}_S(n) = \mathsf{label}_T(m(n))$, and *(ii)* if $n \overset{i}{\rightharpoonup}_S n_i$ then $m(n) \overset{i}{\rightharpoonup}_T m(n_i)$ for all appropriate i. A *morphism* is a mapping $m : S \to T$, which is morphic in all nodes $n \in S$, and $m(\mathsf{root}(S)) = \mathsf{root}(T)$ holds. If there is a morphism $m : S \to T$, then S *shares* to T, denoted by $S \succcurlyeq T$. If $S \succcurlyeq T$ and $T \succcurlyeq S$, then S is *isomorphic* to T, denoted by $S \cong T$.

Reconsidering Example 3, let S be a tree representation of $\mathsf{f}(\mathsf{a},\mathsf{a})$, then $S \succcurlyeq T$. Now recall, that we aim to give a notion of Top, which takes the sharing of successor nodes into account. Thus—with sharing—we can give a definition of Tops for a function symbol f.

Definition 5. Let $f \in \mathcal{F}$, $\triangle$ a fresh symbol wrt. $\mathcal{F}$, and S a tree representation of $f(\triangle,\dots,\triangle)$. Then $\mathsf{Tops}(f) = \{T \mid T \text{ is a termgraph, and } S \succcurlyeq T\}$ and $\mathsf{Tops}(\mathcal{F}) = \bigcup_{f \in \mathcal{F}} \mathsf{Tops}(f)$.

So we can compute the Tops for a function symbol—but we also want to compute the Top from some node in a term dag.

Definition 6. For a term dag $G = (N, \mathsf{succ}, \mathsf{label})$ and a node $n \in G$, we define $\mathsf{Top}(n) := (\{n\} \cup \mathsf{succ}(n), \mathsf{label'}, \mathsf{succ'})$, where *(i)* $\mathsf{label'}(n) = \mathsf{label}(n)$, $\mathsf{succ'}(n) = \mathsf{succ}(n)$, and *(ii)* for $n_i \in \mathsf{succ}(n)$, $\mathsf{label'}(n_i) = \triangle$, and $\mathsf{succ'}(n_i) = [\,]$.

Now, similar to a precedence on function symbols, we define a precedence $\sqsubseteq$ on Tops.

Definition 7. A *precedence* on $\mathcal{F}$ is a transitive relation $\sqsubseteq$ on $\mathsf{Tops}(\mathcal{F})$, where for $S, T \in \mathsf{Tops}(\mathcal{F})$ we have *(i)* $S \cong T$ implies $S \sqsubseteq T$ and $T \sqsubseteq S$, and *(ii)* $T \sqsubseteq S$ implies $|T| \leqslant |S|$.

Condition *(i)* implies reflexivity, but also includes isomorphic copies. Condition *(ii)* hints at a major distinction to the term rewriting setting: We can distinguish the same function symbol with different degrees of sharing—and even embed nodes, which are labelled with function symbols with a smaller arity, in nodes labelled with function symbols with a larger arity. But, to ensure that such an embedding is indeed possible, enough nodes have to present—which is guaranteed by Condition *(ii)*.

Next, we define the embedding of term dags, related to Definition 4. Formally, the embedding relation relies on a *partial* mapping from the embedding term graph S to T. This may seem unnecessarily involved, as one may expect a definition of embedding from T to S that makes use of a mapping from T to S. However, this alternative definition yields to difficulties when we want to embed the tree representation of a term graph into a term graph, which employs more sharing, cf. $\mathsf{f}(\mathsf{a},\mathsf{a})$ in Example 9.

Definition 8. We say that S *embeds* T, denoted as $S \sqsupseteq_{\mathsf{emb}} T$, if there exists a partial, surjective function $m\colon S \to T$, such that for all nodes s in the domain of S, we have *(i)* $\mathsf{Top}_S(s) \sqsupseteq \mathsf{Top}_T(m(s))$, and *(ii)* $m(s) \rightharpoonup_T m(s')$ implies $s \rightharpoonup_S^+ s'$.

The relation $\sqsupseteq_{\mathsf{emb}}$ is transitive, i.e. $S \sqsupseteq_{\mathsf{emb}} T$ and $T \sqsupseteq_{\mathsf{emb}} U$ implies $S \sqsupseteq_{\mathsf{emb}} U$. The proof is straight forward: We construct the embedding $m_3 : S \to U$, based on the implied embeddings $m_2 : S \to T$ and $m_1 : T \to U$, by setting $m_3(n) = m_1(m_2(n))$ and show that m_3 fulfils the conditions in Definition 8. This proof is rather technical, but not difficult, and omitted here.

Example 9. Below we find three term graphs—embedded from the left to the right under the given precedence (for brevity only the label of a node is shown).

$$\begin{array}{c}\mathsf{f}\\ \swarrow\ \searrow\\ \mathsf{a}\quad\mathsf{a}\end{array} \sqsupseteq_{\mathsf{emb}} \begin{array}{c}\mathsf{g}\\ \downarrow\\ \mathsf{a}\end{array} \sqsupseteq_{\mathsf{emb}} \begin{array}{c}\mathsf{f}\\ (\ )\\ \mathsf{a}\end{array} \quad \text{with precedence} \quad \begin{array}{c}\mathsf{f}\\ \swarrow\ \searrow\\ \triangle\quad\triangle\end{array} \sqsupseteq \begin{array}{c}\mathsf{g}\\ \downarrow\\ \triangle\end{array} \sqsupseteq \begin{array}{c}\mathsf{f}\\ (\ )\\ \triangle\end{array}$$

3 Proof: Kruskal's Tree Theorem for Term Graphs

Our proof follows [10] for the term rewrite setting, which in turn follows the minimal bad sequence argument of Nash-Williams [11]: we assume a minimal "bad" infinite sequence of term graphs and construct an even smaller "bad" infinite sequence of their arguments. By minimality we contradict that this sequence of arguments is "bad", and conclude that it is "good". So we start by defining the notions of "good" and "bad".

Definition 10. Assume a reflexive and transitive order $\preccurlyeq$, and an infinite sequence $\mathbf{a}$ with a_i, a_j in $\mathbf{a}$. If for some $i < j$ we have $a_i \preccurlyeq a_j$, then $\mathbf{a}$ is *good*. Otherwise, $\mathbf{a}$ is *bad*. If every infinite sequence is good, then $\preccurlyeq$ is a *well-quasi order* (wqo).

After we determined the sequence of arguments to be good, we want to— roughly speaking—plug the Top back on its argument. For this, we need a wqo on $\mathsf{Tops}(\mathcal{F})$ and the following, well established, lemma.

Lemma 11. *If $\preccurlyeq$ is a wqo then every infinite sequence contains a subsequence—a* chain*—with $a_i \preccurlyeq a_{i+1}$ for all i.*

With this lemma, we can construct witnesses that our original minimal bad sequence of term graphs is good, contradicting its badness and concluding the following theorem.

Theorem 12. *If $\sqsubseteq$ is a wqo on $\mathsf{Tops}(\mathcal{F})$, then $\sqsubseteq_{\mathsf{emb}}$ is a wqo on ground, acyclic term graphs.*

Proof. By definition, $\sqsubseteq_{\mathsf{emb}}$ is a wqo, if every infinite sequence is good, i.e. for every infinite sequence of term graphs, there are two term graphs T_i, T_j, such that $T_i \sqsubseteq_{\mathsf{emb}} T_j$ with $1 \leqslant i < j$. We construct a minimal bad sequence of term graphs $\mathbf{T}$: Assume we picked $T_1, \ldots, T_{n-1}$. We next pick T_n—minimal with respect to $|T_n|$—such that there are bad sequences that start with $T_1, \ldots, T_n$.

Let G_i be the argument graph of the ith term graph T_i. We collect in G the arguments of all term graphs of $\mathbf{T}$, i.e. $G = \bigcup_{i \geqslant 1} G_i$ and show that $\sqsubseteq_{\mathsf{emb}}$ is a wqo on G. For a contradiction, we assume G admits a bad sequence $\mathbf{H}$. We pick $G_k \in G$ with $k \geqslant 1$ such that $H_1 = G_k$. In G' we collect all argument graphs up to G_k, i.e. $G' = \bigcup_{i \geqslant 1}^{k} G_i$. The set G' is finite, hence there exists an index $l > 1$, such that for all H_i with $i \geqslant l$ we have that $H_i \in G$ but $H_i \notin G$'. We write $\mathbf{H}_{\geqslant l}$ for the sequence $\mathbf{H}$ starting at index l. Now consider the sequence $T_1, \ldots, T_{k-1}, G_k, \mathbf{H}_{\geqslant l}$. By minimality of $\mathbf{T}$ this is a good sequence. So we try to find a witness and distinguish on i, j:

$\underbrace{T_1, \ldots, T_{k-1}}_{i,j}, G_k, \mathbf{H}_{\geqslant l}$ For $1 \leqslant i < j \leqslant k-1$, we have $T_i \sqsubseteq_{\mathsf{emb}} T_j$, which contradicts the badness of $\mathbf{T}$.

$\underbrace{T_1, \ldots, T_{k-1}}_{i}, \underbrace{G_k}_{j}, \mathbf{H}_{\geqslant l}$ For $1 \leqslant i \leqslant k-1$ and $j = k$, we have $T_i \sqsubseteq_{\mathsf{emb}} G_k$ and $G_k \sqsubseteq_{\mathsf{emb}} T_k$, where the latter is a direct consequence of the definitions. Hence, by transitivity, $T_i \sqsubseteq_{\mathsf{emb}} T_j$, which contradicts the badness of $\mathbf{T}$.

$\underbrace{T_1, \ldots, T_{k-1}}_{i}, G_k, \underbrace{\mathbf{H}_{\geqslant l}}_{j}$ For $1 \leqslant i \leqslant k-1$ and $j \geqslant l$, we have $H_j \notin G$' by construction, but then $H_j = G_m$ for some $m > k$ and thus $H_j \sqsubseteq_{\mathsf{emb}} T_m$. Together with $T_i \sqsubseteq_{\mathsf{emb}} H_j$, we obtain $T_i \sqsubseteq_{\mathsf{emb}} T_m$ by transitivity, which contradicts the badness of $\mathbf{T}$.

$T_1, \ldots, T_{k-1}, \underbrace{G_k, \mathbf{H}_{\geqslant l}}_{i,j}$ Hence for some $1 \leqslant i < j$, where $i, j \notin \{2, \ldots, l-1\}$, we have some $H_i \sqsubseteq_{\mathsf{emb}} H_j$, which contradicts the badness of $\mathbf{H}$.

We conclude $\mathbf{H}$ is a good sequence and $\sqsubseteq_{\mathsf{emb}}$ is wqo on G.

Next we consider the Tops of $\mathbf{T}$. Let these Tops be $\mathbf{f}$. By assumption, $\sqsubseteq$ is a wqo on $\mathsf{Tops}(\mathcal{F})$, and by Lemma 11, $\mathbf{f}$ contains a chain $\mathbf{f}_\phi$, i.e. $f_{\phi(i)} \sqsubseteq f_{\phi(i+1)}$ for all $i \geqslant 1$. We proved $\sqsubseteq_{\mathsf{emb}}$ to be a wqo on G. Hence we have $G_{\phi(i)} \sqsubseteq_{\mathsf{emb}} G_{\phi(j)}$ for some $1 \leqslant i < j$. It remains to be shown, that

$f_{\phi(i)} \sqsubseteq f_{\phi(j)}$ and $G_{\phi(i)} \sqsubseteq_{\mathsf{emb}} G_{\phi(j)}$ implies $T_{\phi(i)} \sqsubseteq_{\mathsf{emb}} T_{\phi(j)}$. We construct $T_{\phi(i)}$, and analogous $T_{\phi(j)}$, from $f_{\phi(i)} = (n_i, \mathsf{label}_{f\phi(i)}, \mathsf{succ}_{f\phi(i)})$ and $G_{\phi(i)} = (N_{G\phi(i)}, \mathsf{label}_{G\phi(i)}, \mathsf{succ}_{G\phi(i)})$ with $\mathsf{inlets}_{G\phi(i)}$. We have $N_{G\phi(i)} \cap \{n_i\} = \varnothing$. Then $T_{\phi(i)} = (N_{T\phi(i)}, \mathsf{label}_{T\phi(i)}, \mathsf{succ}_{T\phi(i)})$ where *(i)* the nodes $N_{T\phi(i)} := N_{G\phi(i)} \cup \{n_i\}$, *(ii)* $\mathsf{label}_{T\phi(i)} := \mathsf{label}_{G\phi(i)}$ extended by $\mathsf{label}_{T\phi(i)}(n_i) = \mathsf{label}_{f\phi(i)}(n_i)$, and *(iii)* $\mathsf{succ}_{T\phi(i)} := \mathsf{succ}_{G\phi(i)}$ extended by $\mathsf{succ}_{T\phi(i)}(n_i) = \mathsf{inlets}_{G\phi(i)}$.

We aim for $T_{\phi(i)} \sqsubseteq_{\mathsf{emb}} T_{\phi(j)}$ and therefore construct the morphism $m : T_{\phi(j)} \to T_{\phi(i)}$. From $G_{\phi(i)} \sqsubseteq_{\mathsf{emb}} G_{\phi(j)}$, we obtain a morphism $m_G : G_{\phi(j)} \to G_{\phi(i)}$. We set $m(n) = m_G(n)$ for $n \in G_{\phi_j}$, and $m(n_j) = n_i$. It remains to be shown that m fulfils Definition 8. Surjectivity of m follows directly from the surjectivity of m_G. Condition *(i)* holds for all nodes in m_G, and by $f_{\phi(i)} \sqsubseteq f_{\phi(j)}$ also for $\mathsf{root}(T_{\phi(j)}) = n_j$. For Condition *(ii)* we have to show: If $m(n_j) \rightharpoonup_{T_{\phi i}} n'_i = m(n'_j)$ then $n_j \rightharpoonup^+ n'_j$. By definition $n'_i \in \mathsf{inlets}_{G\phi(i)}$ and hence also $n'_i \in G_i$. By surjectivity of m_G exist $m_G(n'_j) = n'_i$. It remains to be shown that $n_j \rightharpoonup^+ n'_j$. By definition $n_j \rightharpoonup u_j$, where $u_j \in \mathsf{inlets}_{G\phi(j)}$. By definition of argument graph, all nodes in $G_{\phi(j)}$ are reachable from nodes in $\mathsf{inlets}_{G\phi(j)}$, and in particular $n_j \rightharpoonup u_j \rightharpoonup^* n'_j$.

Hence we found a $T_{\phi(i)} \sqsubseteq_{\mathsf{emb}} T_{\phi(j)}$, which contradicts the badness of $\mathbf{T}$. Therefore $\mathbf{T}$ is good and $\sqsubseteq_{\mathsf{emb}}$ is a wqo. $\square$

4 Conclusion and Discussion

Inspired by [12] we defined an embedding relation for the term graph rewriting flavour of [1, 2] and reproved Kruskal's Tree Theorem. As opposed to [12], which uses an encoding of Top to function symbols with different arities, we operate directly on term graphs. With a new definition of the embedding relation, based on the notion of morphism and taking sharing into account, and a new definition of arguments we finally showed Kruskal's Tree Theorem for term graphs: A well-quasi order on Tops, i.e. $\sqsubseteq$, induces a well-quasi order $\sqsubseteq_{\mathsf{emb}}$ on ground term graphs. The most important insight from our proof concerns the arguments of a term graph—or rather *the* argument. For a term structure we have several subterms as arguments. For a term graph structure it is beneficial to regard the arguments as only one single argument graph. This preserves sharing. Moreover a single argument simplifies the proof as extending the order to sequences, Higman's Lemma [8], can be omitted.

We conclude our work with a discussion and an outlook on future work. In particular, we investigate how we can define a simplification order based on $\sqsubseteq_{\mathsf{emb}}$. Finally, we inspect $\sqsubseteq_{\mathsf{emb}}$ with respect to the order of arguments.

In the term rewriting setting simplification orders are defined through the embedding relation. That is, a rewrite order $\prec$ is a *simplification order* if $\sqsubset_{\mathsf{emb}} \subseteq \prec$ [10]. Then, if we can orient the rules in a rewrite system with $\prec$, there are no infinite rewrite sequences. We try to directly transfer this idea to the term graph rewriting setting—but this is not sufficient, as the following example shows.

Example 13. We can orient the rule on the left with $\sqsupset_{\mathsf{emb}}$, but still may get an infinite rewrite sequence, as shown on the right.

$$\begin{tikzcd}[column sep=small]
& \mathsf{f} \arrow[dl] \arrow[dr] & & & \mathsf{f} \arrow[d, bend right] \arrow[d, bend left] & & & \mathsf{f} \arrow[dl] \arrow[dr] & & & \mathsf{f} \arrow[d, bend right] \arrow[d, bend left] & & \mathsf{f} \arrow[d, bend right] \arrow[d, bend left] & \\
\mathsf{a} & & \mathsf{a} & \sqsupset_{\mathsf{emb}} & \mathsf{a} & & \mathsf{a} & & \mathsf{a} & \to_{\mathcal{G}} & \mathsf{a} & \to_{\mathcal{G}} & \mathsf{a} & \cdots
\end{tikzcd}$$

Note, that this infinite rewrite sequence is not bad wrt. $\sqsubseteq_{\mathsf{emb}}$.

This problem is *not* caused by our definition of embedding, and also occurs in [12]. Rather, the reason is that from orientation of the rules, we cannot conclude orientation of all rewrite steps. However, it should be noted, that the definition of simplification order in [12] is indeed transferable to our presentation.

Definition 14 ([12]). Let $\sqsubseteq_{\mathsf{emb}}$ be the embedding relation induced by a precedence that is a wqo. A transitive relation $\prec$ is a *simplification order*, if *(i)* $\sqsubset_{\mathsf{emb}} \subset \prec$, and *(ii)* for all S and T, if $S \sqsubseteq_{\mathsf{emb}} T$ and $T \sqsubseteq_{\mathsf{emb}} S$ then $S \not\prec T$.

Then we can apply the proof in [12] to show the absence of infinite sequences in simplification orders based on $\sqsubseteq_{\mathsf{emb}}$. Finally, our Definition 8 of embedding is very permissive, and does not regard the order of the arguments.

Example 15. The two term graphs, representing the terms $\mathsf{f}(\mathsf{a},\mathsf{b})$ and $\mathsf{f}(\mathsf{b},\mathsf{a})$, are mutually embedded:

$$\begin{array}{ccc} & \mathsf{f} & \\ \swarrow & & \searrow \\ \mathsf{a} & & \mathsf{b} \end{array} \;\sqsubseteq_{\mathsf{emb}}\; \begin{array}{ccc} & \mathsf{f} & \\ \swarrow & & \searrow \\ \mathsf{b} & & \mathsf{a} \end{array} \qquad \text{but also} \qquad \begin{array}{ccc} & \mathsf{f} & \\ \swarrow & & \searrow \\ \mathsf{a} & & \mathsf{b} \end{array} \;\sqsupseteq_{\mathsf{emb}}\; \begin{array}{ccc} & \mathsf{f} & \\ \swarrow & & \searrow \\ \mathsf{b} & & \mathsf{a} \end{array}$$

In future work we seek to distill termination orders—similar to simplification orders—based on our rendering of Kruskal. There we will also clarify the implication of our notion of the embedding relation.

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