Recovering Commutation of Logically Constrained Rewriting and Equivalence Transformations

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Abstract

Logically constrained term rewriting is a relatively new rewriting formalism that naturally supports built-in data structures, such as integers and bit vectors. In the analysis of logically constrained term rewrite systems (LCTRSs), rewriting constrained terms plays a crucial role. However, this combines rewrite rule applications and equivalence transformations in a closely intertwined way. This intertwining makes it difficult to establish useful theoretical properties for this kind of rewriting and causes problems in implementations—namely, that impractically large search spaces are often required. To address this issue, we propose in this paper a novel notion of most general constrained rewriting, which operates on existentially constrained terms, a concept recently introduced by the authors. We define a class of left-linear, left-value-free LCTRSs that are general enough to simulate all left-linear LCTRSs and exhibit the desired key property: most general constrained rewriting commutes with equivalence. This property ensures that equivalence transformations can be deferred until after the application of rewrite rules, which helps mitigate the issue of large search spaces in implementations. In addition to that, we show that the original rewriting formalism on constrained terms can be embedded into our new rewriting formalism on existentially constrained terms. Thus, our results are expected to have significant implications for achieving correct and efficient implementations in tools operating on LCTRSs.

CCS Concepts

• **Theory of computation → Equational logic and rewriting**.

Keywords

Logically Constrained Term Rewrite System, Commutation, Equivalence Transformation, Constrained Term

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1 Introduction

The basic formalism of term rewriting is a purely syntactic computational model; due to its simplicity, it is one of the most extensively studied computational models. However, precisely because of this simplicity, it is often not suitable for applications that arise in practical areas, such as in programming languages, formal specifications, etc. One of the main issues of term rewriting and its real-world applications is that the basic formalism lacks painless treatment of built-in data structures, such as integers, bit vectors, etc.

Logically constrained term rewriting [7] is a relatively new extension of term rewriting that intends to overcome such weaknesses of the basic formalism, while keeping succinctness for theoretical analysis as a computational model. Rewrite rules, that are used to model computations, in logically constrained term rewrite systems (LCTRSs) are equipped with constraints over some arbitrary theory, e.g., linear integer arithmetic. Built-in data structures are represented via the satisfiability of constraints within a respective theory. Implementations in LCTRS tools are then able to check these constraints using SMT-solvers and therefore benefit from recent advances in the area of checking *satisfiability modulo theories*. Recent progress on the LCTRS formalism was for example made in confluence analysis [13, 15], (non-)termination analysis [5, 10], completion [18], rewriting induction [3, 8], algebraic semantics [1], and complexity analysis [19].

During the analysis of LCTRSs, not only rewriting of terms but also rewriting of constrained terms, called *constrained rewriting*, is frequently used. Here, a *constrained term* consists of a term and a constraint, which restricts the possibilities in which the term is instantiated. For example, $\mathsf{f}(x)\ [x > 2]$ is a constrained term (in LCTRS notation) which can be intuitively considered as a set of terms $\{\mathsf{f}(x) \mid x > 2\}$. Constrained rewriting is an integral part of many different analysis techniques. For example, in finding a specific joining sequence in confluence analysis you need to deal with two terms under a shared constraint, which results in working on constrained terms. In rewriting induction, rewriting of constrained terms is used for several of its inference steps.

Unfortunately, supporting constrained rewriting completely for LCTRSs is far from practical, as it involves heavy non-determinism. The situation gets even worse as it is also equipped with equivalence transformations before (and/or after) each rewrite step.

Example 1.1. Consider the rewrite rule $\rho\colon \mathsf{f}(x) \to \mathsf{g}(y)\ [x \geqslant 1 \land x + 1 \geqslant y]$ and a constrained term $\mathsf{f}(x)\ [x > 2]$. Because, for any $x > 2$, the instantiation $y := 3$ satisfies the constraint $x \geqslant 1 \land x + 1 \geqslant y$ of the rewrite rule, we obtain a rewrite step

$$\mathsf{f}(x)\ [x > 2] \to_\rho \mathsf{g}(3)\ [x > 2].$$

It is also possible to apply the following rewrite step, because the instantiation $y := x$ satisfies the constraint:

$$\mathsf{f}(x)\ [x > 2] \to_\rho \mathsf{g}(x)\ [x > 2]$$

Actually, by using equivalence transformations (denoted by $\sim$), different variations of rewrite steps are possible, e.g.:

$$\begin{aligned} \mathsf{f}(x)\ [x > 2] &\sim \mathsf{f}(x)\ [x > 2 \land 0 > y] \\ &\to_\rho \mathsf{g}(y)\ [x > 2 \land 0 > y] \\ &\sim \mathsf{g}(y)\ [0 > y] \end{aligned}$$

Note that the resulting constrained terms are often not equivalent, e.g. $\mathsf{g}(3)\ [x > 2] \nsim \mathsf{g}(x)\ [x > 2]$, as $\{\mathsf{g}(3)\} \neq \{\mathsf{g}(x) \mid x > 2\}$.

The question may arise whether restricting to rewriting without equivalence transformations is a good idea. However, it turns out that for some natural computations, we need an equivalence transformation prior to the actual rewrite step:

Example 1.2. Consider the rewrite rule $\rho\colon \mathsf{h}(x, y) \to \mathsf{g}(z)\ [(x + y) + 1 = z]$ and a constrained term $\mathsf{h}(x, y)\ [x < y]$. It is not possible to take any concrete value or variable of x, y for z, and hence the constrained term using the rule ρ cannot be rewritten. However, after the equivalence transformation, the rule becomes applicable:

$$\begin{aligned} \mathsf{h}(x, y)\ [x < y] &\sim \mathsf{h}(x, y)\ [x < y \land x + y + 1 = z] \\ &\to_\rho \mathsf{g}(z)\ [x < y \land x + y + 1 = z] \end{aligned}$$

Clearly, it is not feasible to support the full strength of such an equivalence relation in an implementation.

In this paper, we introduce a novel notion of most general constrained rewriting, which operates on existentially constrained terms, a concept recently introduced by the authors [16]. As seen in the example above, a key source of confusion is that the rewrite step heavily relies on variables in the constraint that do not appear in the term itself. The existentially constrained terms distinguish variables that appear solely in the constraint but not in the term itself by using existential quantifiers. Variables appearing only within the constraint are naturally not allowed to appear in any reduction of a term as they are bound to the scope of the constraint. It turns out that this novel way of rewriting covers the "most general part" of the original rewrite relation, which in practice usually suffices for the analysis of LCTRSs.

Additionally, it fulfills not only a form of uniqueness of reducts but also the commutation property of rewrite steps and equivalence transformations. These features are not supported by the original rewrite relation. The latter property about commutation is very important from an implementation perspective, because as a result one can move intermediate equivalence transformations in rewrite sequences to the end of the sequence. This property reduces huge search spaces for the computations of rewrite sequences. Coincidentally, LCTRS tools such as Ctrl [9], Crisys [4], and crest [14, 15] already implement similar approaches to deal with constrained rewriting. However, this has not formally been defined so far. Our results guarantee the correctness of the approaches in these implementations and provide the foundation for their correctness.

The remainder of the paper is organized as follows. After presenting the necessary background in Section 2, we introduce most general rewrite steps and prove its well-definedness in Section 3. In Sections 4 and 5 we focus on the relation between our new formalism of constrained rewriting and the current state-of-the-art. Then in Section 6, we show two useful properties of most general rewriting: uniqueness of reducts and commutation between rewrite steps and equivalence transformations on pattern-general existentially constrained terms. Subsequently we introduce left-value-free rules [6] within our setting and discuss their rewriting behavior in Section 7. In Section 8 we show a general commutation theorem between rewrite steps using left-value-free rules and the equivalence transformation. Before we conclude in Section 10, we discuss some related work in Section 9.

2 Preliminaries

In this section, we briefly recall the basic notions of LCTRSs [1, 7, 13, 15] and fix additional notations used throughout this paper. Familiarity with the basic notions of term rewriting is assumed (e.g. see [2, 11]).

Logically Constrained Terms. Our signature consists of a set $\mathcal{S}$ of sorts and a set $\mathcal{F}$ of function symbols, where each $f \in \mathcal{F}$ is attached with its sort declaration $f\colon \tau_1 \times \cdots \times \tau_n \to \tau_0$. As in [1], we assume that these sets can be partitioned into two disjoint sets, i.e., $\mathcal{S} = \mathcal{S}_{\mathrm{th}} \uplus \mathcal{S}_{\mathrm{te}}$ and $\mathcal{F} = \mathcal{F}_{\mathrm{th}} \uplus \mathcal{F}_{\mathrm{te}}$, where each $f\colon \tau_1 \times \cdots \times \tau_n \to \tau_0 \in \mathcal{F}_{\mathrm{th}}$ satisfies $\tau_i \in \mathcal{S}_{\mathrm{th}}$ for all $0 \leqslant i \leqslant n$. Elements of $\mathcal{S}_{\mathrm{th}}$ ($\mathcal{F}_{\mathrm{th}}$) and $\mathcal{S}_{\mathrm{te}}$ ($\mathcal{F}_{\mathrm{te}}$) are called theory sorts (symbols) and term sorts (symbols). The sets of variables and terms are denoted by $\mathcal{V}$ and $\mathcal{T}(\mathcal{F}, \mathcal{V})$. We assume a special sort $\mathsf{Bool} \in \mathcal{S}_{\mathrm{th}}$, and call the terms in $\mathcal{T}(\mathcal{F}_{\mathrm{th}}, \mathcal{V})^{\mathsf{Bool}}$ *logical constraints*. We denote the set of variables appearing in terms $t_1, \ldots, t_n$ by $\mathcal{V}(t_1, \ldots, t_n)$. Sometimes sequences of variables and terms are written as $\vec{x}$ and $\vec{t}$. The set of variables occurring in $\vec{x}$ is denoted by $\{\vec{x}\}$. The set of sequences of elements of a set T is denoted by T^*, such that $\vec{x} \in \mathcal{V}^*$.

The set of positions in a term t is denoted by $\mathcal{P}\mathrm{os}(t)$. The symbol and subterm occurring at a position $p \in \mathcal{P}\mathrm{os}(t)$ is denoted by $t(p)$ and $t|_p$, respectively. For $U \subseteq \mathcal{F} \cup \mathcal{V}$, we write $\mathcal{P}\mathrm{os}_U(t) = \{p \in \mathcal{P}\mathrm{os}(t) \mid t(p) \in U\}$ for positions with symbols in U. A term obtained from t by replacing subterms at positions $p_1, \ldots, p_n$ by the terms $t_1, \ldots, t_n$, having the same sort as $t|_{p_1}, \ldots, t|_{p_n}$, is written as $t[t_1, \ldots, t_n]_{p_1,\ldots,p_n}$ or just $t[t_1, \ldots, t_n]$ when no confusions arises. Sometimes we consider an expression obtained by replacing those subterms $t|_{p_1}, \ldots, t|_{p_n}$ in t by holes of the same sorts, which is called a multihole context and denoted by $t[\,]_{p_1,\ldots,p_n}$.

A sort-preserving function σ from $\mathcal{V}$ to $\mathcal{T}(\mathcal{F}, \mathcal{V})$ is called a substitution, where it is identified with its homomorphic extension $\sigma\colon \mathcal{T}(\mathcal{F}, \mathcal{V}) \to \mathcal{T}(\mathcal{F}, \mathcal{V})$. The domain of a substitution σ is given by $\mathcal{D}\mathrm{om}(\sigma) = \{x \in \mathcal{V} \mid x \neq \sigma(x)\}$. A substitution σ is written as $\sigma\colon U \to T$ if $\mathcal{D}\mathsf{om}(\sigma) \subseteq U$ and $\sigma(U) = \{\sigma(x) \mid x \in U\} \subseteq T$. For a set $U \subseteq \mathcal{V}$, a substitution $\sigma|_U$ is given by $\sigma|_U(x) = \sigma(x)$ if $x \in U$ and $\sigma|_U(x) = x$ otherwise. For substitutions σ_1, σ_2 such that $\mathcal{D}\mathsf{om}(\sigma_1) \cap \mathcal{D}\mathsf{om}(\sigma_2) = \varnothing$, the substitution $\sigma_1 \cup \sigma_2$ is given by $(\sigma_1 \cup \sigma_2)(x) = \sigma_i(x)$ if $x \in \mathcal{D}\mathsf{om}(\sigma_i)$ and $(\sigma_1 \cup \sigma_2)(x) = x$ otherwise. A substitution $\sigma\colon \{x_1, \ldots, x_n\} \to \{t_1, \ldots, t_n\}$ such that $\sigma(x_i) = t_i$ is denoted by $\{x_1 \mapsto t_1, \ldots, x_n \mapsto t_n\}$; for brevity sometimes we write just $\{\vec{x} \mapsto \vec{t}\}$. A bijective substitution $\sigma\colon \mathcal{V} \to \mathcal{V}$ is called a renaming, and its inverse is denoted by σ^{-1}.

A model over a sorted signature $\langle \mathcal{S}_{\text{th}}, \mathcal{F}_{\text{th}} \rangle$ consists of the two interpretations $\mathcal{I}$ for sorts and $\mathcal{J}$ for function symbols such that $\mathcal{I}(\tau)$ is a non-empty set of values and $\mathcal{J}$ assigns any function symbol $f\colon \tau_1 \times \cdots \times \tau_n \to \tau_0 \in \mathcal{F}_{\text{th}}$ to a function $\mathcal{J}(f)\colon \mathcal{I}(\tau_1) \times \cdots \times \mathcal{I}(\tau_n) \to \mathcal{I}(\tau_0)$. We assume a fixed model $\mathcal{M} = \langle \mathcal{I}, \mathcal{J} \rangle$ over the sorted signature $\langle \mathcal{S}_{\text{th}}, \mathcal{F}_{\text{th}} \rangle$ such that any element $a \in \mathcal{I}(\tau)$ appears as a constant a^τ in $\mathcal{F}_{\text{th}}$. These constants are called *values* and the set of all values is denoted by $\mathcal{V}\mathsf{al}$. For a term t, we define $\mathcal{V}\mathsf{al}(t) = \{t(p) \mid p \in \mathcal{P}\mathsf{os}_{\mathcal{V}\mathsf{al}}(t)\}$ and for a substitution γ, we define $\mathcal{V}\mathcal{D}\mathsf{om}(\gamma) = \{x \in \mathcal{V} \mid \gamma(x) \in \mathcal{V}\mathsf{al}\}$. Throughout this paper we assume the standard interpretation for the sort Bool $\in \mathcal{S}_{\text{th}}$, namely $\mathcal{I}(\mathsf{Bool}) = \mathbb{B} = \{\mathsf{true}, \mathsf{false}\}$, and the existence of necessary standard theory function symbols such as $\neg, \wedge, \Rightarrow, =^\tau$, etc.

A valuation ρ on the model $\mathcal{M} = \langle \mathcal{I}, \mathcal{J} \rangle$ is a mapping that assigns any $x^\tau \in \mathcal{V}$ with $\rho(x) \in \mathcal{I}(\tau)$. The interpretation of a term t in the model $\mathcal{M}$ over the valuation ρ is denoted by $[\![t]\!]_{\mathcal{M},\rho}$. For a logical constraint φ and a valuation ρ over the model $\mathcal{M}$, we write $\vDash_{\mathcal{M},\rho} \varphi$ if $[\![\varphi]\!]_{\mathcal{M},\rho} = \mathsf{true}$, and $\vDash_{\mathcal{M}} \varphi$ if $\vDash_{\mathcal{M},\rho} \varphi$ for all valuations ρ. For $X \subseteq \mathcal{V}$, a substitution γ is said to be X-valued if $\gamma(X) \subseteq \mathcal{V}\mathsf{al}$. We write $\gamma \vDash_{\mathcal{M}} \varphi$ (and say γ respects φ) if the substitution γ is $\mathcal{V}\mathsf{ar}(\varphi)$-valued (or equivalently, $\mathcal{V}\mathsf{ar}(\varphi) \subseteq \mathcal{V}\mathcal{D}\mathsf{om}(\gamma)$, as well as $\gamma(\mathcal{V}\mathsf{ar}(\varphi)) \subseteq \mathcal{V}\mathsf{al}$) and $\vDash_{\mathcal{M}} \varphi\gamma$. If no confusion arises then we drop the subscript $\mathcal{M}$ in these notations.

Rewriting Logically Constrained Terms. The main focus in this paper lies on rewriting constrained terms, hence we focus solely on introducing this concept. A *constrained term* is a tuple of a term s and a logical constraint φ written as $s\ [\varphi]$. Two constrained terms $s\ [\varphi]$, $t\ [\psi]$ are equivalent, written as $s\ [\varphi] \sim t\ [\psi]$, if for all substitutions σ with $\sigma \vDash \varphi$ there exists a substitution γ with $\gamma \vDash \psi$ such that $s\sigma = t\gamma$, and vice versa.

A constrained rewrite rule is a triple $\ell \to r\ [\varphi]$ of terms ℓ, r with the same sort, and a logical constraint φ. The set of *logical variables* of the rule is given by $\mathcal{L}\mathcal{V}\mathsf{ar}(\ell \to r\ [\varphi]) = (\mathcal{V}\mathsf{ar}(r) \setminus \mathcal{V}\mathsf{ar}(\ell)) \cup \mathcal{V}\mathsf{ar}(\varphi)$. A *logically constrained term rewrite system* (LCTRS, for short) consists of a signature $\Sigma = \langle \mathcal{S}_{\text{th}} \uplus \mathcal{S}_{\text{te}}, \mathcal{F}_{\text{th}} \uplus \mathcal{F}_{\text{te}} \rangle$, a model $\mathcal{M}$ over $\Sigma_{\text{th}} = \langle \mathcal{S}_{\text{th}}, \mathcal{F}_{\text{th}} \rangle$ (which induces the set $\mathcal{V}\mathsf{al} \subseteq \mathcal{F}_{\text{th}}$ of values) and a set $\mathcal{R}$ of constrained rewrite rules over the signature Σ.

Consider a constrained term $s\ [\varphi]$ with a satisfiable logical constraint. Let $\ell \to r\ [\psi]$ be a constrained rewrite rule. If there exists a position $p \in \mathcal{P}\mathsf{os}_{\mathcal{F}}(s)$, a substitution σ with $\ell\sigma = s|_p$, $\sigma(x) \in \mathcal{V}\mathsf{al} \cup \mathcal{V}\mathsf{ar}(\varphi)$ for all $x \in \mathcal{L}\mathcal{V}\mathsf{ar}(\ell \to r\ [\psi])$, and $\vDash_{\mathcal{M}} \varphi \Rightarrow \psi\sigma$, then we have the rewrite step $s\ [\varphi] \to s[r\sigma]_p\ [\varphi]$. The full rewrite relation $\tilde{\to}$ on constrained terms is defined by $\tilde{\to}\ =\ \sim \cdot \to \cdot \sim$. In [15], it is shown that *calculation steps* defined in [7] can be integrated into this rewrite relation by using the constrained rules $\mathcal{R}_{\text{ca}} = \{f(x_1, \ldots, x_n) \to y\ [y = f(x_1, \ldots, x_n)] \mid f \in \mathcal{F}_{\text{th}} \setminus \mathcal{V}\mathsf{al}\}$, including some additional initial $\sim$-steps.

Existentially Constrained Terms and Their Equivalence. We now explain existentially constrained terms and characterization of their equivalence based on [16].

An *existential constraint* is a pair $\langle \vec{x}, \varphi \rangle$ of a sequence of variables $\vec{x}$ and a constraint φ, written as $\exists \vec{x}.\ \varphi$, such that $\{\vec{x}\} \subseteq \mathcal{V}\mathsf{ar}(\varphi)$. The sets of *free* and *bound* variables of $\exists \vec{x}.\ \varphi$ are given by $\mathcal{F}\mathcal{V}\mathsf{ar}(\exists \vec{x}.\ \varphi) = \mathcal{V}\mathsf{ar}(\varphi) \setminus \{\vec{x}\}$ and $\mathcal{B}\mathcal{V}\mathsf{ar}(\exists \vec{x}.\ \varphi) = \{\vec{x}\}$. We write $\vDash_{\mathcal{M},\rho} \exists \vec{x}.\ \varphi$ if there exists $\vec{v} \in \mathcal{V}\mathsf{al}^*$ such that $\vDash_{\mathcal{M},\rho} \varphi\kappa$, where $\kappa = \{\vec{x} \mapsto \vec{v}\}$. An existential constraint $\exists \vec{x}.\ \varphi$ is said to be *valid*, written as $\vDash_{\mathcal{M}} \exists \vec{x}.\ \varphi$, if $\vDash_{\mathcal{M},\rho} \exists \vec{x}.\ \varphi$ for any valuation ρ. An existential constraint $\exists \vec{x}.\ \varphi$ is said to be *satisfiable* if $\vDash_{\mathcal{M},\rho} \exists \vec{x}.\ \varphi$ for some valuation ρ. For any substitution σ, we write $\sigma \vDash_{\mathcal{M}} \exists \vec{x}.\ \varphi$ (and say σ respects $\exists \vec{x}.\ \varphi$) if $\sigma(\mathcal{F}\mathcal{V}\mathsf{ar}(\exists \vec{x}.\ \varphi)) \subseteq \mathcal{V}\mathsf{al}$ and $\vDash_{\mathcal{M}} (\exists \vec{x}.\ \varphi)\sigma$. An *existentially constrained term* is a triple $\langle X, s, \exists \vec{x}.\ \varphi \rangle$, written as $\Pi X.\ s\ [\exists \vec{x}.\ \varphi]$, of a set X of variables, a term s, and an existential constraint $\exists \vec{x}.\ \varphi$ such that $\mathcal{F}\mathcal{V}\mathsf{ar}(\exists \vec{x}.\ \varphi) \subseteq X \subseteq \mathcal{V}\mathsf{ar}(s)$ and $\mathcal{B}\mathcal{V}\mathsf{ar}(\exists \vec{x}.\ \varphi) \cap \mathcal{V}\mathsf{ar}(s) = \varnothing$. Variables in X are called *logical variables* (of $\Pi X.\ s\ [\exists \vec{x}.\ \varphi]$). An existentially constrained term $\Pi X.\ s\ [\exists \vec{x}.\ \varphi]$ is said to be *satisfiable* if $\exists \vec{x}.\ \varphi$ is satisfiable. An existentially constrained term $\Pi X.\ s\ [\exists \vec{x}.\ \varphi]$ is said to be *subsumed by* an existentially constrained term $\Pi Y.\ t\ [\exists \vec{y}.\ \psi]$, denoted by $\Pi X.\ s\ [\exists \vec{x}.\ \varphi] \subsetneqq \Pi Y.\ t\ [\exists \vec{y}.\ \psi]$, if for all X-valued substitutions σ with $\sigma \vDash_{\mathcal{M}} \exists \vec{x}.\ \varphi$ there exists a Y-valued substitution γ with $\gamma \vDash_{\mathcal{M}} \exists \vec{y}.\ \psi$ such that $s\sigma = t\gamma$. Existentially constrained terms $\Pi X.\ s\ [\exists \vec{x}.\ \varphi]$, $\Pi Y.\ t\ [\exists \vec{y}.\ \psi]$ are said to be *equivalent*, denoted by $\Pi X.\ s\ [\exists \vec{x}.\ \varphi] \sim \Pi Y.\ t\ [\exists \vec{y}.\ \psi]$, if $\Pi X.\ s\ [\exists \vec{x}.\ \varphi] \subsetneqq \Pi Y.\ t\ [\exists \vec{y}.\ \psi]$ and $\Pi X.\ s\ [\exists \vec{x}.\ \varphi] \supsetneqq \Pi Y.\ t\ [\exists \vec{y}.\ \psi]$. Throughout this paper, we will use three characterizations of equivalence for existentially constrained terms, presented in [16]. The first one covers variants under renaming.

Proposition 2.1 ([16]). *Let δ be a renaming. Let $\Pi X.\ s\ [\exists \vec{x}.\ \varphi]$, $\Pi Y.\ t\ [\exists \vec{y}.\ \psi]$ be satisfiable existentially constrained terms such that $s\delta = t$. Then, $\Pi X.\ s\ [\exists \vec{x}.\ \varphi] \sim \Pi Y.\ t\ [\exists \vec{y}.\ \psi]$ iff $\delta(X) = Y$ and $\vDash_{\mathcal{M}} (\exists \vec{x}.\ \varphi)\delta \Leftrightarrow (\exists \vec{y}.\ \psi)$.*

The second one considers those that are *pattern-general*, where an existentially constrained term $\Pi X.\ s\ [\exists \vec{x}.\ \varphi]$ is *pattern-general* if s is X-linear and $\mathcal{V}\mathsf{al}(s) = \varnothing$ [16]. Any existentially constrained term can be translated to equivalent pattern-general existentially constrained term by the following translation: $\mathrm{PG}(\Pi X.\ s\ [\exists \vec{x}.\ \varphi]) = \Pi Y.\ t\ [\exists \vec{y}.\ \psi]$ where $t = s[w_1, \ldots, w_n]_{p_1,\ldots,p_n}$ with $\mathcal{P}\mathsf{os}_{X \cup \mathcal{V}\mathsf{al}}(s) = \{p_1, \ldots, p_n\}$ and pairwise distinct fresh variables $w_1, \ldots, w_n$, $Y = \{w_1, \ldots, w_n\}$, $\{\vec{y}\} = \{\vec{x}\} \cup X$, and $\psi = (\varphi \wedge \bigwedge_{i=1}^n (s|_{p_i} = w_i))$.

Proposition 2.2 ([16]). *Let $\Pi X.\ s\ [\exists \vec{x}.\ \varphi], \Pi Y.\ t\ [\exists \vec{y}.\ \psi]$ be satisfiable pattern-general existentially constrained terms. Then, we have $\Pi X.\ s\ [\exists \vec{x}.\ \varphi] \sim \Pi Y.\ t\ [\exists \vec{y}.\ \psi]$ iff there exists a renaming ρ such that $s\rho = t$, $\rho(X) = Y$, and $\vDash_{\mathcal{M}} (\exists \vec{x}.\ \varphi)\rho \Leftrightarrow (\exists \vec{y}.\ \psi)$.*

The last characterization does not impose any restriction on the existentially constrained terms. In order to present this, we need a couple of notions and results from [16]. Let $\Pi X.\ s\ [\exists \vec{x}.\ \varphi]$ be an existentially constrained term. Then, we define a binary relation $\sim_{\mathcal{P}\mathsf{os}_{X \cup \mathcal{V}\mathsf{al}}(s)}$ over the positions in $\mathcal{P}\mathsf{os}_{X \cup \mathcal{V}\mathsf{al}}(s)$ as follows: $p \sim_{\mathcal{P}\mathsf{os}_{X \cup \mathcal{V}\mathsf{al}}(s)} q$ iff $\vDash_{\mathcal{M}} ((\exists \vec{x}.\ \varphi) \Rightarrow s|_p = s|_q)$. Then, $\sim_{\mathcal{P}\mathsf{os}_{X \cup \mathcal{V}\mathsf{al}}(s)}$ is an equivalence relation over the positions in $\mathcal{P}\mathsf{os}_{X \cup \mathcal{V}\mathsf{al}}(s)$. The equivalence class of a position $p \in \mathcal{P}\mathsf{os}_{X \cup \mathcal{V}\mathsf{al}}(s)$ w.r.t. $\sim_{\mathcal{P}\mathsf{os}_{X \cup \mathcal{V}\mathsf{al}}(s)}$ is denoted by $[p]_{\sim_{\mathcal{P}\mathsf{os}_{X \cup \mathcal{V}\mathsf{al}}(s)}}$. If it is clear from the context then we may simply denote it by $[p]_\sim$. We further denote the representative of $[p]_\sim$ by $\hat{p}$. Let $\Pi X.\ s\ [\exists\vec{x}.\ \varphi]$ be a satisfiable existentially constrained term. We define $\mathcal{P}\mathrm{os}_{\mathcal{V}\mathrm{al}!}(s) = \{p \in \mathcal{P}\mathrm{os}_{X\cup\mathcal{V}\mathrm{al}}(s) \mid$ there exists $v \in \mathcal{V}\mathrm{al}$ such that $\models_\mathcal{M} ((\exists\vec{x}.\ \varphi) \Rightarrow (s|_p = v))\}$. For each $p \in \mathcal{P}\mathrm{os}_{\mathcal{V}\mathrm{al}!}(s)$, there exists a unique value v such that $\mathcal{M} \models (\exists\vec{x}.\ \varphi) \Rightarrow (s|_p = v)$ [16]; we denote such a v by $\mathcal{V}\mathrm{al}!(p)$. We then define a *representative substitution* $\mu_X : X \to X \cup \mathcal{V}\mathrm{al}$ of $\Pi X.\ s\ [\exists\vec{x}.\ \varphi]$ as follows:

$$\mu_X(z) = \begin{cases} \mathcal{V}\mathrm{al}!(p) & \text{if } s(p) = z \text{ for some } p \in \mathcal{P}\mathrm{os}_{\mathcal{V}\mathrm{al}!}(s), \\ s(\hat{p}) & \text{otherwise,} \end{cases}$$

where $\hat{p}$ is the representative of the equivalence class $[p]_\sim$.

Proposition 2.3 ([16]). *Let $\Pi X.\ s\ [\exists\vec{x}.\ \varphi], \Pi Y.\ t\ [\exists\vec{y}.\ \psi]$ be satisfiable existentially constrained terms. Then, $\Pi X.\ s\ [\exists\vec{x}.\ \varphi] \sim \Pi Y.\ t\ [\exists\vec{y}.\ \psi]$ iff the following statements hold:*

1. *$\mathcal{P}\mathrm{os}_{X\cup\mathcal{V}\mathrm{al}}(s) = \mathcal{P}\mathrm{os}_{Y\cup\mathcal{V}\mathrm{al}}(t)$ $(= \{p_1, \ldots, p_n\})$,*
2. *there exists a renaming $\rho\colon \mathcal{V}\mathrm{ar}(s) \setminus X \to \mathcal{V}\mathrm{ar}(t) \setminus Y$ such that $\rho(s[\,]_{p_1,\ldots,p_n}) = t[\,]_{p_1,\ldots,p_n}$,*
3. *for any $i, j \in \{1, \ldots, n\}$, $\models_\mathcal{M} (\exists\vec{x}.\ \varphi) \Rightarrow (s|_{p_i} = s|_{p_j})$ iff $\models_\mathcal{M} (\exists\vec{y}.\ \psi) \Rightarrow (t|_{p_i} = t|_{p_j})$,*
4. *for any $i \in \{1, \ldots, n\}$ and $v \in \mathcal{V}\mathrm{al}$, $\models_\mathcal{M} (\exists\vec{x}.\ \varphi) \Rightarrow (s|_{p_i} = v)$ iff $\models_\mathcal{M} (\exists\vec{y}.\ \psi) \Rightarrow (t|_{p_i} = v)$, and*
5. *let $\sim\ = \sim_{\mathcal{P}\mathrm{os}_{X\cup\mathcal{V}\mathrm{al}}(s)} = \sim_{\mathcal{P}\mathrm{os}_{X\cup\mathcal{V}\mathrm{al}}(t)}$, and μ_X, μ_Y be representative substitutions of $\Pi X.\ s\ [\exists\vec{x}.\ \varphi]$ and $\Pi Y.\ t\ [\exists\vec{y}.\ \psi]$, respectively, based on the same representative for each equivalence class $[p_i]_\sim$ $(1 \leqslant i \leqslant n)$, $\theta = \{\langle s|_{p_i}, t|_{p_i}\rangle \mid 1 \leqslant i \leqslant n\}$, $\tilde{X} = \hat{X} \cap X$, and $\tilde{Y} = \hat{Y} \cap Y$. Then, $\models_\mathcal{M} (\exists\vec{x}.\ \varphi)\mu_X\theta|_{\tilde{X}} \Leftrightarrow (\exists\vec{y}.\ \psi)\mu_Y$, with the renaming $\theta|_{\tilde{X}}\colon \tilde{X} \to \tilde{Y}$.*

3 Most General Constrained Rewriting

In this section, we present the definition of most general rewriting and prove its well-definedness. Furthermore, we also show some properties needed in later sections.

We start by devising the notion of constrained rewrite rules tailored to our needs. Similar to constrained equations [1], we attach the respective set of logical variables explicitly to constrained rewrite rules. Previously to this, rewriting relied heavily on the notion of logical variables ($\mathcal{LV}\mathrm{ar}(\ldots)$) for this.

Definition 3.1 (Constrained Rewrite Rule). A *constrained rewrite rule* is a quadruple of two terms ℓ, r, a set of variables Z and a logical constraint π, written as $\Pi Z.\ \ell \to r\ [\pi]$, where ℓ and r are of the same sort and Z satisfies $(\mathcal{V}\mathrm{ar}(r) \setminus \mathcal{V}\mathrm{ar}(\ell)) \cup \mathcal{V}\mathrm{ar}(\pi) \subseteq Z \subseteq \mathcal{V}\mathrm{ar}(\ell, r, \pi)$. When no confusion arises, we abbreviate constrained rewrite rules by *constrained rules* or just *rules*; sometimes we attach a label to it, as $\rho\colon \Pi Z.\ \ell \to r\ [\pi]$, in order to ease referencing it. Let $\rho\colon \Pi Z.\ \ell \to r\ [\pi]$ be a constrained rule. Then, ρ is *left-linear* if ℓ is linear; the set of *extra-variables* of ρ is given by $\mathcal{E}x\mathcal{V}\mathrm{ar}(\rho) = \mathcal{V}\mathrm{ar}(r) \setminus \mathcal{V}\mathrm{ar}(\ell)$[1].

It follows naturally, that a constrained rule $\ell \to r\ [\varphi]$ as defined previously can easily be transformed into $\Pi X.\ \ell \to r\ [\varphi]$ by taking $X = \mathcal{LV}\mathrm{ar}(\ell \to r\ [\varphi])$ $(:= \mathcal{V}\mathrm{ar}(\pi) \cup (\mathcal{V}\mathrm{ar}(r) \setminus \mathcal{V}\mathrm{ar}(\ell)))$. We also do not impose the constraint $\ell|_\epsilon \in \mathcal{F}_{\mathrm{te}}$ as it is done in [7], in order to deal with calculation steps via rule steps as in [15].

[1]Do not confuse the notation $\mathcal{E}x\mathcal{V}\mathrm{ar}(\rho)$ with $\mathcal{EV}\mathrm{ar}(\rho) = \mathcal{V}\mathrm{ar}(r) \setminus (\mathcal{V}\mathrm{ar}(\ell) \cup \mathcal{V}\mathrm{ar}(\pi))$ from [13, 15].

Example 3.2. For example, $\Pi\{x, y\}.\ \mathsf{f}(x) \to \mathsf{g}(y)\ [x \geqslant 1]$ and $\Pi\{x\}.\ \mathsf{h}(x, y) \to \mathsf{g}(y)\ [x \geqslant 1]$ are constrained rewrite rules. However, none of these are: $\Pi\{x, y, z\}.\ \mathsf{f}(x) \to \mathsf{g}(y)\ [x \geqslant 1]$ (z does not appear in $\mathcal{V}\mathrm{ar}(\mathsf{f}(x), \mathsf{g}(y), x \geqslant 1)$) and $\Pi\{x\}.\ \mathsf{f}(x) \to \mathsf{g}(y)\ [x \geqslant 1]$ ($y \in \mathcal{V}\mathrm{ar}(r) \setminus \mathcal{V}\mathrm{ar}(\ell)$ does not appear in $\{x\}$). The rewrite rule $\rho\colon \mathsf{f}(x) \to \mathsf{g}(y)\ [x \geqslant 1 \wedge x + 1 \geqslant y]$ in Theorem 1.1 is written as $\Pi\{x, y\}.\ \mathsf{f}(x) \to \mathsf{g}(y)\ [x \geqslant 1 \wedge x + 1 \geqslant y]$.

It turns out that certain complications arise if we deal with non-left-linear rules, *hence in the following we consider only left-linear constrained rules and leave the extension to non-left-linear rules for future work.* Before presenting the definition of most general constrained rewrite steps, we introduce the notion of ρ-redex for a rule ρ.

Definition 3.3 (ρ-Redex). Let $\Pi X.\ s\ [\exists\vec{x}.\ \varphi]$ be a satisfiable existentially constrained term. Suppose that $\rho\colon \Pi Z.\ \ell \to r\ [\pi]$ is a left-linear constrained rule satisfying $\mathcal{V}\mathrm{ar}(\rho) \cap \mathcal{V}\mathrm{ar}(s, \varphi) = \emptyset$. We say $\Pi X.\ s\ [\exists\vec{x}.\ \varphi]$ *has a ρ-redex at position $p \in \mathcal{P}\mathrm{os}(s)$ using substitution γ* if the following is satisfied: (1.) $\mathcal{D}\mathrm{om}(\gamma) = \mathcal{V}\mathrm{ar}(\ell)$, (2.) $s|_p = \ell\gamma$, (3.) $\gamma(x) \in \mathcal{V}\mathrm{al} \cup X$ for all $x \in \mathcal{V}\mathrm{ar}(\ell) \cap Z$, and (4.) $\models_\mathcal{M} (\exists\vec{x}.\ \varphi) \Rightarrow (\exists\vec{z}.\ \pi\gamma)$, where $\{\vec{z}\} = \mathcal{V}\mathrm{ar}(\pi) \setminus \mathcal{V}\mathrm{ar}(\ell)$.

Example 3.4. Consider a constrained rule $\rho\colon \Pi\{x', y'\}.\ \mathsf{f}(x') \to \mathsf{g}(y')\ [x' \geqslant 0 \wedge y' > x']$. The constrained term $\Pi\{x\}.\ \mathsf{f}(x)\ [x > 0]$ has a ρ-redex at the root position using the substitution $\gamma = \{x' \mapsto x\}$.

Definition 3.5 (Most General Rewrite Steps for Constrained Terms). Let $\Pi X.\ s\ [\exists\vec{x}.\ \varphi]$ be a satisfiable existentially constrained term. Suppose that $\rho\colon \Pi Z.\ \ell \to r\ [\pi]$ is a left-linear constrained rule satisfying $\mathcal{V}\mathrm{ar}(\rho) \cap \mathcal{V}\mathrm{ar}(s, \varphi) = \emptyset$. Suppose $\Pi X.\ s\ [\exists\vec{x}.\ \varphi]$ has a ρ-redex at position $p \in \mathcal{P}\mathrm{os}(s)$ using substitution γ. Then we have a rewrite step $\Pi X.\ s\ [\exists\vec{x}.\ \varphi] \to_\rho \Pi Y.\ t\ [\exists\vec{y}.\ \psi]$ (or $\Pi X.\ s\ [\exists\vec{x}.\ \varphi] \to^p_{\rho,\gamma} \Pi Y.\ t\ [\exists\vec{y}.\ \psi]$ with explicit p and γ) where (1.) $t = s[r\gamma]$, (2.) $\psi = \varphi \wedge \pi\gamma$, (3.) $\{\vec{y}\} = \mathcal{V}\mathrm{ar}(\psi) \setminus \mathcal{V}\mathrm{ar}(t)$, and (4.) $Y = \mathcal{E}x\mathcal{V}\mathrm{ar}(\rho) \cup (X \cap \mathcal{V}\mathrm{ar}(t))$. We call this notion *most general rewrite steps*; the reason behind this naming will become clear in the remainder of the paper.

Example 3.6 (Cont'd of Theorem 3.4). Observe that $\gamma(x') = x \in \{x\}$ and $\models_\mathcal{M} x > 0 \Rightarrow (\exists y'.\ x \geqslant 0 \wedge y' > x)$. Therefore, we obtain a rewrite step

$$\begin{array}{l} \Pi\{x\}.\ \mathsf{f}(x)\ [x > 0] \\ \quad \to_{\rho,\gamma} \Pi\{y'\}.\ \mathsf{g}(y')\ [\exists x.\ x > 0 \wedge x \geqslant 0 \wedge y' > x] \end{array}$$

Note that $(x' \geqslant 0 \wedge y' > x')\gamma = (x \geqslant 0 \wedge y' > x)$, $\mathcal{V}\mathrm{ar}(x \geqslant 0 \wedge y' > x) \setminus \mathcal{V}\mathrm{ar}(\mathsf{g}(y')) = \{x\}$, and $\mathcal{E}x\mathcal{V}\mathrm{ar}(\rho) \cup (\{x\} \cap \mathcal{V}\mathrm{ar}(\mathsf{g}(y'))) = \{y'\}$.

In a constrained rewrite step $\Pi X.\ s\ [\exists\vec{x}.\ \varphi] \to \Pi Y.\ t\ [\exists\vec{y}.\ \psi]$, variables of X will sometimes either be moved to Y or $\{\vec{y}\}$, or even be eliminated such that they do not appear in $\Pi Y.\ t\ [\exists\vec{y}.\ \psi]$ anymore. The latter behavior may happen in rewrite rules with eliminating variables, i.e., where $\mathcal{V}\mathrm{ar}(\ell) \setminus \mathcal{V}\mathrm{ar}(r) \neq \emptyset$. The following example illustrates such situations.

Example 3.7. Let $\rho\colon \Pi\{v', w', x', y', z'\}.\, \mathsf{f}(v', w', x', y', z') \to v'\ [y' \geqslant 0] (= \Pi Z.\, \ell \to r\ [\pi])$ be a constrained rule. Consider a rewrite step

$$\begin{array}{rcl} \Pi X.\ s\ [\exists \vec{x}.\ \varphi] & = & \Pi\{v,w,x,y,z\}.\ \mathsf{g}(\mathsf{f}(v,w,x,y,z),w)\ [x \geqslant 0] \\ & \to^1_{\rho,\gamma} & \Pi\{v,w\}.\ \mathsf{g}(v,w)\ [\exists x,y.\ x > 1 \wedge y \geqslant 0] \\ & = & \Pi Y.\ t\ [\exists \vec{y}.\ \psi] \end{array}$$

where and $\gamma = \{v' \mapsto v, \ldots, z' \mapsto z\}$. We trace now where the variables in X move during this rewrite step:

- We have $v \in Y$ because $v' \in \mathcal{V}\mathrm{ar}(r)$, $\gamma(v') = v$ and $v \in \mathcal{V}\mathrm{ar}(t)$.
- We have $w \in Y$ because $s|_2 = t|_2 = w$, hence $w \in \mathcal{V}\mathrm{ar}(t)$.
- We have $x \in \{\vec{y}\}$. Note $x \in \mathcal{V}\mathrm{ar}(s)$, but $x \notin \mathcal{V}\mathrm{ar}(r\gamma)$ and $x \notin \mathcal{V}\mathrm{ar}(t)$; however, since $x \in \mathcal{V}\mathrm{ar}(\varphi)$, we have $x \in \{\vec{y}\}$.
- We have $y \in \{\vec{y}\}$; note that $y \in \mathcal{V}\mathrm{ar}(s)$, but $y \notin \mathcal{V}\mathrm{ar}(r\gamma)$, $y \notin \mathcal{V}\mathrm{ar}(t)$ and $y \notin \mathcal{V}\mathrm{ar}(\varphi)$; however, $y \in \mathcal{V}\mathrm{ar}(\pi\gamma)$ and hence $y \in \mathcal{V}\mathrm{ar}(\psi)$.
- We have $z \notin Y \cup \{\vec{y}\}$; although $z \in \mathcal{V}\mathrm{ar}(s)$, we have that $z \notin \mathcal{V}\mathrm{ar}(t)$, $z \notin \mathcal{V}\mathrm{ar}(r)$ and $z \notin \mathcal{V}\mathrm{ar}(\psi)$.

In the following we show that rewriting existentially constrained terms is well-defined, in the sense that each rewrite step results in an existentially constrained term. Before that we give certain characterizations which are used in later proofs; proofs of these lemmas are presented in the full version of this paper [17].

Lemma 3.8 (Characterization of Free Variables in Reducts). *Let $\Pi X.\ s\ [\exists \vec{x}.\ \varphi]$ be a satisfiable existentially constrained term. Suppose that $\rho\colon \Pi Z.\ \ell \to r\ [\pi]$ is a left-linear constrained rule satisfying $\mathcal{V}\mathrm{ar}(\rho) \cap \mathcal{V}\mathrm{ar}(s,\varphi) = \varnothing$. If $\Pi X.\ s\ [\exists \vec{x}.\ \varphi] \to_\rho \Pi Y.\ t\ [\exists \vec{y}.\ \psi]$ then $\mathcal{F}\mathcal{V}\mathrm{ar}(\exists \vec{y}.\ \psi) \subseteq \mathcal{E}x\mathcal{V}\mathrm{ar}(\rho) \cup (X \cap \mathcal{V}\mathrm{ar}(t))$.*

Lemma 3.9 (Characterization of Bound Variables). *Suppose the satisfiable existentially constrained term $\Pi X.\ s\ [\exists \vec{x}.\ \varphi]$ such that $\Pi X.\ s\ [\exists \vec{x}.\ \varphi] \to_{\rho,\gamma} \Pi Y.\ t\ [\exists \vec{y}.\ \psi]$, where $\rho\colon \Pi Z.\ \ell \to r\ [\pi]$ is a left-linear constrained rewrite rule satisfying $\mathcal{V}\mathrm{ar}(\rho) \cap \mathcal{V}\mathrm{ar}(s,\varphi) = \varnothing$. Then the following statements hold:*

1. *$\mathcal{B}\mathcal{V}\mathrm{ar}(\exists \vec{x}.\ \varphi) \subseteq \mathcal{B}\mathcal{V}\mathrm{ar}(\exists \vec{y}.\ \psi)$.*
2. *$\mathcal{B}\mathcal{V}\mathrm{ar}(\exists \vec{x}.\ \varphi) \cap \mathcal{V}\mathrm{ar}(l\gamma, r\gamma, \pi\gamma) = \varnothing$.*
3. *$Y \cup \mathcal{B}\mathcal{V}\mathrm{ar}(\exists \vec{y}.\ \psi) = \mathcal{E}x\mathcal{V}\mathrm{ar}(\rho) \cup (X \cap \mathcal{V}\mathrm{ar}(t)) \cup \mathcal{V}\mathrm{ar}(\psi)$.*

We are ready to show that our rewrite steps are well-defined.

Theorem 3.10 (Well-Definedness of Rewrite Steps). *Let ρ be a left-linear constrained rewrite rule and $\Pi X.\ s\ [\exists \vec{x}.\ \varphi]$ a satisfiable existentially constrained term such that $\mathcal{V}\mathrm{ar}(\rho) \cap \mathcal{V}\mathrm{ar}(s,\varphi) = \varnothing$. If $\Pi X.\ s\ [\exists \vec{x}.\ \varphi] \to_\rho \Pi Y.\ t\ [\exists \vec{y}.\ \psi]$ then $\Pi Y.\ t\ [\exists \vec{y}.\ \psi]$ is a satisfiable existentially constrained term.*

Proof. Suppose $\Pi X.\ s\ [\exists \vec{x}.\ \varphi] \to^p_{\rho,\gamma} \Pi Y.\ t\ [\exists \vec{y}.\ \psi]$, where $\rho\colon \Pi Z.\ \ell \to r\ [\pi]$. Then we have (1) $\mathcal{D}\mathrm{om}(\gamma) = \mathcal{V}\mathrm{ar}(\ell)$, (2) $s|_p = \ell\gamma$, (3) $\gamma(x) \in \mathcal{V}\mathrm{al} \cup X$ for any $x \in \mathcal{V}\mathrm{ar}(\ell) \cap Z$, (4) $\vDash_\mathcal{M} (\exists \vec{x}.\ \varphi) \Rightarrow (\exists \vec{z}.\ \pi\gamma)$, where $\{\vec{z}\} = \mathcal{V}\mathrm{ar}(\pi) \setminus \mathcal{V}\mathrm{ar}(\ell)$, and $t = s[r\gamma]$, $\psi = \varphi \wedge \pi\gamma$, $\{\vec{y}\} = \mathcal{V}\mathrm{ar}(\psi) \setminus \mathcal{V}\mathrm{ar}(t)$, and $Y = \mathcal{E}x\mathcal{V}\mathrm{ar}(\rho) \cup (X \cap \mathcal{V}\mathrm{ar}(t))$. Clearly, $\exists \vec{y}.\ \psi$ is an existential quantified constraint.

We show $\mathcal{F}\mathcal{V}\mathrm{ar}(\exists \vec{y}.\ \psi) \subseteq Y \subseteq \mathcal{V}\mathrm{ar}(t)$ and $\mathcal{B}\mathcal{V}\mathrm{ar}(\exists \vec{y}.\ \psi) \cap \mathcal{V}\mathrm{ar}(t) = \varnothing$. From Theorem 3.8, $\mathcal{F}\mathcal{V}\mathrm{ar}(\exists \vec{y}.\ \psi) \subseteq \mathcal{E}x\mathcal{V}\mathrm{ar}(\rho) \cup (X \cap \mathcal{V}\mathrm{ar}(t)) = Y$ follows. Since $\{\vec{y}\} = \mathcal{V}\mathrm{ar}(\psi) \setminus \mathcal{V}\mathrm{ar}(t)$, $\{\vec{y}\} \cap \mathcal{V}\mathrm{ar}(t) = \varnothing$ clearly holds. It remains to show $Y = \mathcal{E}x\mathcal{V}\mathrm{ar}(\rho) \cup (X \cap \mathcal{V}\mathrm{ar}(t)) \subseteq \mathcal{V}\mathrm{ar}(t)$. However, as $X \cap \mathcal{V}\mathrm{ar}(t) \subseteq \mathcal{V}\mathrm{ar}(t)$, it suffices to show $\mathcal{E}x\mathcal{V}\mathrm{ar}(\rho) \subseteq \mathcal{V}\mathrm{ar}(t)$. Let $x \in \mathcal{E}x\mathcal{V}\mathrm{ar}(\rho) = \mathcal{V}\mathrm{ar}(r) \setminus \mathcal{V}\mathrm{ar}(\ell)$. Then we obtain by $\gamma(x) = x$ that $x \notin \mathcal{V}\mathrm{ar}(\ell) = \mathcal{D}\mathrm{om}(\gamma)$. Hence, from $x \in \mathcal{V}\mathrm{ar}(r)$, it follows that $x \in \mathcal{V}\mathrm{ar}(r\gamma) \subseteq \mathcal{V}\mathrm{ar}(t)$.

Finally, we show that $\Pi Y.\ t\ [\exists \vec{y}.\ \psi]$ is satisfiable. By the satisfiability of $\Pi X.\ s\ [\exists \vec{x}.\ \varphi]$, there exists a valuation ξ such that $\vDash_{\mathcal{M},\xi} \exists \vec{x}.\ \varphi$. Hence by (4), $\vDash_{\mathcal{M},\xi} \exists \vec{z}.\ \pi\gamma$. Since $\{\vec{x}\} \subseteq \mathcal{V}\mathrm{ar}(\varphi)$ and $\{\vec{z}\} \subseteq \mathcal{V}\mathrm{ar}(\rho)$, we have $\{\vec{x}\} \cap \{\vec{z}\} = \varnothing$. Thus, one can extend the valuation ξ to ξ' such that $\vDash_{\mathcal{M},\xi'} \varphi \wedge \pi\gamma$. This also gives $\vDash_{\mathcal{M},\xi'} \exists \vec{y}.\ \psi$. Hence it follows that $\Pi Y.\ t\ [\exists \vec{y}.\ \psi]$ is satisfiable. ◻

Remark 3.11. In our new notion an instance of the constraint of the used rewrite rule is attached to the constraint of the constrained term. Previously, the constraint remained unchanged [7]. This implies that in each rewrite step a fresh variant of a constrained rewrite rule is needed in order to prevent variable clashes. Therefore, the constraint part of a constrained term may grow along rewriting sequences. However, we expect that this does not cause any troubles in actual implementations as rewrite steps can be combined with simplification of constraints. For example, in Theorem 3.6 one may perform simplifications after the rewrite step:

$$\begin{array}{l} \Pi\{x\}.\ \mathsf{f}(x)\ [x > 0] \\ \to_\rho \Pi\{y'\}.\ \mathsf{g}(y')\ [\exists x.\ x > 0 \wedge x \geqslant 0 \wedge y' > x] \\ \ \sim\ \Pi\{y'\}.\ \mathsf{g}(y')\ [y' > 1] \end{array}$$

In the following, as we focus on the theoretical aspects of rewrite steps, we do not consider simplifications of constraints.

4 Simulating the State-of-the-Art of Non-Quantified Constrained Rewriting

Our new formalism for constrained rewriting is closely related to the original definition which was introduced in [7]. In this and the following sections, we formally describe this relation. To reflect the idea behind the original constrained terms of [7] and to prepare it for a suitable comparison to our existentially constrained terms, we introduce the concept of *non-quantified* constrained terms extended with the Π-notation as follows:

Definition 4.1 (Non-Quantified Terms Extended with Π-notation). A *non-quantified term (extended with Π-notation)* is a triple $\langle X, s, \varphi\rangle$, written as $\Pi X.\ s\ [\varphi]$, of a set X of theory variables, a term s, and a logical constraint φ such that $\mathcal{V}\mathrm{ar}(\varphi) \subseteq X \subseteq \mathcal{V}\mathrm{ar}(\varphi, s)$.

Example 4.2. For example, $\Pi\{x,y\}.\ \mathsf{f}(x)\ [x < 2 \wedge 0 > y]$ and $\Pi\{x,y,z\}.\ \mathsf{h}(x,y)\ [x < y \wedge x + y + 1 = z]$ are non-quantified constrained terms.

For non-quantified constrained terms, satisfiability and equivalence are defined in the usual way. A constrained term $s\ [\varphi]$ without Π-notation can be easily lifted to one with Π-notation as $\Pi\mathcal{V}\mathrm{ar}(\varphi).\ s\ [\varphi]$. Throughout this paper, we disambiguate the two notions of constrained terms by explicitly naming them *non-quantified constrained terms* and existentially constrained terms.

We focus now on rewrite steps of non-quantified constrained terms (extended with Π-notation), which reflects the original notion of rewriting constrained terms [7]. For equational theories, Aoto et al. in [1] introduced special Π-notations in order to have explicit sets of logical variables which the ones that need to be instantiated by values. The definition of constrained equation $\Pi Z.\ \ell \approx r\ [\pi]$ is very similar to $\Pi Z.\ \ell \to r\ [\pi]$, albeit a slight difference is added to the definition of constrained equations in [1]. In particular, we request that $Z \subseteq \mathcal{V}\mathrm{ar}(\ell, r, \pi)$ for a constrained rule $\Pi Z.\ \ell \to r\ [\pi]$ in order to avoid redundant variables in Z. The variables in Z of $\Pi Z.\ \ell \to r\ [\pi]$ and $\Pi Z.\ \ell \approx r\ [\pi]$ are the ones that need to be instantiated by values in rewrite steps. Rewrite steps using a constrained rewrite rule $\rho\colon \Pi Z.\ \ell \to r\ [\pi]$ on a non-quantified constrained term are performed as follows:

Definition 4.3 (Constrained Rewriting of Non-Quantified Terms). Let $\Pi X.\ s\ [\varphi]$ be a satisfiable non-quantified constrained term and suppose $\mathcal{V}\mathrm{ar}(\rho) \cap (X \cup \mathcal{V}\mathrm{ar}(s, \varphi)) = \varnothing$. Then we obtain the rewrite step $\Pi X.\ s\ [\varphi] \to^{p}_{\rho,\sigma} \Pi Y.\ t\ [\varphi]$ if there exists a position $p \in \mathcal{P}\mathrm{os}_{\mathcal{F}}(s)$, a substitution σ with $\mathcal{D}\mathrm{om}(\sigma) = \mathcal{V}\mathrm{ar}(\ell, r, \pi)$, $\ell\sigma = s|_p$, $\sigma(x) \in \mathcal{V}\mathrm{al} \cup X$ for all $x \in Z$, and $\vDash_{\mathcal{M}} \varphi \Rightarrow \pi\sigma$, where $t = s[r\sigma]_p$ and $Y = X \cap \mathcal{V}\mathrm{ar}(t, \varphi)$.

It is easy to check that non-quantified constrained rules in [7] are covered by taking $X = Y = \mathcal{V}\mathrm{ar}(\varphi)$ and $Z := \mathcal{L}\mathcal{V}\mathrm{ar}(\ell \to r\ [\pi])$.

Example 4.4. We revisit Theorems 1.1 and 1.2. Consider the constrained rewrite rule $\rho\colon \Pi\{x', y'\}.\ \mathsf{f}(x') \to \mathsf{g}(y')\ [x' \geqslant 1 \wedge x' + 1 \geqslant y']$. Then the rewrite steps in Theorem 1.1 are encoded as follows.

- Take $p = \varepsilon$ and $\sigma = \{x' \mapsto x, y' \mapsto 3\}$. We obtain the rewrite step $\Pi\{x\}.\ \mathsf{f}(x)\ [x > 2] \to^{p}_{\rho,\sigma} \Pi\{x\}.\ \mathsf{g}(3)\ [x > 2]$. Note here that $\sigma(x') = x \in \mathcal{V}\mathrm{al} \cup X$ and $\sigma(y') = 3 \in \mathcal{V}\mathrm{al} \cup X$. Also, we have $(x' \geqslant 1 \wedge x' + 1 \geqslant y')\sigma = (x \geqslant 1 \wedge x + 1 \geqslant 3)$ and $\vDash_{\mathcal{M}} (x > 2 \Rightarrow (x \geqslant 1 \wedge x + 1 \geqslant 3)$.
- Take $p = \varepsilon$ and $\sigma = \{x' \mapsto x, y' \mapsto x\}$. We obtain the rewrite step $\Pi\{x\}.\ \mathsf{f}(x)\ [x > 2] \to^{p}_{\rho,\sigma} \Pi\{x\}.\ \mathsf{g}(x)\ [x > 2]$. Note here that $\sigma(x') = \sigma(y') = x \in \mathcal{V}\mathrm{al} \cup X$. Also, we have $(x' \geqslant 1 \wedge x' + 1 \geqslant y')\sigma = (x \geqslant 1 \wedge x + 1 \geqslant x)$ and $\vDash_{\mathcal{M}} (x > 2 \Rightarrow (x \geqslant 1 \wedge x + 1 \geqslant x)$.
- We have a rewrite step $\Pi\{x, y\}.\ \mathsf{f}(x)\ [x > 2 \wedge 0 > y] \to^{p}_{\rho,\sigma} \Pi\{x, y\}.\ \mathsf{g}(y)\ [x > 2 \wedge 0 > y]$, where $p = \varepsilon$ and $\sigma = \{x' \mapsto x, y' \mapsto y\}$. Note here that $\sigma(x') = x \in \mathcal{V}\mathrm{al} \cup X$ and $\sigma(y') = y \in \mathcal{V}\mathrm{al} \cup X$. Also, we have $(x' \geqslant 1 \wedge x' + 1 \geqslant y')\sigma = (x \geqslant 1 \wedge x + 1 \geqslant y)$ and $\vDash_{\mathcal{M}} (x > 2 \wedge 0 > y \Rightarrow (x \geqslant 1 \wedge x + 1 \geqslant y)$.

Similarly, let $\rho\colon \Pi\{x', y', z'\}.\ \mathsf{h}(x', y') \to \mathsf{g}(z')\ [(x' + y') + 1 = z']$ from which we obtain $\Pi\{x, y, z\}.\ \mathsf{h}(x, y)\ [x < y \wedge x + y + 1 = z] \to^{p}_{\rho,\sigma} \Pi\{x, y, z\}.\ \mathsf{g}(z)\ [x < y \wedge x + y + 1 = z]$ as in Theorem 1.2.

Below, we show the relation between the current state-of-the-art of constrained rewrite steps and our new definition. We begin by defining translations between them.

Definition 4.5 (Existential Extension and Existential Removing Translations).

(1) An *existential extension* ext of a non-quantified constrained term is defined as $\mathrm{ext}(\Pi X.\ s\ [\varphi]) = \Pi Y.\ s\ [\exists \vec{x}.\ \varphi]$ where $\{\vec{x}\} = \mathcal{V}\mathrm{ar}(\varphi) \setminus \mathcal{V}\mathrm{ar}(s)$ and $Y = X \setminus \{\vec{x}\}$.

(2) An *existential removing* rmv of an existentially constrained term is defined as $\mathrm{rmv}(\Pi X.\ s\ [\exists \vec{x}.\ \varphi]) = \Pi X \cup \{\vec{x}\}.\ s\ [\varphi]$.

Example 4.6 (Cont'd of Theorem 4.2). The two non-quantified constrained terms in Theorem 4.2 are translated to existentially constrained terms as follows: $\mathrm{ext}(\Pi\{x, y\}.\ \mathsf{f}(x)\ [x < 2 \wedge 0 > y]) = \Pi\{x\}.\ \mathsf{f}(x)\ [\exists\{y\}.\ x < 2 \wedge 0 > y]$ and $\mathrm{ext}(\Pi\{x, y, z\}.\ \mathsf{h}(x, y)\ [x < y \wedge x + y + 1 = z]) = \Pi\{x, y\}.\ \mathsf{h}(x, y)\ [\exists\{z\}.\ x < y \wedge x + y + 1 = z]$. Similarly, we have $\mathrm{rmv}(\Pi\{x\}.\ \mathsf{f}(x)\ [\exists\{y\}.\ x < 2 \wedge 0 > y]) = \Pi\{x, y\}.\ \mathsf{f}(x)\ [x < 2 \wedge 0 > y]$ and $\mathrm{rmv}(\Pi\{x, y\}.\ \mathsf{h}(x, y)\ [\exists\{z\}.\ x < y \wedge x + y + 1 = z]) = \Pi\{x, y, z\}.\ \mathsf{h}(x, y)\ [x < y \wedge x + y + 1 = z]$.

The next lemmata straightforwardly follow from Theorem 4.5.

Lemma 4.7. *For any non-quantified constrained term $\Pi X.\ s\ [\varphi]$, $\mathrm{ext}(\Pi X.\ s\ [\varphi])$ is an existentially constrained term; for any existentially constrained term $\Pi X.\ s\ [\exists \vec{x}.\ \varphi]$ $\mathrm{rmv}(\Pi X.\ s\ [\exists \vec{x}.\ \varphi])$ is a non-quantified constrained term.*

Lemma 4.8. *The translation* $\mathrm{rmv} \circ \mathrm{ext}$ *is the identity translation on non-quantified constrained terms; the translation* $\mathrm{ext} \circ \mathrm{rmv}$ *is the identity translation on existentially constrained terms.*

We show that any rewrite step on existentially constrained terms results in a $\tilde{\to}$-rewrite step on non-quantified constrained terms obtained by existential removing. We need the following lemma.

Lemma 4.9. *Let $\Pi X.\ s\ [\varphi]$ be a non-quantified constrained term. Suppose $\vDash_{\mathcal{M}} \varphi \Rightarrow \exists \vec{z}.\ \pi$ with $\{\vec{z}\} = \mathcal{V}\mathrm{ar}(\pi) \setminus (X \cup \mathcal{V}\mathrm{ar}(\varphi))$ and $\mathcal{V}\mathrm{ar}(s) \cap \{\vec{z}\} = \varnothing$. Then $\Pi X.\ s\ [\varphi] \sim \Pi X \cup \mathcal{V}\mathrm{ar}(\pi).\ s\ [\varphi \wedge \pi]$.*

Proof. ($\subsetneqq$) Let σ be an X-valued substitution such that $\sigma \vDash_{\mathcal{M}} \varphi$. Then we have $X \cup \mathcal{V}\mathrm{ar}(\varphi) \subseteq \mathcal{V}\mathcal{D}\mathrm{om}(\sigma)$ and $\vDash_{\mathcal{M}} \varphi\sigma$. Let γ be a valuation such that $\gamma(x) = \sigma(x)$ for any $x \in X \cup \mathcal{V}\mathrm{ar}(\varphi)$. From the latter, we have $\vDash_{\mathcal{M},\gamma} \varphi$ and hence $\vDash_{\mathcal{M},\gamma} \exists \vec{z}.\ \pi$ by our assumption. Thus, there exists a sequence $\vec{v} \in \mathcal{V}\mathrm{al}^{*}$ of values so that $\vDash_{\mathcal{M},\gamma} \pi\kappa$, where $\kappa = \{\vec{z} \mapsto \vec{v}\}$. As $\{\vec{z}\} = \mathcal{V}\mathrm{ar}(\pi) \setminus (X \cup \mathcal{V}\mathrm{ar}(\varphi))$, we have $\mathcal{V}\mathrm{ar}(\pi\kappa\sigma) = \varnothing$ and $\vDash_{\mathcal{M}} \pi\kappa\sigma$. Take a substitution $\delta = \sigma \circ \kappa$. Then, it follows δ is $\mathcal{V}\mathrm{ar}(\pi)$-valued and $\vDash_{\mathcal{M}} \varphi\delta$. For any $x \in X$, if $x \in \mathcal{D}\mathrm{om}(\kappa)$, then $\delta(x) = \sigma(\kappa(x)) = \kappa(x) \in \mathcal{V}\mathrm{al}$. If $x \notin \mathcal{D}\mathrm{om}(\kappa)$, then $\delta(x) = \sigma(\kappa(x)) = \sigma(x) \in \mathcal{V}\mathrm{al}$, because σ is X-valued. Thus, δ is also X-valued. By our assumption $\mathcal{V}\mathrm{ar}(s) \cap \{\vec{z}\} = \varnothing$, and thus $s\delta = s\kappa\sigma = s\sigma$. Also, as $\{\vec{z}\} = \mathcal{V}\mathrm{ar}(\pi) \setminus (X \cup \mathcal{V}\mathrm{ar}(\varphi))$, we have $\{\vec{z}\} \cap \mathcal{V}\mathrm{ar}(\varphi) = \varnothing$, and therefore, $\varphi\delta = \varphi\kappa\sigma = \varphi\sigma$. Hence, $\mathcal{V}\mathrm{ar}(\varphi\delta) = \varnothing$ and $\vDash_{\mathcal{M}} \varphi\delta$. All in all, δ is $X \cup \mathcal{V}\mathrm{ar}(\pi)$-valued, $s\sigma = s\delta$, and $\delta \vDash_{\mathcal{M}} \varphi \wedge \pi$. The ($\supsetneqq$)-part is trivial. □

Theorem 4.10 (Simulation of Rewrite Steps by Existential Removing). *Let ρ be a left-linear constrained rewrite rule. If we have $\Pi X.\ s\ [\exists \vec{x}.\ \varphi] \to_{\rho} \Pi Y.\ t\ [\exists \vec{y}.\ \psi]$ then $\mathrm{rmv}(\Pi X.\ s\ [\exists \vec{x}.\ \varphi]) \tilde{\to} \mathrm{rmv}(\Pi Y.\ t\ [\exists \vec{y}.\ \psi])$.*

Proof. Let $\rho\colon \Pi Z.\ \ell \to r\ [\pi]$ be a left-linear constrained rewrite rule and suppose that $\Pi X.\ s\ [\exists \vec{x}.\ \varphi] \to^{p}_{\rho,\gamma} \Pi Y.\ t\ [\exists \vec{y}.\ \psi]$ such that $\mathcal{V}\mathrm{ar}(s, \varphi) \cap \mathcal{V}\mathrm{ar}(\rho) = \varnothing$. Then we have (1) $\mathcal{D}\mathrm{om}(\gamma) = \mathcal{V}\mathrm{ar}(\ell)$, (2) $s|_p = \ell\gamma$, (3) $\gamma(x) \in \mathcal{V}\mathrm{al} \cup X$ for any $x \in \mathcal{V}\mathrm{ar}(\ell) \cap Z$, and (4) $\vDash_{\mathcal{M}} (\exists \vec{x}.\ \varphi) \Rightarrow (\exists \vec{z}.\ \pi\gamma)$, where $\{\vec{z}\} = \mathcal{V}\mathrm{ar}(\pi) \setminus \mathcal{V}\mathrm{ar}(\ell)$, $t = s[r\gamma]$, $\psi = \varphi \wedge \pi\gamma$, $\{\vec{y}\} = \mathcal{V}\mathrm{ar}(\psi) \setminus \mathcal{V}\mathrm{ar}(t)$, and $Y = \mathcal{E}\mathrm{x}\mathcal{V}\mathrm{ar}(\rho) \cup (X \cap \mathcal{V}\mathrm{ar}(t))$.

Note that we have $\mathrm{rmv}(\Pi X.\ s\ [\exists \vec{x}.\ \varphi]) = \Pi X \cup \{\vec{x}\}.\ s\ [\varphi]$. We begin by showing the following equivalence: (5) $\Pi X \cup \{\vec{x}\}.\ s\ [\varphi] \sim \Pi X \cup \{\vec{x}\} \cup \mathcal{V}\mathrm{ar}(\pi\gamma) \cup \mathcal{E}\mathrm{x}\mathcal{V}\mathrm{ar}(\rho).\ s\ [\varphi \wedge \pi\gamma \wedge \bigwedge_{z \in \mathcal{E}\mathrm{x}\mathcal{V}\mathrm{ar}(\rho)} (z = z)]$. For this, we use Theorem 4.9. By (4), we have that $\vDash_{\mathcal{M}} \varphi \Rightarrow \exists \vec{z}.\ \pi\gamma$ holds. Since we have $\vDash_{\mathcal{M}} \bigwedge_{z \in \mathcal{E}\mathrm{x}\mathcal{V}\mathrm{ar}(\rho)} (z = z)$, also $\vDash_{\mathcal{M}} \varphi \Rightarrow \exists \vec{z}.\ \pi\gamma \wedge \bigwedge_{z \in \mathcal{E}\mathrm{x}\mathcal{V}\mathrm{ar}(\rho)} (z = z)$ holds. Now, we have

$$
\begin{aligned}
&\mathcal{V}\mathrm{ar}(\pi\gamma \wedge \textstyle\bigwedge_{z \in \mathcal{E}\mathrm{x}\mathcal{V}\mathrm{ar}(\rho)} (z = z)) \setminus (X \cup \{\vec{x}\} \cup \mathcal{V}\mathrm{ar}(\varphi)) \\
&= ((\mathcal{V}\mathrm{ar}(\pi) \setminus \mathcal{V}\mathrm{ar}(\ell)) \cup (\textstyle\bigcup_{y \in \mathcal{V}\mathrm{ar}(\ell) \cap \mathcal{V}\mathrm{ar}(\pi)} \mathcal{V}\mathrm{ar}(\gamma(y))) \\
&\quad \cup \mathcal{E}\mathrm{x}\mathcal{V}\mathrm{ar}(\rho)) \setminus (X \cup \{\vec{x}\} \cup \mathcal{V}\mathrm{ar}(\varphi)) \\
&= ((\mathcal{V}\mathrm{ar}(\pi) \setminus \mathcal{V}\mathrm{ar}(\ell)) \cup \mathcal{E}\mathrm{x}\mathcal{V}\mathrm{ar}(\rho)) \setminus (X \cup \{\vec{x}\} \cup \mathcal{V}\mathrm{ar}(\varphi)) \\
&= \mathcal{V}\mathrm{ar}(r, \pi) \setminus \mathcal{V}\mathrm{ar}(\ell).
\end{aligned}
$$

Take $\{\vec{z}'\} = \mathcal{V}\mathrm{ar}(r,\pi) \setminus \mathcal{V}\mathrm{ar}(\ell) = \mathcal{V}\mathrm{ar}(\pi\gamma \wedge \bigwedge_{z \in \mathcal{E}x\mathcal{V}\mathrm{ar}(\rho)}(z = z)) \setminus (X \cup \{\vec{x}\} \cup \mathcal{V}\mathrm{ar}(\varphi))$. By $\mathcal{V}\mathrm{ar}(\rho) \cap \mathcal{V}\mathrm{ar}(s) = \varnothing$, we have $\{\vec{z}'\} \cap \mathcal{V}\mathrm{ar}(s) = \varnothing$. Furthermore, since $\{\vec{z}'\} \supseteq \mathcal{E}x\mathcal{V}\mathrm{ar}(\rho) = \{\vec{z}\}$, we have $\vDash_{\mathcal{M}} \varphi \Rightarrow \exists \vec{z}'.\ \pi\gamma \wedge \bigwedge_{z \in \mathcal{E}x\mathcal{V}\mathrm{ar}(\rho)}(z = z)$. Thus, by Theorem 4.9, we conclude the equivalence (5).

We proceed to show the rewrite step $\Pi X \cup \{\vec{x}\} \cup \mathcal{V}\mathrm{ar}(\pi\gamma) \cup \mathcal{E}x\mathcal{V}\mathrm{ar}(\rho).\ s\ [\varphi \wedge \pi\gamma \wedge \bigwedge_{z \in \mathcal{E}x\mathcal{V}\mathrm{ar}(\rho)}(z = z)] \to^{p}_{\rho'} \Pi Y \cup \{\vec{y}\} \cup \mathcal{E}x\mathcal{V}\mathrm{ar}(\rho).\ t\ [\varphi \wedge \pi\gamma \wedge \bigwedge_{z \in \mathcal{E}x\mathcal{V}\mathrm{ar}(\rho)}(z = z)]$ over non-quantified constrained terms, by a variant $\rho' \colon \Pi Z'.\ \ell' \to r'\ [\pi']$ of the constrained rewrite rule ρ such that $\mathcal{V}\mathrm{ar}(\rho) \cap \mathcal{V}\mathrm{ar}(\rho') = \varnothing$. Accordingly, we have a variable renaming ξ such that $Z = \xi(Z')$, $\ell = \ell'\xi$, $r = r'\xi$, and $\pi = \pi'\xi$. We also suppose that $\mathcal{V}\mathrm{ar}(\rho') \cap \mathcal{V}\mathrm{ar}(s, \varphi \wedge \pi\gamma \wedge \bigwedge_{z \in \mathcal{E}x\mathcal{V}\mathrm{ar}(\rho)}(z = z)) = \varnothing$.

Take $\delta = \gamma \circ \xi$. Then, by (1) and our assumption on ξ, we have $\mathcal{D}\mathrm{om}(\delta) = \mathcal{V}\mathrm{ar}(\ell', r', \pi')$. Also, $\ell'\delta = \ell'\xi\gamma = \ell\gamma = s|_p$ holds by (2) and the definition of δ.

We now show $\delta(x) \in \mathcal{V}\mathrm{al} \cup X \cup \{\vec{x}\} \cup \mathcal{V}\mathrm{ar}(\pi\gamma) \cup \mathcal{E}x\mathcal{V}\mathrm{ar}(\rho)$ for all $x \in Z'$. To this end assume that $x \in Z'$. If $x \in \mathcal{V}\mathrm{ar}(\ell')$, then $\delta(x) = x\xi\gamma = \gamma(y)$ for some $y \in \mathcal{V}\mathrm{ar}(\ell) \cap Z$. Hence, $\delta(x) \in \mathcal{V}\mathrm{al} \cup X$ by (3). Thus, suppose $x \in Z' \setminus \mathcal{V}\mathrm{ar}(\ell')$. Then, because of $Z' \subseteq \mathcal{V}\mathrm{ar}(\ell', r', \pi')$, we know $x \in \mathcal{V}\mathrm{ar}(r', \pi') \setminus \mathcal{V}\mathrm{ar}(\ell')$. Thus, $\delta(x) = x\xi\gamma = \gamma(y)$ for some $y \in \mathcal{V}\mathrm{ar}(r,\pi) \setminus \mathcal{V}\mathrm{ar}(\ell)$. Hence, if $y \in \mathcal{V}\mathrm{ar}(\pi) \setminus \mathcal{V}\mathrm{ar}(\ell)$ then $\delta(x) = \gamma(y) \in \mathcal{V}\mathrm{ar}(\pi\gamma)$, and if $y \in \mathcal{V}\mathrm{ar}(r) \setminus \mathcal{V}\mathrm{ar}(\ell)$ then $\delta(x) = \gamma(y) = y \in \mathcal{E}x\mathcal{V}\mathrm{ar}(\rho)$.

Furthermore, we have $\vDash_{\mathcal{M}} (\varphi \wedge \pi\gamma \wedge \bigwedge_{z \in \mathcal{E}x\mathcal{V}\mathrm{ar}(\rho)}(z = z)) \Rightarrow \pi'\delta$, because $\pi'\delta = \pi'\xi\gamma = \pi\gamma$. We also have $t = s[r\gamma]_p = s[r'\xi\gamma]_p = s[r'\delta]_p$. Thus, it remains to show that $Y \cup \{\vec{y}\} \cup \mathcal{E}x\mathcal{V}\mathrm{ar}(\rho) = (X \cup \{\vec{x}\} \cup \mathcal{V}\mathrm{ar}(\pi\gamma) \cup \mathcal{E}x\mathcal{V}\mathrm{ar}(\rho)) \cap \mathcal{V}\mathrm{ar}(t, \varphi \wedge \pi\gamma \wedge \bigwedge_{z \in \mathcal{E}x\mathcal{V}\mathrm{ar}(\rho)}(z = z))$. We denote the right-hand side of the equation by *rhs*, and obtain: $(rhs) = ((X \cup \{\vec{x}\} \cup \mathcal{V}\mathrm{ar}(\pi\gamma) \cup \mathcal{E}x\mathcal{V}\mathrm{ar}(\rho)) \cap \mathcal{V}\mathrm{ar}(t)) \cup ((X \cup \{\vec{x}\} \cup \mathcal{V}\mathrm{ar}(\pi\gamma) \cup \mathcal{E}x\mathcal{V}\mathrm{ar}(\rho)) \cap \mathcal{V}\mathrm{ar}(\varphi \wedge \pi\gamma \wedge \bigwedge_{z \in \mathcal{E}x\mathcal{V}\mathrm{ar}(\rho)}(z = z))) = (X \cap \mathcal{V}\mathrm{ar}(t)) \cup (\mathcal{V}\mathrm{ar}(\pi\gamma) \cap \mathcal{V}\mathrm{ar}(t)) \cup \mathcal{V}\mathrm{ar}(\varphi \wedge \pi\gamma) \cup \mathcal{E}x\mathcal{V}\mathrm{ar}(\rho) = (X \cap \mathcal{V}\mathrm{ar}(t)) \cup (\mathcal{V}\mathrm{ar}(\varphi \wedge \pi\gamma \wedge \bigwedge_{z \in \mathcal{E}x\mathcal{V}\mathrm{ar}(\rho)}(z = z))) \cup \mathcal{E}x\mathcal{V}\mathrm{ar}(\rho) = Y \cup \{\vec{y}\} \cup \mathcal{E}x\mathcal{V}\mathrm{ar}(\rho)$ (by Theorem 3.9). Putting all of this together gives the claimed rewrite step. To complete the proof, we need to show the equivalence $\Pi Y \cup \{\vec{y}\} \cup \mathcal{E}x\mathcal{V}\mathrm{ar}(\rho).\ t\ [\varphi \wedge \pi\gamma \wedge \bigwedge_{z \in \mathcal{E}x\mathcal{V}\mathrm{ar}(\rho)}(z = z)] \sim \Pi Y \cup \{\vec{y}\}.\ t\ [\exists \vec{y}.\ \varphi \wedge \pi\gamma]) = \mathrm{rmv}(\Pi Y.\ t\ [\exists \vec{y}.\ \psi])$ which should be trivial to see. ◻

Example 4.11. Let us consider the most general rewrite step $\Pi\{x\}.\ \mathsf{f}(x)\ [x > 0] \to_\rho \Pi\{y'\}.\ \mathsf{g}(y')\ [\exists x.\ x > 0 \wedge x \geqslant 0 \wedge y' > x]$ of Theorem 3.6 where $\rho \colon \Pi\{x', y'\}.\ \mathsf{f}(x') \to \mathsf{g}(y')\ [x' \geqslant 0 \wedge y' > x']$. We obtain the rewrite step over the non-quantified constrained terms $\mathrm{rmv}(\Pi\{x\}.\ \mathsf{f}(x)\ [x > 0]) \xrightarrow{\sim} \mathrm{rmv}(\Pi\{y'\}.\ \mathsf{g}(y')\ [\exists x.\ x > 0 \wedge x \geqslant 0 \wedge y' > x])$ as follows:

$$
\begin{aligned}
&\mathrm{rmv}(\Pi\{x\}.\ \mathsf{f}(x)\ [x > 0]) \\
&\quad = \Pi\{x\}.\ \mathsf{f}(x)\ [x > 0] \\
&\quad \sim \Pi\{x, y'\}.\ \mathsf{f}(x)\ [x > 0 \wedge x \geqslant 0 \wedge y' > x] \\
&\quad \to_\rho \Pi\{x, y'\}.\ \mathsf{g}(y')\ [x > 0 \wedge x \geqslant 0 \wedge y' > x] \\
&\quad = \mathrm{rmv}(\Pi\{y'\}.\ \mathsf{g}(y')\ [\exists x.\ x > 0 \wedge x \geqslant 0 \wedge y' > x])
\end{aligned}
$$

5 Embedding Non-Quantified Constrained Rewriting into the Most General Form

In this section, we concern ourselves with the problem of characterizing the opposite direction. That is, whether any rewrite step on non-quantified constrained terms results in a rewrite step on existentially constrained terms that are obtained by the existential extension. Naively this is not the case as depicted by the following example.

Example 5.1. In Theorem 4.4, we encoded the rewrite steps of Theorem 1.1 by rewriting of non-quantified constrained terms with the constrained rewrite rule $\rho \colon \Pi\{x', y'\}.\ \mathsf{f}(x') \to \mathsf{g}(y')\ [x' \geqslant 1 \wedge x' + 1 \geqslant y']$. We had $\Pi\{x\}.\ \mathsf{f}(x)\ [x > 2] \to_\rho \Pi\{x\}.\ \mathsf{g}(3)\ [x > 2]$ and $\Pi\{x\}.\ \mathsf{f}(x)\ [x > 2] \to_\rho \Pi\{x\}.\ \mathsf{g}(x)\ [x > 2]$. Applying the following most general rewrite steps to $\mathrm{ext}(\Pi\{x\}.\ \mathsf{f}(x)\ [x > 2]) = \Pi\{x\}.\ \mathsf{f}(x)\ [x > 2])$ yields $\Pi\{x\}.\ \mathsf{f}(x)\ [x > 2] \to_\rho \Pi\{x, y'\}.\ \mathsf{g}(y')\ [x > 2 \wedge x \geqslant 1 \wedge x + 1 \geqslant y']$. Unfortunately, neither $\mathrm{ext}(\Pi\{x\}.\ \mathsf{g}(3)\ [x > 2]) = \Pi\{x\}.\ \mathsf{g}(3)\ [x > 2])$ nor $\mathrm{ext}(\Pi\{x\}.\ \mathsf{f}(x)\ [x > 2]) = \Pi\{x\}.\ \mathsf{f}(x)\ [x > 2])$ is obtained. However, note that $\Pi\{x, y'\}.\ \mathsf{g}(y')\ [x > 2 \wedge x \geqslant 1 \wedge x + 1 \geqslant y'] \gtrsim \Pi\{x\}.\ \mathsf{g}(3)\ [x > 2]$, as well as $\Pi\{x, y'\}.\ \mathsf{g}(y')\ [x > 2 \wedge x \geqslant 1 \wedge x + 1 \geqslant y'] \gtrsim \Pi\{x\}.\ \mathsf{g}(x)\ [x > 2]$ holds.

In fact, as the example above demonstrates, we can give the following characterization of non-quantified constrained rewriting in our formalism of most general rewriting. Namely most general rewrite steps subsume any reduct of non-quantified rewrite steps—this relation serves as the main motivation for us to call our new definition *most general rewrite steps.*

Theorem 5.2 (Simulation of Rewrite Steps by Existential Extension). *Let ρ be a left-linear constrained rewrite rule. If we have $\Pi X.\ s\ [\varphi] \to_\rho \Pi Y.\ t\ [\varphi]$, then $\mathrm{ext}(\Pi X.\ s\ [\varphi]) \to \cdot \gtrsim \mathrm{ext}(\Pi Y.\ t\ [\varphi])$.*

Proof. Assume $\Pi X.\ s\ [\varphi] \to^{p}_{\rho,\gamma} \Pi Y.\ t\ [\varphi]$ where $\rho \colon \Pi Z.\ \ell \to r\ [\pi]$ is a left-linear constrained rewrite rule with $\mathcal{V}\mathrm{ar}(\rho) \cap (X \cup \mathcal{V}\mathrm{ar}(s, \varphi)) = \varnothing$. Note that this is a rewrite step for non-quantified constrained terms. Thus, we have (1) $\mathcal{D}\mathrm{om}(\gamma) = \mathcal{V}\mathrm{ar}(\ell, r, \pi)$, (2) $s|_p = \ell\gamma$, (3) $\gamma(x) \in \mathcal{V}\mathrm{al} \cup X$ for all $x \in Z$, and (4) $\vDash_{\mathcal{M}} \varphi \Rightarrow \pi\gamma$. Furthermore, $t = s[r\gamma]_p$ and $Y = X \cap \mathcal{V}\mathrm{ar}(t, \varphi)$. W.l.o.g. we assume $\mathcal{V}\mathrm{ar}(\rho) \cap (Y \cup \mathcal{V}\mathrm{ar}(t, \varphi)) = \varnothing$. Let $\mathrm{ext}(\Pi X.\ s\ [\varphi]) = \Pi X \setminus \{\vec{x}\}.\ s\ [\exists \vec{x}.\ \varphi]$ with $\{\vec{x}\} = \mathcal{V}\mathrm{ar}(\varphi) \setminus \mathcal{V}\mathrm{ar}(s)$ and $\mathrm{ext}(\Pi Y.\ t\ [\psi]) = \Pi Y \setminus \{\vec{y}\}.\ t\ [\exists \vec{y}.\ \psi]$, with $\vec{y} = \mathcal{V}\mathrm{ar}(\psi) \setminus \mathcal{V}\mathrm{ar}(t)$.

We first show that $\Pi X \setminus \{\vec{x}\}.\ s\ [\exists \vec{x}.\ \varphi]$ has a ρ-redex at position p using a substitution $\sigma = \gamma|_{\mathcal{V}\mathrm{ar}(\ell)}$. For this, we need to show (5) $\mathcal{D}\mathrm{om}(\sigma) = \mathcal{V}\mathrm{ar}(\ell)$, (6) $s|_p = \ell\sigma$, (7) $\sigma(x) \in \mathcal{V}\mathrm{al} \cup (X \setminus \{\vec{x}\})$ for any $x \in \mathcal{V}\mathrm{ar}(\ell) \cap Z$, and (8) $\vDash_{\mathcal{M}} (\exists \vec{x}.\ \varphi) \Rightarrow (\exists \vec{z}.\ \pi\sigma)$, where $\{\vec{z}\} = \mathcal{V}\mathrm{ar}(\pi) \setminus \mathcal{V}\mathrm{ar}(\ell)$. (5) is obvious by definition of σ. Similarly, (6) follows as $s|_p = \ell\gamma = \ell\sigma$ by (2). We have $\sigma(x) \in \mathcal{V}\mathrm{al} \cup X$ for any $x \in \mathcal{V}\mathrm{ar}(\ell) \cap Z$, because $\sigma(x) = \gamma(x)$ for all $x \in \mathcal{V}\mathrm{ar}(\ell)$ and $\gamma(x) \in \mathcal{V}\mathrm{al} \cup X$ by (3); moreover, if $\sigma(x) \in X$, then, as $\sigma(x) \in \mathcal{V}\mathrm{ar}(\ell\sigma) \subseteq \mathcal{V}\mathrm{ar}(s)$ by $x \in \mathcal{V}\mathrm{ar}(\ell)$, we have $\sigma(x) \in X \cap \mathcal{V}\mathrm{ar}(s)$, which implies that $\sigma(x) \notin \{\vec{x}\} = \mathcal{V}\mathrm{ar}(\varphi) \setminus \mathcal{V}\mathrm{ar}(s)$. Hence (7) $\sigma(x) \in \mathcal{V}\mathrm{al} \cup (X \setminus \{\vec{x}\})$ for any $x \in \mathcal{V}\mathrm{ar}(\ell) \cap Z$.

It remains to show (8) $\vDash_{\mathcal{M}} (\exists \vec{x}.\ \varphi) \Rightarrow (\exists \vec{z}.\ \pi\sigma)$. First of all, note that by definition of σ and $\mathcal{V}\mathrm{ar}(\rho) \cap \mathcal{V}\mathrm{ar}(s) = \varnothing$, we have $\gamma = \gamma' \circ \sigma$ where $\gamma' = \{z \mapsto \gamma(z) \mid z \in \mathcal{V}\mathrm{ar}(r, \pi) \setminus \mathcal{V}\mathrm{ar}(\ell)\}$. Suppose $\vDash_{\mathcal{M},\xi} \exists \vec{x}.\ \varphi$ for a valuation ξ. Then, for some $\kappa = \{\vec{x} \mapsto \vec{v}\}$ with $\vec{v} \in \mathcal{V}\mathrm{al}^*$ we have $\vDash_{\mathcal{M},\xi\circ\kappa} \varphi$. Hence, from (4) it follows $\vDash_{\mathcal{M},\xi\circ\kappa} \pi\gamma$. Then, by $\gamma = \gamma' \circ \sigma$, we have $\vDash_{\mathcal{M},\xi\circ\kappa} \exists \vec{z}.\ \pi\sigma$ where $\{\vec{z}\} = \mathcal{V}\mathrm{ar}(\pi) \setminus \mathcal{V}\mathrm{ar}(\ell)$. Also, $\mathcal{F}\mathcal{V}\mathrm{ar}(\exists \vec{z}.\ \pi\sigma) \subseteq \mathcal{V}\mathrm{ar}(\ell\gamma) \subseteq \mathcal{V}\mathrm{ar}(s)$. Since $\mathcal{D}\mathrm{om}(\kappa) = \{\vec{x}\} = \mathcal{V}\mathrm{ar}(\varphi) \setminus \mathcal{V}\mathrm{ar}(s)$, we know $\mathcal{D}\mathrm{om}(\kappa) \cap \mathcal{F}\mathcal{V}\mathrm{ar}(\exists \vec{z}.\ \pi\sigma) = \varnothing$, and thus $(\exists\vec{z}.\ \pi\sigma)\kappa = \exists\vec{z}.\ \pi\sigma$. Hence, $\vDash_{\mathcal{M},\xi} \exists\vec{z}.\ \pi\sigma$ follows and therefore we have shown that (8) $\vDash_{\mathcal{M}} (\exists\vec{x}.\ \varphi) \Rightarrow (\exists\vec{z}.\ \pi\sigma)$ holds.

We conclude that $\Pi X\setminus\{\vec{x}\}.\ s\ [\exists\vec{x}.\ \varphi]$ has a ρ-redex at position p using the substitution σ. Consequently, we obtain the rewrite step $\Pi X\setminus\{\vec{x}\}.\ s\ [\exists\vec{x}.\ \varphi] \to^{p}_{\rho,\sigma} \Pi Y'.\ t'\ [\exists\vec{y}\,'.\ \psi']$ where $t' = s[r\sigma]_p$, $\psi' = \varphi \wedge \pi\sigma$, $\{\vec{y}\,'\} = \mathcal{V}\mathrm{ar}(\psi')\setminus\mathcal{V}\mathrm{ar}(t')$, and $Y' = \mathcal{E}x\mathcal{V}\mathrm{ar}(\rho) \cup ((X\setminus\{\vec{x}\}) \cap \mathcal{V}\mathrm{ar}(t'))$.

It remains to show $\Pi Y'.\ t'\ [\exists\vec{y}\,'.\ \psi'] \underset{\sim}{\supset} \Pi Y\setminus\{\vec{y}\}.\ t\ [\exists\vec{y}.\ \psi]$.

Prior to that, let us show $\{\vec{y}\} \subseteq \{\vec{y}\,'\}$. Let $w \in \{\vec{y}\} = \mathcal{V}\mathrm{ar}(\varphi)\setminus\mathcal{V}\mathrm{ar}(t)$. Since $\{\vec{y}\,'\} = \mathcal{V}\mathrm{ar}(\varphi\wedge\pi\gamma)\setminus\mathcal{V}\mathrm{ar}(t')$, it suffices to show that for any $w \in \mathcal{V}\mathrm{ar}(\varphi)$, it holds that $w \in \mathcal{V}\mathrm{ar}(t')$ implies $w \in \mathcal{V}\mathrm{ar}(t)$. Since $t' = s[r\sigma]$ and $t = s[r\gamma]$, the case $w \in \mathcal{V}\mathrm{ar}(s[\,])$ is trivial. Thus, suppose $w \in \mathcal{V}\mathrm{ar}(r\sigma)$. Then either $w \in \mathcal{E}x\mathcal{V}\mathrm{ar}(\rho)$ or $w \in \mathcal{V}\mathrm{ar}(x\sigma)$ for some $x \in \mathcal{V}\mathrm{ar}(\ell)\cap\mathcal{V}\mathrm{ar}(r)$. The former case contradicts with $w \in \mathcal{V}\mathrm{ar}(\varphi)$ because $\mathcal{V}\mathrm{ar}(\varphi)\cap\mathcal{V}\mathrm{ar}(\rho) = \varnothing$. In the latter case, $w \in \mathcal{V}\mathrm{ar}(r\gamma)$ follows, as $\sigma(x) = \gamma(x)$ for $x \in \mathcal{V}\mathrm{ar}(\ell)$.

We proceed to show that $\Pi Y'.\ t'\ [\exists\vec{y}\,'.\ \psi'] \underset{\sim}{\supset} \Pi Y\setminus\{\vec{y}\}.\ t\ [\exists\vec{y}.\ \psi]$. Let θ be a $(Y\setminus\{\vec{y}\})$-valued substitution such that $\theta \vDash_{\mathcal{M}} \exists\vec{y}.\ \psi$, i.e. $((\mathcal{V}\mathrm{ar}(\psi)\cup Y)\setminus\{\vec{y}\}) \subseteq \mathcal{VD}\mathrm{om}(\theta)$ and $\vDash_{\mathcal{M}} (\exists\vec{y}.\ \psi)\theta$. We will prove there exists a substitution θ' such that it is Y'-valued and $\theta' \vDash \exists\vec{y}\,'.\ \psi'$, i.e. $(Y' \cup (\mathcal{V}\mathrm{ar}(\psi')\setminus\{\vec{y}\,'\})) \subseteq \mathcal{VD}\mathrm{om}(\theta')$ and $\vDash_{\mathcal{M}} (\exists\vec{y}\,'.\ \psi')\theta'$, which satisfies $t\theta = t'\theta'$. For this purpose, we assume w.l.o.g.

- $\mathcal{D}\mathrm{om}(\theta)\cap\mathcal{V}\mathrm{ar}(\rho) = \varnothing$,
- $\mathcal{D}\mathrm{om}(\theta)\cap\{\vec{y}\} = \varnothing$, and
- $\mathcal{V}\mathrm{ar}(\theta(z))\cap\{\vec{y}\} = \varnothing$ for any $z \in \mathcal{FV}\mathrm{ar}(\exists\vec{y}.\ \psi)$.

Note here also $\{\vec{y}\}\cap\mathcal{V}\mathrm{ar}(t) = \varnothing$, as $\{\vec{y}\} = \mathcal{V}\mathrm{ar}(\varphi)\setminus\mathcal{V}\mathrm{ar}(t)$. Let us define $\theta' = \theta \cup \{z \mapsto z\gamma\theta \mid z \in \mathcal{V}\mathrm{ar}(r,\pi)\setminus\mathcal{V}\mathrm{ar}(\ell)\}$. As a first step, we show that $\theta(\gamma(x)) = \theta'(\sigma(x))$ for any $x \in \mathcal{V}\mathrm{ar}(\rho)$. Suppose that $x \in \mathcal{V}\mathrm{ar}(\ell)$. Then $\gamma(x) = \sigma(x)$ and since $x\gamma$ is a subterm of $\ell\gamma$, we have that $\mathcal{V}\mathrm{ar}(\gamma(x))\cap\mathcal{V}\mathrm{ar}(\rho) = \varnothing$. In particular, $\mathcal{V}\mathrm{ar}(\gamma(x))\cap(\mathcal{V}\mathrm{ar}(r,\pi)\setminus\mathcal{V}\mathrm{ar}(\ell)) = \varnothing$, thus, by definition of θ', we have $\theta'(\gamma(x)) = \theta(\gamma(x))$. Hence $\theta(\gamma(x)) = \theta'(\gamma(x)) = \theta'(\sigma(x))$. Otherwise, suppose that $x \in (\mathcal{V}\mathrm{ar}(r,\pi)\setminus\mathcal{V}\mathrm{ar}(\ell))$. Then, by definition of θ', we have $\theta'(x) = \theta(\gamma(x))$. Since $\mathcal{D}\mathrm{om}(\sigma) = \mathcal{V}\mathrm{ar}(\ell)$, we have $\sigma(x) = x$ and thus $\theta(\gamma(x)) = \theta'(x) = \theta'(\sigma(x))$.

We conclude that $\theta(\gamma(x)) = \theta'(\sigma(x))$ for any $x \in \mathcal{V}\mathrm{ar}(\rho)$. From this, it follows that $\pi\gamma\theta = \pi\sigma\theta'$ and $r\gamma\theta = r\sigma\theta'$. We further have $\theta'(x) = \theta(x)$ for any $x \in \mathcal{V}\mathrm{ar}(s)$ by definition of θ' as $\mathcal{V}\mathrm{ar}(s)\cap\mathcal{V}\mathrm{ar}(\rho) = \varnothing$. Therefore, we have $t\theta = s[r\gamma]\theta = s\theta[r\gamma\theta] = s\theta'[r\sigma\theta'] = s[r\sigma]\theta' = t'\theta'$.

Let us claim that $\theta'(Y') \subseteq \mathcal{V}\mathrm{al}$. Assume $y \in Y'$. We are now going to show that $\theta'(y) \in \mathcal{V}\mathrm{al}$ by distinguishing two cases.

- Assume $y \in \mathcal{E}x\mathcal{V}\mathrm{ar}(\rho)$. Then, $y \in \mathcal{V}\mathrm{ar}(r)\setminus\mathcal{V}\mathrm{ar}(\ell)$ and hence $\theta'(y) = \theta(\gamma(y))$. Since $\mathcal{V}\mathrm{ar}(r,\pi)\setminus\mathcal{V}\mathrm{ar}(\ell) \subseteq Z$ by the definition of ρ, we have that $\gamma(y) \in \mathcal{V}\mathrm{al}\cup X$ by our condition (3). In case $\gamma(y) \in \mathcal{V}\mathrm{al}$, clearly, $\theta'(y) = \theta(\gamma(y)) = \gamma(y) \in \mathcal{V}\mathrm{al}$. Otherwise, we have $\gamma(y) \in X$. As $y \in \mathcal{E}x\mathcal{V}\mathrm{ar}(\rho)$ and $\mathcal{D}\mathrm{om}(\gamma) = \mathcal{V}\mathrm{ar}(\ell,r,\pi)$, $\gamma(y)$ appears in $t = s[r\gamma]$. Thus, we have $\gamma(y) \in \mathcal{V}\mathrm{ar}(t)$. Moreover, by $\{\vec{y}\} = \mathcal{V}\mathrm{ar}(\varphi)\setminus\mathcal{V}\mathrm{ar}(t)$, we know that $\gamma(y) \notin \{\vec{y}\}$. Thus, $\gamma(y) \in (X\cap\mathcal{V}\mathrm{ar}(t,\varphi))\setminus\{\vec{y}\} = Y\setminus\{\vec{y}\}$. As θ is $(Y\setminus\{\vec{y}\})$-valued, we conclude $\theta'(y) = \theta(\gamma(y)) \in \mathcal{V}\mathrm{al}$.
- Assume $y \notin \mathcal{E}x\mathcal{V}\mathrm{ar}(\rho)$. As $Y' = \mathcal{E}x\mathcal{V}\mathrm{ar}(\rho)\cup((X\setminus\{\vec{x}\})\cap\mathcal{V}\mathrm{ar}(t'))$, we have $y \in ((X\setminus\{\vec{x}\})\cap\mathcal{V}\mathrm{ar}(t'))$. If $y \notin \mathcal{V}\mathrm{ar}(t) = \mathcal{V}\mathrm{ar}(s[r\gamma]_p)$, then, as $y \in \mathcal{V}\mathrm{ar}(t') = \mathcal{V}\mathrm{ar}(s[r\sigma]_p)$ and $\sigma = \gamma|_{\mathcal{V}\mathrm{ar}(\ell)}$, we have $y \in \mathcal{E}x\mathcal{V}\mathrm{ar}(\rho)$. This contradicts our assumption and therefore $y \in \mathcal{V}\mathrm{ar}(t)$. Then, since $y \in X$, we have $y \in (X\cap\mathcal{V}\mathrm{ar}(t,\varphi)) = Y$. Also, by $\{\vec{y}\} = \mathcal{V}\mathrm{ar}(\varphi)\setminus\mathcal{V}\mathrm{ar}(t)$, we know that $y \notin \{\vec{y}\}$. Thus, it follows that $y \in Y\setminus\{\vec{y}\}$, and moreover, that $\theta(y) \in \mathcal{V}\mathrm{al}$ as θ is $(Y\setminus\{\vec{y}\})$-valued. As $y \notin \mathcal{E}x\mathcal{V}\mathrm{ar}(\rho)$, we have $\theta'(y) = \theta(y)$ and we obtain $\theta'(y) \in \mathcal{V}\mathrm{al}$.

It remains to show that $\theta' \vDash_{\mathcal{M}} \exists\vec{y}\,'.\ \psi'$. First, $\mathcal{FV}\mathrm{ar}(\exists\vec{y}\,'.\ \psi') \subseteq \mathcal{VD}\mathrm{om}(\theta')$ follows from $\mathcal{FV}\mathrm{ar}(\exists\vec{y}\,'.\ \psi') \subseteq Y'$ and the fact that θ' is Y'-valued as shown above. We are now going to show $\vDash_{\mathcal{M}} (\exists\vec{y}\,'.\ \psi')\theta'$. From $\vDash_{\mathcal{M}} (\exists\vec{y}.\ \varphi)\theta$, there exists a valuation $\eta = \{\vec{y} \mapsto \vec{v}\}$ with $\vec{v} \in \mathcal{V}\mathrm{al}^*$ such that $\vDash_{\mathcal{M}} \varphi\eta\theta$. Using the second and third assumption on θ which we have assumed above w.l.o.g., it follows that $\vDash_{\mathcal{M}} \varphi\theta\eta$. Thus, from (4), we have that $\vDash_{\mathcal{M}} \pi\gamma\theta\eta$, and hence $\vDash_{\mathcal{M}} \pi\sigma\theta'\eta$. Again, using the second and third assumptions on θ, it follows that $\vDash_{\mathcal{M}} \pi\sigma\eta\theta'$. Also, from $\mathcal{V}\mathrm{ar}(\rho)\cap\mathcal{V}\mathrm{ar}(\varphi) = \varnothing$, we have that $\varphi\theta = \varphi\theta'$, and using our assumptions on θ it also follows that $\varphi\eta\theta = \varphi\eta\theta'$. Therefore, $\vDash_{\mathcal{M}} \varphi\eta\theta'$ holds and we conclude that $\vDash_{\mathcal{M}} (\varphi\wedge\pi\sigma)\eta\theta'$. We obtain that $\vDash_{\mathcal{M}} \exists\vec{y}.\ (\varphi\wedge\pi\sigma)\theta'$, and $\{\vec{y}\} \subseteq \{\vec{y}\,'\}$ implies that $\vDash_{\mathcal{M}} \exists\vec{y}\,'.\ (\varphi\wedge\pi\sigma)\theta'$. We ultimately have $\theta' \vDash_{\mathcal{M}} \exists\vec{y}\,'.\ \psi'$ which concludes the proof. ◻

6 Uniqueness of Reducts and Partial Commutation of Rewriting and Equivalence

The aim of this section is to show two useful properties of most general rewriting. First, we show that applying the same constrained rewrite rule at the same position yields a unique term. Second, we show that our new notion of rewriting commutes with the equivalence transformation for pattern-general constrained terms.

Theorem 6.1 (Uniqueness of Reducts). *Let $\Pi X.\ s\ [\exists\vec{x}.\ \varphi]$ be an existentially constrained term and $p \in \mathcal{P}\mathrm{os}(s)$. Suppose that ρ, ρ' are renamed variants of the same left-linear constrained rule satisfying $\mathcal{V}\mathrm{ar}(\rho)\cap\mathcal{V}\mathrm{ar}(s,\varphi) = \varnothing$ and $\mathcal{V}\mathrm{ar}(\rho')\cap\mathcal{V}\mathrm{ar}(s,\varphi) = \varnothing$. If $\Pi X.\ s\ [\exists\vec{x}.\ \varphi] \to^{p}_{\rho} \Pi Y.\ t\ [\exists\vec{y}.\ \psi]$ and $\Pi X.\ s\ [\exists\vec{x}.\ \varphi] \to^{p}_{\rho'} \Pi Y'.\ t'\ [\exists\vec{y}\,'.\ \psi']$ then $\Pi Y.\ t\ [\exists\vec{y}.\ \psi] \sim \Pi Y'.\ t'\ [\exists\vec{y}\,'.\ \psi']$.*

Proof. Suppose that $\Pi X.\ s\ [\exists\vec{x}.\ \varphi] \to^{p}_{\rho,\gamma} \Pi Y.\ t\ [\exists\vec{y}.\ \psi]$ and $\Pi X.\ s\ [\exists\vec{x}.\ \varphi] \to^{p}_{\rho',\gamma'} \Pi Y'.\ t'\ [\exists\vec{y}\,'.\ \psi']$, where $\rho\colon \Pi Z.\ \ell \to r\ [\pi]$ and $\rho'\colon \Pi Z'.\ \ell' \to r'\ [\pi']$. Then we have (1) $\mathcal{D}\mathrm{om}(\gamma) = \mathcal{V}\mathrm{ar}(\ell)$, (2) $s|_p = \ell\gamma$, (3) $\gamma(x) \in \mathcal{V}\mathrm{al}\cup X$ for any $x \in \mathcal{V}\mathrm{ar}(\ell)\cap Z$, and (4) $\vDash_{\mathcal{M}} (\exists\vec{x}.\ \varphi) \Rightarrow (\exists\vec{z}.\ \pi\gamma)$, where $\{\vec{z}\} = \mathcal{V}\mathrm{ar}(\pi)\setminus\mathcal{V}\mathrm{ar}(\ell)$, $t = s[r\gamma]$, $\psi = \varphi\wedge\pi\gamma$, $\{\vec{y}\} = \mathcal{V}\mathrm{ar}(\psi)\setminus\mathcal{V}\mathrm{ar}(t)$, and $Y = \mathcal{E}x\mathcal{V}\mathrm{ar}(\rho)\cup(X\cap\mathcal{V}\mathrm{ar}(t))$. Similarly, we have (1') $\mathcal{D}\mathrm{om}(\gamma') = \mathcal{V}\mathrm{ar}(\ell')$, (2') $s|_p = \ell'\gamma'$, (3') $\gamma'(x) \in \mathcal{V}\mathrm{al}\cup X$ for any $x \in \mathcal{V}\mathrm{ar}(\ell')\cap Z'$, and (4') $\vDash_{\mathcal{M}} (\exists\vec{x}.\ \varphi) \Rightarrow (\exists\vec{z}\,'.\ \pi'\gamma')$, where $\{\vec{z}\,'\} = \mathcal{V}\mathrm{ar}(\pi')\setminus\mathcal{V}\mathrm{ar}(\ell')$, $t' = s[r'\gamma']$, $\psi' = \varphi\wedge\pi'\gamma'$, $\{\vec{y}\,'\} = \mathcal{V}\mathrm{ar}(\psi')\setminus\mathcal{V}\mathrm{ar}(t')$, and $Y' = \mathcal{E}x\mathcal{V}\mathrm{ar}(\rho')\cup(X\cap\mathcal{V}\mathrm{ar}(t'))$.

Since ρ, ρ' are renamed variants of the same rule, we have renaming $\sigma\colon \mathcal{V}\mathrm{ar}(\rho) \to \mathcal{V}\mathrm{ar}(\rho')$ such that $\sigma(Z) = Z'$, $\sigma(\ell) = \ell'$, $\sigma(r) = r'$, and $\sigma(\pi) = \pi'$.

For each $x \in \mathcal{V}\mathrm{ar}(r,\pi)\setminus\mathcal{V}\mathrm{ar}(\ell)$, we have by $x \notin \mathcal{V}\mathrm{ar}(\ell) = \mathcal{D}\mathrm{om}(\gamma)$ that $\gamma(x) = x$, and by $\sigma(x) \notin \mathcal{V}\mathrm{ar}(\ell') = \mathcal{D}\mathrm{om}(\gamma')$ that $\gamma'(\sigma(x)) = \sigma(x)$. Therefore, $\sigma(\gamma(x)) = \sigma(x) = \gamma'(\sigma(x))$ for all $x \in \mathcal{V}\mathrm{ar}(r,\pi)\setminus\mathcal{V}\mathrm{ar}(\ell)$. Suppose $x \in \mathcal{V}\mathrm{ar}(\ell)$. Then from $\gamma(x) \in \mathcal{V}\mathrm{ar}(s)$ and $\mathcal{V}\mathrm{ar}(s)\cap\mathcal{V}\mathrm{ar}(\rho) = \varnothing$, we know $\gamma(x) \notin \mathcal{V}\mathrm{ar}(\rho)$, and hence $\gamma(x) \notin \mathcal{D}\mathrm{om}(\sigma)$. Thus, $\sigma(\gamma(x)) = \gamma(x)$. Moreover, since

$\ell\gamma = s|_p = \ell'\gamma' = \ell\sigma\gamma'$, by fixing $x = \ell|_q$ for a position q, we obtain $\gamma(x) = s|_{pq} = \gamma'(\sigma(x))$. Thus, $\sigma(\gamma(x)) = \gamma(x) = \gamma'(\sigma(x))$ for all $x \in \mathcal{V}\mathrm{ar}(\ell)$.

Putting all of this together gives $\sigma(\gamma(x)) = \gamma'(\sigma(x))$ for all $x \in \mathcal{V}\mathrm{ar}(\rho)$. Hence, we have $r\gamma\sigma = r\sigma\gamma' = r'\gamma'$ and $\pi\gamma\sigma = \pi\sigma\gamma' = \pi'\gamma'$. Therefore, we have $t\sigma = s[r\gamma]_p\sigma = s[r\gamma\sigma]_p = s[r'\gamma']_p = t'$ and $\psi\sigma = (\varphi \wedge \pi\gamma)\sigma = \varphi \wedge \pi\gamma\sigma = \varphi \wedge \pi'\gamma' = \psi'$. In particular, it also follows that $\sigma(\mathcal{V}\mathrm{ar}(t)) = \mathcal{V}\mathrm{ar}(t\sigma) = \mathcal{V}\mathrm{ar}(t')$ and $\sigma(\mathcal{V}\mathrm{ar}(\psi)) = \mathcal{V}\mathrm{ar}(\psi\sigma) = \mathcal{V}\mathrm{ar}(\psi')$. Thus, as σ is bijective on $\mathcal{V}\mathrm{ar}(\rho)$, we obtain $\sigma(\{\vec{y}\}) = \sigma(\mathcal{V}\mathrm{ar}(\psi) \setminus \mathcal{V}\mathrm{ar}(t)) = \sigma(\{x \in \mathcal{V}\mathrm{ar}(\rho) \mid x \in \mathcal{V}\mathrm{ar}(\psi) \wedge x \notin \mathcal{V}\mathrm{ar}(t)\} = \{\sigma(x) \in \mathcal{V}\mathrm{ar}(\rho) \mid x \in \mathcal{V}\mathrm{ar}(\psi) \wedge x \notin \mathcal{V}\mathrm{ar}(t)\} = \{\sigma(x) \in \mathcal{V}\mathrm{ar}(\rho) \mid \sigma(x) \in \mathcal{V}\mathrm{ar}(\psi') \wedge \sigma(x) \notin \mathcal{V}\mathrm{ar}(t')\} = \mathcal{V}\mathrm{ar}(\psi') \setminus \mathcal{V}\mathrm{ar}(t') = \{\vec{y}\,'\}$. From $X \cap \mathcal{V}\mathrm{ar}(\rho) = \varnothing$, we know that $X \cap \mathcal{D}\mathrm{om}(\sigma) = \varnothing$. Thus, $\sigma(X) = X$. Therefore, $\sigma(Y) = \sigma(\mathcal{E}\mathrm{x}\mathcal{V}\mathrm{ar}(\rho) \cup (X \cap \mathcal{V}\mathrm{ar}(t))) = \sigma((\mathcal{V}\mathrm{ar}(r) \setminus \mathcal{V}\mathrm{ar}(\ell)) \cup (X \cap \mathcal{V}\mathrm{ar}(t))) = (\mathcal{V}\mathrm{ar}(r\sigma) \setminus \mathcal{V}\mathrm{ar}(\ell\sigma)) \cup (X \cap \mathcal{V}\mathrm{ar}(t\sigma)) = \mathcal{E}\mathrm{x}\mathcal{V}\mathrm{ar}(\rho') \cup (X \cap \mathcal{V}\mathrm{ar}(t')) = Y'$. Since σ is a bijective renaming by Theorem 2.1 we have $\Pi Y.\ t\ [\exists \vec{y}.\ \psi] \sim \Pi Y'.\ t'\ [\exists \vec{y}\,'.\ \psi']$. □

Uniqueness of reducts implies that for LCTRSs with a finite number of constrained rewrite rules, there exists only a finite number of reducts for any existentially constrained term (modulo equivalence). This stands in sharp contrast to the original notion of rewriting non-quantified terms and represents a useful property to check convergence of constrained terms.

Theorem 6.2 (Commutation of Rewrite Steps and Equivalence for Pattern-General Terms). *Let ρ be a left-linear constrained rewrite rule and $\Pi X.\ s\ [\exists \vec{x}.\ \varphi], \Pi Y.\ t\ [\exists \vec{y}.\ \psi]$ be existentially constrained terms that are satisfiable and pattern-general. If $\Pi X'.\ s'\ [\exists \vec{x}\,'.\ \varphi']\ {}_\rho\!\leftarrow \Pi X.\ s\ [\exists \vec{x}.\ \varphi] \sim \Pi Y.\ t\ [\exists \vec{y}.\ \psi]$ then $\Pi X'.\ s'\ [\exists \vec{x}\,'.\ \varphi'] \sim \Pi Y'.\ t'\ [\exists \vec{y}\,'.\ \psi']\ {}_\rho\!\leftarrow \Pi Y.\ t\ [\exists \vec{y}.\ \psi]$ for some $\Pi Y'.\ t'\ [\exists \vec{y}\,'.\ \psi']$ (see below).*

$$\begin{tikzcd} \Pi Y.\ t\ [\exists \vec{y}.\ \psi] \arrow[r, leftrightsquigarrow] \arrow[d, dashed] & \Pi X.\ s\ [\exists \vec{x}.\ \varphi] \arrow[d] \\ \Pi Y'.\ t'\ [\exists \vec{y}\,'.\ \psi'] \arrow[r, leftrightsquigarrow, dashed] & \Pi X'.\ s'\ [\exists \vec{x}\,'.\ \varphi'] \end{tikzcd}$$

Proof. From Theorem 2.2 it follows that there is a renaming $\sigma\colon \mathcal{V}\mathrm{ar}(s,\varphi) \to \mathcal{V}\mathrm{ar}(t,\psi)$ such that (1) $s\sigma = t$, (2) $Y = \sigma(X)$, and (3) $\vDash_{\mathcal{M}} ((\exists \vec{x}.\ \varphi)\sigma \Leftrightarrow (\exists \vec{y}.\ \psi))$.

Suppose that $\Pi X.\ s\ [\exists \vec{x}.\ \varphi] \to^p_{\rho,\gamma} \Pi X'.\ s'\ [\exists \vec{x}\,'.\ \varphi']$, where $\rho\colon \Pi Z.\ \ell \to r\ [\pi]$ such that $\mathcal{V}\mathrm{ar}(\rho) \cap \mathcal{V}\mathrm{ar}(s,\varphi) = \varnothing$. W.l.o.g. we assume $\mathcal{V}\mathrm{ar}(\rho) \cap \mathcal{V}\mathrm{ar}(s,\varphi,t,\psi) = \varnothing$. By definition, we have (4) $\mathcal{D}\mathrm{om}(\gamma) = \mathcal{V}\mathrm{ar}(\ell)$, (5) $s|_p = \ell\gamma$, (6) $\gamma(x) \in \mathcal{V}\mathrm{al} \cup X$ for any $x \in \mathcal{V}\mathrm{ar}(\ell) \cap Z$, and (7) $\vDash_{\mathcal{M}} (\exists \vec{x}.\ \varphi) \Rightarrow (\exists \vec{z}.\ \pi\gamma)$, where $\{\vec{z}\} = \mathcal{V}\mathrm{ar}(\pi) \setminus \mathcal{V}\mathrm{ar}(\ell)$, $s' = s[r\gamma]$, $\varphi' = \varphi \wedge \pi\gamma$, $\{\vec{x}\,'\} = \mathcal{V}\mathrm{ar}(\varphi') \setminus \mathcal{V}\mathrm{ar}(s')$, and $X' = \mathcal{E}\mathrm{x}\mathcal{V}\mathrm{ar}(\rho) \cup (X \cap \mathcal{V}\mathrm{ar}(s'))$.

We show a rewrite step $\Pi Y.\ t\ [\exists \vec{y}.\ \psi] \to^p_{\rho,\delta} \Pi Y'.\ t'\ [\exists \vec{y}\,'.\ \psi']$, where $\delta = \{x \mapsto \sigma(\gamma(x)) \mid x \in \mathcal{V}\mathrm{ar}(\ell)\}$. First we show that $\Pi Y.\ t\ [\exists \vec{y}.\ \psi]$ has a ρ-redex at p by δ, that is, (4') $\mathcal{D}\mathrm{om}(\delta) = \mathcal{V}\mathrm{ar}(\ell)$, (5') $t|_p = \ell\delta$, (6') $\delta(x) \in \mathcal{V}\mathrm{al} \cup Y$ for any $x \in \mathcal{V}\mathrm{ar}(\ell) \cap Z$, and (7') $\vDash_{\mathcal{M}} (\exists \vec{y}.\ \psi) \Rightarrow (\exists \vec{z}.\ \pi\delta)$, where $\{\vec{z}\} = \mathcal{V}\mathrm{ar}(\pi) \setminus \mathcal{V}\mathrm{ar}(\ell)$.

(4') is satisfied by definition of δ. (5') follows as $\ell\delta = \ell\gamma\sigma = (s|_p)\sigma = (s\sigma)|_p = t|_p$, using (1) and (5). To show (6'), assume $x \in \mathcal{V}\mathrm{al}(\ell) \cap Z$. Then $\gamma(x) \in \mathcal{V}\mathrm{al} \cup X$ by (6), and $\delta(x) = \sigma(\gamma(x)) \in \mathcal{V}\mathrm{al} \cup \sigma(X) = \mathcal{V}\mathrm{al} \cup Y$, by using (2). It remains to show (7'). From (7) we have $\vDash_{\mathcal{M}} (\exists \vec{x}.\ \varphi) \Rightarrow (\exists \vec{z}.\ \pi\gamma)$, thus $\vDash_{\mathcal{M}} (\exists \vec{x}.\ \varphi)\sigma \Rightarrow (\exists \vec{z}.\ \pi\gamma)\sigma$. From (3) follows that $\vDash_{\mathcal{M}} (\exists \vec{y}.\ \psi) \Rightarrow (\exists \vec{z}.\ \pi\gamma)\sigma$. By $\sigma\colon \mathcal{V}\mathrm{ar}(s,\varphi) \to \mathcal{V}\mathrm{ar}(t,\psi)$ and $\{\vec{z}\} \subseteq \mathcal{V}\mathrm{ar}(\rho)$, we conclude from $\mathcal{V}\mathrm{ar}(\rho) \cap \mathcal{V}\mathrm{ar}(s,\varphi) = \varnothing$ that $\{\vec{z}\} \cap \mathcal{V}\mathrm{ar}(s,\varphi) = \varnothing$, and further from $\mathcal{V}\mathrm{ar}(\rho) \cap \mathcal{V}\mathrm{ar}(t,\psi) = \varnothing$ that $\{\vec{z}\} \cap \mathcal{V}\mathrm{ar}(t,\psi) = \varnothing$. Hence, by definition of δ, $(\exists \vec{z}.\ \pi\gamma)\sigma = \exists \vec{z}.\ \pi\gamma\sigma = \exists \vec{z}.\ \pi\delta$, and therefore $\vDash_{\mathcal{M}} (\exists \vec{y}.\ \psi) \Rightarrow (\exists \vec{z}.\ \pi\delta)$ is satisfied. Let $t' = s[r\delta]_p$, $\psi' = \psi \wedge \pi\delta$, $\{\vec{y}\,'\} = \mathcal{V}\mathrm{ar}(\psi') \setminus \mathcal{V}\mathrm{ar}(t')$, and $Y' = \mathcal{E}\mathrm{x}\mathcal{V}\mathrm{ar}(\rho) \cup (Y \cap \mathcal{V}\mathrm{ar}(t'))$, we obtain the rewrite step $\Pi Y.\ t\ [\exists \vec{y}.\ \psi] \to^p_{\rho,\delta} \Pi Y'.\ t'\ [\exists \vec{y}\,'.\ \psi']$.

It remains to show that $\Pi X'.\ s'\ [\exists \vec{x}\,'.\ \varphi'] \sim \Pi Y'.\ t'\ [\exists \vec{y}\,'.\ \psi']$. To that end we use Theorem 2.3. We show (8) $\mathcal{P}\mathrm{os}_{X' \cup \mathcal{V}\mathrm{al}}(s') = \mathcal{P}\mathrm{os}_{Y' \cup \mathcal{V}\mathrm{al}}(t')\ (= \{p_1, \dots, p_n\})$, (9) $\rho(s'[\,]_{p_1,\dots,p_n}) = t'[\,]_{p_1,\dots,p_n}$ for some renaming $\rho\colon \mathcal{V}\mathrm{ar}(s') \setminus X' \to \mathcal{V}\mathrm{ar}(t') \setminus Y'$, (10) for any $i,j \in \{1,\dots,n\}$, $\vDash_{\mathcal{M}} (\exists \vec{x}.\ \varphi) \Rightarrow (s'|_{p_i} = s'|_{p_j})$ iff $\vDash_{\mathcal{M}} (\exists \vec{y}.\ \psi) \Rightarrow (t'|_{p_i} = t'|_{p_j})$, (11) for any $i \in \{1,\dots,n\}$ and $v \in \mathcal{V}\mathrm{al}$, $\vDash_{\mathcal{M}} (\exists \vec{x}.\ \varphi) \Rightarrow (s'|_{p_i} = v)$ iff $\vDash_{\mathcal{M}} (\exists \vec{y}.\ \psi) \Rightarrow (t'|_{p_i} = v)$, and (12) let $\sim\, = \,\sim_{\mathcal{P}\mathrm{os}_{X' \cup \mathcal{V}\mathrm{al}}(s')} = \,\sim_{\mathcal{P}\mathrm{os}_{X' \cup \mathcal{V}\mathrm{al}}(t')}$ and $\mu_{X'}, \mu_{Y'}$ be representative substitutions of $\Pi X'.\ s'\ [\exists \vec{x}\,'.\ \varphi']$ and $\Pi Y'.\ t'\ [\exists \vec{y}\,'.\ \psi']$, respectively, based on the same representative for each equivalence class $[p_i]_\sim$ $(1 \leqslant i \leqslant n)$, and we have $\vDash_{\mathcal{M}} (\exists \vec{x}\,'.\ \varphi')\mu_{X'}\theta|_{\tilde{X}'} \Leftrightarrow (\exists \vec{y}\,'.\ \psi')\mu_{Y'}$ with a renaming $\theta|_{\tilde{X}'}\colon \tilde{X}' \to \tilde{Y}'$, where $\theta = \{\langle s'|_{p_i}, t'|_{p_i}\rangle \mid 1 \leqslant i \leqslant n\}$, $\tilde{X}' = \hat{X}' \cap X'$, and $\tilde{Y}' = \hat{Y}' \cap Y'$.

We start by showing (8). Consequently observe that

$$s'\sigma = s[r\gamma]_p\sigma = s\sigma[r\gamma\sigma]_p = t[r\delta]_p = t' \tag{13}$$

Hence, $\sigma(X') = \sigma(\mathcal{E}\mathrm{x}\mathcal{V}\mathrm{ar}(\rho) \cup \sigma(X \cap \mathcal{V}\mathrm{ar}(s')) = \sigma(\mathcal{E}\mathrm{x}\mathcal{V}\mathrm{ar}(\rho)) \cup \sigma(X \cap \mathcal{V}\mathrm{ar}(s')) = \mathcal{E}\mathrm{x}\mathcal{V}\mathrm{ar}(\rho) \cup (\sigma(X) \cap \mathcal{V}\mathrm{ar}(s'\sigma)) = \mathcal{E}\mathrm{x}\mathcal{V}\mathrm{ar}(\rho) \cup (Y \cap \mathcal{V}\mathrm{ar}(t')) = Y'$ follows. This implies that $\mathcal{P}\mathrm{os}_{X' \cup \mathcal{V}\mathrm{al}}(s') = \mathcal{P}\mathrm{os}_{\sigma(X') \cup \mathcal{V}\mathrm{al}}(s'\sigma) = \mathcal{P}\mathrm{os}_{Y' \cup \mathcal{V}\mathrm{al}}(t')$, which shows (8). In the following let $\mathcal{P}\mathrm{os}_{X' \cup \mathcal{V}\mathrm{al}}(s') = \{p_1, \dots, p_n\}$.

We proceed to show (9). We let $s' = s'[s'_1, \dots, s'_n]_{p_1,\dots,p_n}$ and $t' = t'[t'_1, \dots, t'_n]_{p_1,\dots,p_n}$, using the fact (8). Then, from (13), it follows that $s'\sigma[\,]_{p_1,\dots,p_n} = t'[\,]_{p_1,\dots,p_n}$ and $s'_i\sigma = t'_i$ for each $1 \leqslant i \leqslant n$. Note here that $s'_i \in \mathcal{V}\mathrm{al} \cup X'$ and $t'_i \in \mathcal{V}\mathrm{al} \cup Y'$. Thus, since σ is a renaming, we have $s'_i \in X'$ iff $t'_i \in Y'$, $s'_i \in \mathcal{V}\mathrm{al}$ iff $t'_i \in \mathcal{V}\mathrm{al}$, and $s'_i = s'_j$ iff $t'_i = t'_j$ for $i,j \in \{1,\dots,n\}$. Hence $\{s'_1, \dots, s'_n\} \cap X' = \{t'_1, \dots, t'_n\} \cap Y'$. Since $\sigma(X') = Y'$, it implies that $\eta = \{x \mapsto \sigma(x) \mid x \in \mathcal{V}\mathrm{ar}(s') \setminus X'\}$ is a bijection from $\mathcal{V}\mathrm{ar}(s') \setminus X'$ to $\mathcal{V}\mathrm{ar}(t') \setminus Y'$ such that $\eta(s'[\,]_{p_1,\dots,p_n}) = t'[\,]_{p_1,\dots,p_n}$.

Before we proceed to the remainder of the proof, let us introduce a useful temporary notation and show some related properties. For $\exists \vec{x}.\ \varphi$ and $\exists \vec{x}\,'.\ \varphi' = \exists \vec{x}\,'.\ \varphi \wedge \pi\gamma$, we have, by Theorem 3.9, that $\{\vec{x}\} \subseteq \{\vec{x}\,'\}$. Let $\{\vec{x}\,''\} = \{\vec{x}\,'\} \setminus \{\vec{x}\}$ and let us write $\varphi = \varphi(\vec{x}, \vec{x}\,'')$. Also, for $\pi\gamma$, because $\{\vec{x}\} \cap \mathcal{V}\mathrm{ar}(\pi\gamma) = \varnothing$ by Theorem 3.9, we have $\mathcal{V}\mathrm{ar}(\pi\gamma) \cap \{\vec{x}\,'\} = \{\vec{x}\,'''\} \subseteq \{\vec{x}\,''\}$. Thus, using the subsequence $\vec{x}\,'''$ of $\vec{x}\,''$, we write $\pi\gamma = \pi\gamma(\vec{x}\,''')$. Similarly, for $\exists \vec{y}.\ \psi$ and $\exists \vec{y}\,'.\ \psi' = \exists \vec{y}\,'.\ \psi \wedge \pi\delta$, let us write $\psi = \psi(\vec{y}, \vec{y}\,'')$ and $\pi\delta = \pi\delta(\vec{y}\,''')$ where $\{\vec{y}\,''\} = \{\vec{y}\,'\} \setminus \{\vec{y}\}$ and $\{\vec{y}\,'''\} \subseteq \{\vec{y}\,''\}$. Note that $\mathcal{V}\mathrm{ar}(\pi\gamma) = (\bigcup_{x \in \mathcal{V}\mathrm{ar}(\ell) \cap \mathcal{V}\mathrm{ar}(\pi)} \mathcal{V}\mathrm{ar}(x\gamma)) \cup (\mathcal{V}\mathrm{ar}(\pi) \setminus \mathcal{V}\mathrm{ar}(\ell))$. Moreover, we obtain that $\sigma(\{\vec{x}\,'''\}) = \sigma(\mathcal{V}\mathrm{ar}(\pi\gamma) \cap \{\vec{x}\,'\}) = \sigma(((\bigcup_{x \in \mathcal{V}\mathrm{ar}(\ell) \cap \mathcal{V}\mathrm{ar}(\pi)} \mathcal{V}\mathrm{ar}(x\gamma)) \cup (\mathcal{V}\mathrm{ar}(\pi) \setminus \mathcal{V}\mathrm{ar}(\ell))) \setminus \mathcal{V}\mathrm{ar}(s')) = ((\bigcup_{x \in \mathcal{V}\mathrm{ar}(\ell) \cap \mathcal{V}\mathrm{ar}(\pi)} \mathcal{V}\mathrm{ar}(x\delta)) \cup (\mathcal{V}\mathrm{ar}(\pi) \setminus \mathcal{V}\mathrm{ar}(\ell))) \setminus \mathcal{V}\mathrm{ar}(t') = \mathcal{V}\mathrm{ar}(\pi\delta) \cap \{\vec{y}\} = \{\vec{y}\,'''\}$.

From (3) $\vDash_{\mathcal{M}} ((\exists \vec{x}.\ \varphi)\sigma \Leftrightarrow (\exists \vec{y}.\ \psi))$, we have that, for any valuation ξ, $\vDash_{\mathcal{M},\xi} \varphi\sigma(\vec{a}, \vec{x}\,''\sigma)$ for some $\vec{a} \in |\mathcal{M}|^*$ iff $\vDash_{\mathcal{M},\xi} \psi(\vec{b}, \vec{y}\,'')$ for some $\vec{b} \in |\mathcal{M}|^*$. Thus, for any valuation ξ, $\vDash_{\mathcal{M},\xi} \varphi\sigma(\vec{a}, \vec{x}\,''\sigma) \wedge \pi\gamma\sigma(\vec{x}\,'''\sigma)$ for some $\vec{a} \in |\mathcal{M}|^*$ iff $\vDash_{\mathcal{M},\xi} \psi(\vec{b}, \vec{y}\,'') \wedge \pi\delta(\vec{y}\,''')$ for some $\vec{b} \in |\mathcal{M}|^*$, as we know $\pi\gamma\sigma = \pi\delta$ and $\sigma(\{\vec{x}'''\}) = \{\vec{y}'''\}$. In particular, we have

$$\vDash_{\mathcal{M}} ((\exists\vec{x}'.\,\varphi')\sigma \Leftrightarrow (\exists\vec{y}'.\,\psi')) \tag{14}$$

Using (14) and our assumption, $\vDash_{\mathcal{M}} \exists x'.\,\varphi' \Rightarrow (s'|_{p_i} = s'|_{p_j})$ iff $\vDash_{\mathcal{M}} \exists x'.\,\varphi'\sigma \Rightarrow (\sigma(s'|_{p_i}) = \sigma(s'|_{p_j}))$ iff $\vDash_{\mathcal{M}} \exists y'.\,\psi' \Rightarrow (s'\sigma|_{p_i} = s'\sigma|_{p_j})$ iff $\vDash_{\mathcal{M}} \exists y'.\,\psi' \Rightarrow (t'|_{p_i}) = t'|_{p_j})$ for $i, j \in \{1, \ldots, n\}$. Similarly, $\vDash_{\mathcal{M}} \exists x'.\,\varphi' \Rightarrow (s'|_{p_i} = v)$ iff $\vDash_{\mathcal{M}} \exists x'.\,\varphi'\sigma \Rightarrow (\sigma(s'|_{p_i}) = v\sigma)$ iff $\vDash_{\mathcal{M}} \exists y'.\,\psi' \Rightarrow (s'\sigma|_{p_i} = v)$ iff $\vDash_{\mathcal{M}} \exists y'.\,\psi' \Rightarrow (t'|_{p_i} = v)$ for $i, j \in \{1, \ldots, n\}$. This shows (10) and (11).

Finally, take $\theta = \{\langle s'|_{p_i}, t'|_{p_i}\rangle \mid 1 \leqslant i \leqslant n\}$. As $\theta|_{\tilde{X}'} \subseteq \sigma$, from (10), (11) and $s'\sigma = t'$, it follows that $\theta|_{\tilde{X}'} \circ \mu_{X'} = \mu_{Y'} \circ \sigma$. From (14), we have $\vDash_{\mathcal{M}} (\exists\vec{x}'.\,\varphi')\sigma\mu_{Y'} \Leftrightarrow (\exists\vec{y}'.\,\psi')\mu_{Y'}$, and thus, $\vDash_{\mathcal{M}} (\exists\vec{x}'.\,\varphi')\mu_{X'}\theta|_{\tilde{X}'} \Leftrightarrow (\exists\vec{y}'.\,\psi')\mu_{Y'}$. As $\mu_{X'}, \mu_{Y'}$ are based on the same representative for each equivalence class $[p_i]_\sim$ ($1 \leqslant i \leqslant n$), $\theta|_{\tilde{X}'}$ is a variable renaming from $\tilde{X}'$ to $\tilde{Y}'$. This shows (12) and completes the proof. ◻

In general, the commutation property allows us to postpone all equivalence translations, depicted by the following diagram:

$$\begin{tikzcd}
\Pi X.\, s\,[\exists\vec{x}.\,\varphi] \arrow[r, squiggly] \arrow[d, dashed] & \cdot \arrow[d] & & & \\
\Pi X'.\, s'\,[\exists\vec{x}'.\,\varphi'] \arrow[r, squiggly, dashed] \arrow[d, dashed] & \cdot \arrow[r, squiggly] & \cdot \arrow[d] & & \\
\Pi X''.\, s''\,[\exists\vec{x}''.\,\varphi''] \arrow[rr, squiggly, dashed] \arrow[d, dashed] & & \cdot \arrow[r, squiggly] & \cdot \arrow[d] & \\
\Pi X'''.\, s'''\,[\exists\vec{x}'''.\,\varphi''] \arrow[rrr, squiggly, dashed] & & & \cdot \arrow[r, squiggly] & \Pi Y.\, t\,[\exists\vec{y}.\,\psi]
\end{tikzcd}$$

Here, the rewrite steps $\Pi X.\, s\,[\exists\vec{x}.\,\varphi] \rightsquigarrow^* \Pi Y.\, t\,[\exists\vec{y}.\,\psi]$ can be obtained via the rewrite steps $\Pi X.\, s\,[\exists\vec{x}.\,\varphi] \to \Pi X'.\, s'\,[\exists\vec{x}'.\,\varphi'] \to \Pi X''.\, s''\,[\exists\vec{x}''.\,\varphi''] \to \Pi X'''.\, s'''\,[\exists\vec{x}'''.\,\varphi'''] \sim \Pi Y.\, t\,[\exists\vec{y}.\,\psi]$ if repeated applications of commutation are possible. However, Theorem 6.2 does not allow repeated applications, because, in general, reducts of most general rewrite steps do not necessarily lead again to pattern-general constrained terms. Hence, its restriction to pattern-general terms prevents the possibility of repeated applications.

7 Simulation via Left-Value-Free Rules

In Theorem 6.2, we assume that the equivalent existentially constrained terms are pattern-general. A natural question is, what happens if we drop this condition?

Example 7.1. Let $\rho\colon \Pi\varnothing.\, \mathsf{f}(0) \to 0\,[\mathsf{true}]$ be a constrained rule. Consider the equivalence $\Pi\varnothing.\, \mathsf{f}(0) \sim \Pi\varnothing.\, \mathsf{f}(x)\,[x = 0]$ and a rewrite step $\Pi\varnothing.\, \mathsf{f}(0) \to_\rho \Pi\varnothing.\, 0$. However, $\Pi\varnothing.\, \mathsf{f}(x)\,[x = 0]$ cannot be rewritten by ρ, because there does not exist a γ such that $\mathsf{f}(x) = \mathsf{f}(0)\gamma$. Thus constrained terms that are not pattern-general, may not commute with rewrite steps in general.

However, we can recover commutation using a reasonable transformation on constrained rewrite rules. In this section, we introduce this transformation and show that any constrained rewrite rule can be simulated by using it.

The counterexample of commutation in Theorem 7.1 uses the rule $\rho\colon \Pi\varnothing.\, \mathsf{f}(0) \to 0\,[\mathsf{true}]$, which is not able to reduce the constrained term $\Pi\varnothing.\, \mathsf{f}(x)\,[x = 0]$. However, the rule $\rho'\colon \Pi\{x\}.\, \mathsf{f}(x) \to 0\,[x = 0]$ has the same effect as ρ, but can reduce the term as follows: $\Pi\varnothing.\, \mathsf{f}(x)\,[x = 0] \to_{\rho'} \Pi\varnothing.\, 0\,[\exists x.\, x = 0] \sim \Pi\varnothing.\, 0$. In the following, we show how this can be achieved for arbitrary constrained rewrite rules. We define left-value-free rewrite rules [6] in our setting as follows.

Definition 7.2 (Left-Value-Free Rewrite Rule). A left-linear constrained rewrite rule $\Pi Z.\, \ell \to r\,[\pi]$ is *left-value-free* if $\mathcal{V}\mathrm{al}(\ell) = \varnothing$.

Definition 7.3 (Left-Value-Free Transformation). Let $\rho\colon \Pi Z.\, \ell \to r\,[\pi]$ be a left-linear constrained rule, $\mathcal{P}\mathrm{os}_{\mathcal{V}\mathrm{al}}(\ell) = \{p_1, \ldots, p_n\}$, and $\ell|_{p_i} = v_i$ for $1 \leqslant i \leqslant n$. Taking the fresh variables $x_1, \ldots, x_n$, we define the *left-value-free transformation* of ρ as $\mathrm{lvf}(\rho) = \Pi\hat{Z}.\, \hat{\ell} \to r\,[\hat{\pi}]$ where $\hat{Z} = Z \cup \{x_1, \ldots, x_n\}$, $\hat{\ell} = \ell[x_1, \ldots, x_n]_{p_1,\ldots,p_n}$, and $\hat{\pi} = \pi \wedge \bigwedge_{i=1}^n (x_i = v_i)$.

We demonstrate that restricting rewrite rules to left-value-free ones preserves generality.

Theorem 7.4. *Let $\rho\colon \Pi Z.\, \ell \to r\,[\pi]$ be a left-linear constrained rewrite rule, $\mathrm{lvf}(\rho) = \hat{\rho}\colon \Pi\hat{Z}.\, \hat{\ell} \to r\,[\hat{\pi}]$, and $\Pi X.\, s\,[\exists\vec{x}.\,\varphi]$ an existentially constrained term such that $\mathcal{V}\mathrm{ar}(\hat{\rho}) \cap \mathcal{V}\mathrm{ar}(s, \varphi) = \varnothing$. If $\Pi X.\, s\,[\exists\vec{x}.\,\varphi] \to_\rho^p \Pi Y.\, t\,[\exists\vec{y}.\,\psi]$ then there exists a constrained term $\Pi Y'.\, t'\,[\exists\vec{y}'.\,\psi']$ such that $\Pi X.\, s\,[\exists\vec{x}.\,\varphi] \to_{\hat{\rho}}^p \Pi Y'.\, t'\,[\exists\vec{y}'.\,\psi'] \sim \Pi Y.\, t\,[\exists\vec{y}.\,\psi]$.*

Proof. Suppose $\ell = \ell[v_1, \ldots, v_n]_{p_1,\ldots,p_n}$, where $\{p_1, \ldots, p_n\} = \mathcal{P}\mathrm{os}_{\mathcal{V}\mathrm{al}}(\ell)$. Let $y_1, \ldots, y_n$ are fresh variables, $\hat{Y} = \{y_1, \ldots, y_n\}$, $\hat{Z} = Z \cup \hat{Y}$, $\hat{\ell} = \ell[y_1, \ldots, y_n]_{p_1,\ldots,p_n}$, and $\hat{\pi} = \pi \wedge \bigwedge_{i=1}^n (y_i = v_i)$. We obtain $\mathcal{V}\mathrm{ar}(\hat{\ell}) \cap \hat{Z} = (\mathcal{V}\mathrm{ar}(\ell) \cup \hat{Y}) \cap (Z \cup \hat{Y}) = (\mathcal{V}\mathrm{ar}(\ell) \cap Z) \cup \hat{Y}$ and $\mathcal{V}\mathrm{ar}(\hat{\pi}) \setminus \mathcal{V}\mathrm{ar}(\hat{\ell}) = (\mathcal{V}\mathrm{ar}(\pi) \cup \hat{Y}) \setminus (\mathcal{V}\mathrm{ar}(\ell) \cup \hat{Y}) = \mathcal{V}\mathrm{ar}(\pi) \setminus \mathcal{V}\mathrm{ar}(\ell)$. Suppose $\Pi X.\, s\,[\exists\vec{x}.\,\varphi] \to_{\rho,\gamma}^p \Pi Y.\, t\,[\exists\vec{y}.\,\psi]$. Then we have (1) $\mathcal{D}\mathrm{om}(\gamma) = \mathcal{V}\mathrm{ar}(\ell)$, (2) $s|_p = \ell\gamma$, (3) $\gamma(x) \in \mathcal{V}\mathrm{al} \cup X$ for any $x \in \mathcal{V}\mathrm{ar}(\ell) \cap Z$, and (4) $\vDash_{\mathcal{M}} (\exists\vec{x}.\,\varphi) \Rightarrow (\exists\vec{z}.\,\pi\gamma)$, where $\{\vec{z}\} = \mathcal{V}\mathrm{ar}(\pi) \setminus \mathcal{V}\mathrm{ar}(\ell)$, $t = s[r\gamma]$, $\psi = \varphi \wedge \pi\gamma$, $\{\vec{y}\} = \mathcal{V}\mathrm{ar}(\psi) \setminus \mathcal{V}\mathrm{ar}(t)$, and $Y = \mathcal{E}x\mathcal{V}\mathrm{ar}(\rho) \cup (X \cap \mathcal{V}\mathrm{ar}(t))$.

Let $\delta = \gamma \cup \{y_i \mapsto v_i \mid 1 \leqslant i \leqslant n\}$. We first show that the term $\Pi X.\, s\,[\exists\vec{x}.\,\varphi]$ has a $\hat{\rho}$-redex at $p \in \mathcal{P}\mathrm{os}(s)$ using δ. By (1), we have $\mathcal{D}\mathrm{om}(\delta) = \mathcal{D}\mathrm{om}(\gamma) \cup \hat{Y} = \mathcal{V}\mathrm{ar}(\ell) \cup \hat{Y}$. From (2), we have $s|_p = \ell\gamma = \ell[v_1, \ldots, v_n]\gamma = \ell\gamma[v_1, \ldots, v_n] = \ell\delta[\delta(y_1), \ldots, \delta(y_n)] = \ell[y_1, \ldots, y_n]\delta = \hat{\ell}\delta$. We proceed to show that $\delta(x) \in \mathcal{V}\mathrm{al} \cup X$ for any $x \in \mathcal{V}\mathrm{ar}(\hat{\ell}) \cap \hat{Z}$. Let $x \in \mathcal{V}\mathrm{ar}(\hat{\ell}) \cap \hat{Z} = (\mathcal{V}\mathrm{ar}(\ell) \cap Z) \cup \{y_1, \ldots, y_n\}$. If $x \in \mathcal{V}\mathrm{ar}(\ell) \cap Z$, then $\delta(x) = \gamma(x) \in \mathcal{V}\mathrm{al} \cup X$. Otherwise, $x \in \hat{Y}$, and thus $\delta(x) = \delta(y_i) = v_i \in \mathcal{V}\mathrm{al} \subseteq \mathcal{V}\mathrm{al} \cup X$ for some $1 \leqslant i \leqslant n$. Using the fact that $\mathcal{V}\mathrm{ar}(\hat{\pi}) \setminus \mathcal{V}\mathrm{ar}(\hat{\ell}) = \mathcal{V}\mathrm{ar}(\pi) \setminus \mathcal{V}\mathrm{ar}(\ell)$, we show that $\vDash_{\mathcal{M}} (\exists\vec{x}.\,\varphi) \Rightarrow (\exists\vec{z}.\,\hat{\pi}\delta)$. Here we have $\hat{\pi}\gamma = (\pi \wedge \bigwedge_{i=1}^n (y_i = v_i))\gamma = \pi\gamma \wedge \bigwedge_{i=1}^n (v_i = v_i) = \pi\delta \wedge \bigwedge_{i=1}^n (v_i = v_i)$. Clearly $\vDash_{\mathcal{M}} \pi\gamma \Leftrightarrow \hat{\pi}\delta$. Hence, from (4), we have $\vDash_{\mathcal{M}} (\exists\vec{x}.\,\varphi) \Rightarrow (\exists\vec{z}.\,\hat{\pi}\delta)$. We conclude that $\Pi X.\, s\,[\exists\vec{x}.\,\varphi]$ has a $\hat{\rho}$-redex at $p \in \mathcal{P}\mathrm{os}(s)$ using δ.

Hence, we obtain $\Pi X.\, s\,[\exists\vec{x}.\,\varphi] \to_{\hat{\rho},\delta}^p \Pi Y'.\, t'\,[\exists\vec{y}'.\,\psi']$, where $t' = s[r\delta]$, $\psi' = \varphi \wedge \hat{\pi}\delta$, $\{\vec{y}'\} = \mathcal{V}\mathrm{ar}(\psi') \setminus \mathcal{V}\mathrm{ar}(t')$, and $Y' = \mathcal{E}x\mathcal{V}\mathrm{ar}(\hat{\rho}) \cup (X \cap \mathcal{V}\mathrm{ar}(t'))$. From $\mathcal{E}x\mathcal{V}\mathrm{ar}(\hat{\rho}) \cap \hat{Y} = \varnothing$, we have $r\delta = r\gamma$ and $\pi\delta = \pi\gamma$. Thus, (5) $t' = s[r\delta] = s[r\gamma] = t$. It also follows from $\psi = \varphi \wedge \pi\gamma$, $\psi' = \varphi \wedge \hat{\pi}\delta$, and $\vDash_{\mathcal{M}} \pi\gamma \Leftrightarrow \hat{\pi}\delta$ (shown above), that (6) $\vDash_{\mathcal{M}} \psi' \Leftrightarrow \psi$. Using (5) and $\mathcal{V}\mathrm{ar}(\hat{\pi}\delta) = \mathcal{V}\mathrm{ar}(\pi\delta \wedge \bigwedge_{i=1}^n (v_i = v_i)) = \mathcal{V}\mathrm{ar}(\pi\gamma)$, we obtain that, (7) $\{\vec{y}'\} = \mathcal{V}\mathrm{ar}(\psi') \setminus \mathcal{V}\mathrm{ar}(t') = \mathcal{V}\mathrm{ar}(\varphi \wedge \hat{\pi}\delta) \setminus \mathcal{V}\mathrm{ar}(t') = \mathcal{V}\mathrm{ar}(\varphi \wedge \pi\gamma) \setminus \mathcal{V}\mathrm{ar}(t) = \mathcal{V}\mathrm{ar}(\psi) \setminus \mathcal{V}\mathrm{ar}(t) = \{\vec{y}\}$.

By $\mathcal{E}x\mathcal{V}\mathrm{ar}(\hat{\rho}) \cap \hat{Y} = \emptyset$, we have $\mathcal{E}x\mathcal{V}\mathrm{ar}(\rho) = \mathcal{E}x\mathcal{V}\mathrm{ar}(\hat{\rho})$. (8) $Y' = \mathcal{E}x\mathcal{V}\mathrm{ar}(\hat{\rho}) \cup (X \cap \mathcal{V}\mathrm{ar}(t')) = \mathcal{E}x\mathcal{V}\mathrm{ar}(\rho) \cup (X \cap \mathcal{V}\mathrm{ar}(t)) = Y$. Having (5)–(8), it is easy to check from the definition of equivalence that $\Pi Y'.\ t'\ [\exists \vec{y'}.\ \psi'] \sim \Pi Y.\ t\ [\exists \vec{y}.\ \psi]$ holds. ◻

Example 7.5. Let us revisit Theorem 7.1. There we consider the constrained rewrite rule $\rho\colon \Pi\emptyset.\ \mathsf{f}(0) \to 0\ [\mathsf{true}]$ and the equivalence $\Pi\emptyset.\ \mathsf{f}(0) \sim \Pi\emptyset.\ \mathsf{f}(x)\ [x = 0]$. Let $\hat{\rho} = \mathrm{lvf}(\rho) = \Pi\{y\}.\ \mathsf{f}(y) \to 0\ [\mathsf{true} \wedge y = 0]$. Using the rule $\hat{\rho}$, instead of ρ, yields $\Pi\emptyset.\ \mathsf{f}(0) \to_{\hat{\rho}} \Pi\emptyset.\ 0\ [\mathsf{true} \wedge 0 = 0]$ and $\Pi\emptyset.\ \mathsf{f}(x)\ [x = 0] \to_{\hat{\rho}} \Pi\emptyset.\ 0\ [\exists x.\ \mathsf{true} \wedge (x = 0) \wedge (x = 0)]$. In this case, clearly, $\Pi\emptyset.\ 0\ [\mathsf{true} \wedge 0 = 0] \sim \Pi\emptyset.\ 0\ [\exists x.\ \mathsf{true} \wedge (x = 0) \wedge (x = 0)]$.

In the next section we show that, if we only consider left-value-free constrained rewrite rules, then the commutation property holds for all existentially constrained terms that are not necessarily pattern-general. Considering left-value-free rewrite rules does not weaken the applicability of the commutation property, because every constrained rewrite rule can be translated to a left-value-free rule, while preserving its effect, using Theorem 7.4.

8 General Commutation for Rewrite Steps with Left-Value-Free Rules and Equivalence

By focusing on pattern-general terms, we have shown in Theorem 6.2 that rewrite steps commute with the equivalence transformation. However, this commutation property inhabits issues w.r.t. to repeated applications as explained at the end of Section 6.

In the following we show that commutation holds without that restriction if the employed rule is left-value-free. Since any rule has an equivalent left-value-free rule, by Theorem 7.4, one can repeatedly apply the commutation property. This guarantees that the equivalence transformations can be postponed till the end of the rewrite sequence. We first introduce a useful notation that is used in the following proofs.

Definition 8.1. Let X be a set of variables and s a term. Suppose $\{p_1, \ldots, p_n\} = \mathcal{P}\mathrm{os}_{X \cup \mathcal{V}\mathrm{al}}(s)$. We define $s^{\bullet_X} = s[x_1, \ldots, x_n]_{p_1,\ldots,p_n}$ for some pairwise distinct fresh variables $x_1, \ldots, x_n$. We denote by $u^{\bullet_X} = s^{\bullet_X}|_p$ each subterm $u = s|_p$ of s. Consider a linear term ℓ together with a substitution γ such that $s = \ell\gamma$ with $\mathcal{D}\mathrm{om}(\gamma) = \mathcal{V}\mathrm{ar}(\ell)$. We define a substitution $\gamma^{\bullet_X}$ by $\gamma^{\bullet_X}(x) = \gamma(x)^{\bullet_X}$. When no confusion arises, the superscript $\bullet_X$ will be abbreviated by $\bullet$.

We now develop several properties of $\bullet$-translation; all proofs are given in the full version of this paper [17].

Lemma 8.2. *Let X be a set of variables and s a term. Let $s^{\bullet_X} = s[x_1, \ldots, x_n]_{p_1,\ldots,p_n}$ and $\sigma = \{x_i \mapsto s|_{p_i} \mid 1 \leqslant i \leqslant n\}$. Then, $u^{\bullet_X}\sigma = u$ holds for any subterm u of s.*

Lemma 8.3. *Let X be a set of variables, s a term, and ℓ a linear term such that $s|_p = \ell\gamma$ with $\mathcal{D}\mathrm{om}(\gamma) = \mathcal{V}\mathrm{ar}(\ell)$ for a substitution γ. If ℓ is value-free then $s^{\bullet_X}|_p = \ell\gamma^{\bullet_X}$.*

Lemma 8.4. *Let X be a set of variables and s a term. Suppose that $\mathcal{P}\mathrm{os}_{X \cup \mathcal{V}\mathrm{al}}(s) = \{p_1, \ldots, p_n\}$ and $s^{\bullet_X} = s[x_1, \ldots, x_n]_{p_1,\ldots,p_n}$. Then, for any subterm u of s, we have $\vDash_{\mathcal{M}} (\bigwedge_{i=1}^{n}(s(p_i) = x_i)) \Rightarrow u = u^{\bullet_X}$.*

Below we use yet another specific characterization of equivalence given as follows, whose proof is given in the full version of this paper [17].

Lemma 8.5. *Let $\Pi X.\ s\ [\exists \vec{x}.\ \varphi], \Pi Y.\ t\ [\exists \vec{y}.\ \psi]$ be satisfiable existentially constrained terms. Suppose (1) $\sigma\colon V \to \mathcal{V} \cup \mathcal{V}\mathrm{al}$ with $V \subseteq X$ and (2) $s\sigma = t$, (3) for any $x, y \in V$, $\sigma(x) = \sigma(y)$ implies $\vDash_{\mathcal{M}} (\exists \vec{x}.\ \varphi) \Rightarrow x = y$. (4) $X \setminus V = Y \setminus \sigma(V)$ and (5) $\sigma(V) \cap \mathcal{V} \subseteq Y$, (6) $\mathcal{F}\mathcal{V}\mathrm{ar}((\exists \vec{x}.\ \varphi)\sigma) = \mathcal{F}\mathcal{V}\mathrm{ar}(\exists \vec{y}.\ \psi)$ and $\vDash_{\mathcal{M}} (\exists \vec{x}.\ \varphi)\sigma \Leftrightarrow (\exists \vec{y}.\ \psi)$. Then $\Pi X.\ s\ [\exists \vec{x}.\ \varphi] \sim \Pi Y.\ t\ [\exists \vec{y}.\ \psi]$.*

We show a key lemma for the theorem that follows.

Lemma 8.6. *Let ρ be a left-value-free constrained rewrite rule. If $\Pi X.\ s\ [\exists \vec{x}.\ \varphi] \to_\rho \Pi Y.\ t\ [\exists \vec{y}.\ \psi]$ then $\mathrm{PG}(\Pi X.\ s\ [\exists \vec{x}.\ \varphi]) \to_\rho \Pi Y'.\ t'\ [\exists \vec{y}'.\ \psi'] \sim \Pi Y.\ t\ [\exists \vec{y}.\ \psi]$ for some $\Pi Y'.\ t'\ [\exists \vec{y}'.\ \psi']$.*

Proof. Let $p \in \mathcal{P}\mathrm{os}(s)$ and $\rho\colon \Pi Z.\ \ell \to r\ [\pi]$ be a left-linear constrained rewrite rule such that $\mathcal{V}\mathrm{al}(\ell) = \emptyset$ and $\mathcal{V}\mathrm{ar}(\rho) \cap \mathcal{V}\mathrm{ar}(\varphi, s) = \emptyset$. Suppose $\Pi X.\ s\ [\exists \vec{x}.\ \varphi] \to^{p}_{\rho,\gamma} \Pi Y.\ t\ [\exists \vec{y}.\ \psi]$. Then we have (1) $\mathcal{D}\mathrm{om}(\gamma) = \mathcal{V}\mathrm{ar}(\ell)$, (2) $s|_p = \ell\gamma$, (3) $\gamma(x) \in \mathcal{V}\mathrm{al} \cup X$ for any $x \in \mathcal{V}\mathrm{ar}(\ell) \cap Z$, and (4) $\vDash_{\mathcal{M}} (\exists \vec{x}.\ \varphi) \Rightarrow (\exists \vec{z}.\ \pi\gamma)$, where $\{\vec{z}\} = \mathcal{V}\mathrm{ar}(\pi) \setminus \mathcal{V}\mathrm{ar}(\ell)$, $t = s[r\gamma]$, $\psi = \varphi \wedge \pi\gamma$, $\{\vec{y}\} = \mathcal{V}\mathrm{ar}(\psi) \setminus \mathcal{V}\mathrm{ar}(t)$, and $Y = \mathcal{E}x\mathcal{V}\mathrm{ar}(\rho) \cup (X \cap \mathcal{V}\mathrm{ar}(t))$.

Using the notation of Theorem 8.1, define $\mathrm{PG}(\Pi X.\ s\ [\exists \vec{x}.\ \varphi]) = \Pi X'.\ s^{\bullet_X}\ [\exists \vec{x}'.\ \varphi']$, i.e. $s^{\bullet_X} = s[x_1, \ldots, x_n]_{p_1,\ldots,p_n}$ for set of positions $\{p_1, \ldots, p_n\} = \mathcal{P}\mathrm{os}_{X \cup \mathcal{V}\mathrm{al}}(s)$, some pairwise distinct fresh variables $x_1, \ldots, x_n$, $X' = \{x_1, \ldots, x_n\}$ and $\{\vec{x}'\} = \{\vec{x}\} \cup X$ and $\varphi' = (\varphi \wedge \bigwedge_{i=1}^{n}(s|_{p_i} = x_i))$.

We first show $\Pi X'.\ s^\bullet\ [\exists \vec{x}'.\ \varphi']$ has a ρ-redex at p by $\gamma^\bullet$. Clearly, (1') $\mathcal{D}\mathrm{om}(\gamma^\bullet) = \mathcal{D}\mathrm{om}(\gamma) = \mathcal{V}\mathrm{ar}(\ell)$. By Theorem 8.3, we have (2') $s^\bullet|_p = \ell\gamma^\bullet$. Suppose $x \in \mathcal{V}\mathrm{ar}(\ell) \cap Z$. Then $\gamma(x) \in \mathcal{V}\mathrm{al} \cup X$, which implies $\gamma(x) = s|_{p_i}$ for some $1 \leqslant i \leqslant n$. Then, $\gamma^\bullet(x) = \gamma(x)^\bullet = (s|_{p_i})^\bullet = s^\bullet|_{p_i} = (s[x_1, \ldots, x_n]_{p_1,\ldots,p_n})|_{p_i} = x_i \in X' \subseteq \mathcal{V}\mathrm{al} \cup X'$. Thus, (3') $\gamma^\bullet(x) \in \mathcal{V}\mathrm{al} \cup X'$ for any $x \in \mathcal{V}\mathrm{ar}(\ell) \cap Z$ follows. It remains to show (4') $\vDash_{\mathcal{M}} (\exists \vec{x}'.\ \varphi') \Rightarrow (\exists \vec{z}.\ \pi\gamma^\bullet)$.

For this, let $\vDash_{\mathcal{M},\xi} \exists \vec{x}'.\ \varphi'$ for a valuation ξ. This means, $\vDash_{\mathcal{M},\xi} \exists \vec{x}, \vec{x}''.\ \varphi \wedge \bigwedge_{i=1}^{n}(s|_{p_i} = x_i))$, where $\{\vec{x}''\} = X$. By Theorem 8.4, $\vDash_{\mathcal{M}} \bigwedge_{i=1}^{n}(s(p_i) = x_i)) \Rightarrow (\gamma(x) = \gamma(x)^\bullet)$ holds for any $x \in \mathcal{V}$. For, it trivially follows if $x \notin \mathcal{D}\mathrm{om}(\gamma)$, and otherwise, $\gamma(x)$ is a subterm of s, and thus Theorem 8.4 applies. Note that $s(p_i) = s|_{p_i}$ for all $1 \leqslant i \leqslant n$. This implies that $\vDash_{\mathcal{M},\xi} \exists \vec{x}, \vec{x}''.\ \varphi \wedge (\bigwedge_{i=1}^{n}(s|_{p_i} = x_i)) \wedge (\bigwedge_{x \in \mathcal{V}\mathrm{ar}(\pi)}(\gamma(x) = \gamma(x)^\bullet))$. Hence, we obtain $\vDash_{\mathcal{M},\xi} \exists \vec{x}, \vec{x}''.\ \varphi \wedge (\bigwedge_{i=1}^{n}(s|_{p_i} = x_i)) \wedge (\pi\gamma \Leftrightarrow \pi\gamma^\bullet)$. Now, by definition, there exists a sequence $\vec{v} \in \mathcal{V}\mathrm{al}^*$ such that $\vDash_{\mathcal{M},\xi} (\exists \vec{x}.\ \varphi \wedge (\bigwedge_{i=1}^{n}(s|_{p_i} = x_i)) \wedge (\pi\gamma \Leftrightarrow \pi\gamma^\bullet))\kappa$, where $\kappa = \{\vec{x}'' \mapsto \vec{v}\}$. By $\mathcal{D}\mathrm{om}(\kappa) = \{\vec{x}''\} = X$, it follows that $\vDash_{\mathcal{M},\xi} \exists \vec{x}.\ \varphi\kappa \wedge (\bigwedge_{i=1}^{n}(s|_{p_i}\kappa = x_i)) \wedge (\pi\gamma\kappa = \pi\gamma^\bullet)$. In particular, $\vDash_{\mathcal{M},\xi} \exists \vec{x}.\ \varphi\kappa$ holds and then by $\mathcal{D}\mathrm{om}(\kappa) \cap \{\vec{x}\} = \emptyset$, we have $\vDash_{\mathcal{M},\xi} (\exists \vec{x}.\ \varphi)\kappa$, i.e., $\vDash_{\mathcal{M},\xi\circ\kappa} \exists \vec{x}.\ \varphi$. Then, by (4), it follows $\vDash_{\mathcal{M},\xi\circ\kappa} \exists \vec{z}.\ \pi\gamma$, i.e., $\vDash_{\mathcal{M},\xi} (\exists \vec{z}.\ \pi\gamma)\kappa$. As $\{\vec{z}\} \cap X = \emptyset$, this implies $\vDash_{\mathcal{M},\xi} \exists \vec{z}.\ \pi\gamma\kappa$. Moreover, by $\{\vec{x}\} \cap \mathcal{V}\mathrm{ar}(\pi\gamma\kappa, \pi\gamma^\bullet) = \emptyset$, from $\vDash_{\mathcal{M},\xi} \exists \vec{x}.\ (\pi\gamma\kappa \Leftrightarrow \pi\gamma^\bullet)$ it follows that $\vDash_{\mathcal{M},\xi} (\pi\gamma\kappa \Leftrightarrow \pi\gamma^\bullet)$. Thus, we obtain $\vDash_{\mathcal{M},\xi} \exists \vec{z}.\ \pi\gamma^\bullet$ and have proven (4').

Hence, we obtain the rewrite step $\Pi X'.\ s^\bullet\ [\exists \vec{x}'.\ \varphi'] \to^{p}_{\rho,\gamma^\bullet} \Pi Y'.\ t'\ [\exists \vec{y}'.\ \psi']$, where $t' = s^\bullet[r\gamma^\bullet]$, $\psi' = \varphi' \wedge \pi\gamma^\bullet$, $\{\vec{y}'\} = \mathcal{V}\mathrm{ar}(\psi') \setminus \mathcal{V}\mathrm{ar}(t')$, and $Y' = \mathcal{E}x\mathcal{V}\mathrm{ar}(\rho) \cup (X' \cap \mathcal{V}\mathrm{ar}(t'))$.

It remains to show $\Pi Y'.\ t'\ [\exists \vec{y}'.\ \psi'] \sim \Pi Y.\ t\ [\exists \vec{y}.\ \psi]$. For this, we use Theorem 8.5 by taking $\sigma = \{x_i \mapsto s|_{p_i} \mid 1 \leqslant i \leqslant n, x_i \in Y'\}$ and $V = \{x_i \mid 1 \leqslant i \leqslant n, x_i \in Y'\}$. We refer to the conditions (1)–(6) of Theorem 8.5 by (C1)–(C6). Clearly, (C1) $\sigma\colon V \to \mathcal{V} \cup \mathcal{V}\mathrm{al}$ with $V \subseteq Y'$. (C2) holds as $t'\sigma = s^\bullet[r\gamma^\bullet]\sigma = s^\bullet\sigma[r\gamma^\bullet\sigma] = s[r\gamma] = t$, using Theorem 8.2. Let $x_i, x_j \in Y'$ and

$\sigma(x_i) = \sigma(x_j)$. Then, $s|_{p_i} = s|_{p_j}$ and hence $\vDash_{\mathcal{M}} \psi' \Rightarrow (s|_{p_i} = s|_{p_j})$. By $x_i, x_j \in Y'$, we know that $x_i, x_j \notin \{\vec{y}'\}$, and hence it follows $\vDash_{\mathcal{M}} \exists\vec{y}'.\ \psi' \Rightarrow (x_i = x_j)$. From this we have (C3). Note that $Y' = \mathcal{E}x\mathcal{V}ar(\rho) \cup (X' \cap \mathcal{V}ar(t')) = \mathcal{E}x\mathcal{V}ar(\rho) \cup (\{x_1, \ldots, x_n\} \cap \mathcal{V}ar(t'))$ and hence $V = \{x_1, \ldots, x_n\} \cap \mathcal{V}ar(t')$ and that $Y = \mathcal{E}x\mathcal{V}ar(\rho) \cup (\{s|_{p_1}, \ldots, s|_{p_n}\} \cap X \cap \mathcal{V}ar(t))$. Since $Y' \setminus V = \mathcal{E}x\mathcal{V}ar(\rho) = Y \setminus \sigma(V)$, we have (C4). For (C5), let $x \in Y'$ such that $\sigma(x) \in \mathcal{V}$. If $x \in V$ then $\sigma(x) \in Y$. Otherwise $\sigma(x) = x$, and by $s\sigma = t$, we know $x \in \mathcal{V}ar(t)$. Hence $x \in \mathcal{E}x\mathcal{V}ar(\rho) \subseteq Y$. Thus, (C5) holds. In order to show (C6), note first that $\mathcal{FV}ar(\exists\vec{y}'.\ \psi') = \mathcal{V}ar(\psi') \cap \mathcal{V}ar(t')$ and $\mathcal{FV}ar(\exists\vec{y}.\ \psi) = \mathcal{V}ar(\psi) \cap \mathcal{V}ar(t)$. Note that $\psi'\sigma = (\varphi' \wedge \pi\gamma^\bullet)\sigma = (\varphi \wedge \bigwedge_i (s|_{p_i} = x_i) \wedge \pi\gamma^\bullet)\sigma = (\varphi \wedge \bigwedge_i (s|_{p_i} = s|_{p_i}) \wedge \pi\gamma)$ and $\psi = (\varphi \wedge \pi\gamma)$, and hence $\mathcal{V}ar(\psi'\sigma) = \mathcal{V}ar(\psi)$. By $t'\sigma = t$ it follows $\mathcal{FV}ar((\exists\vec{y}'.\ \psi')\sigma) = \sigma(\mathcal{FV}ar(\exists\vec{y}'.\ \psi')) = \mathcal{FV}ar(\exists\vec{y}.\ \psi)$. Finally,

$$
\begin{aligned}
&\vDash_{\mathcal{M}} (\exists\vec{y}'.\ \psi')\sigma \\
&\iff \vDash_{\mathcal{M}} (\exists\vec{y}'.\ (\varphi \wedge (\textstyle\bigwedge_{i=1}^n (s|_{p_i} = x_i))) \wedge \pi\gamma^\bullet)\sigma \\
&\iff \vDash_{\mathcal{M}} \exists\vec{y}'.\ (\varphi \wedge (\textstyle\bigwedge_{i=1}^n (s|_{p_i} = x_i\sigma))) \wedge \pi\gamma^\bullet\sigma \\
&\iff \vDash_{\mathcal{M}} \exists\vec{y}'.\ (\varphi \wedge (\textstyle\bigwedge_{i=1}^n (s|_{p_i} = s|_{p_i})) \wedge \pi\gamma \\
&\iff \vDash_{\mathcal{M}} \exists\vec{y}'.\ (\psi \wedge (\textstyle\bigwedge_{i=1}^n (s|_{p_i} = s|_{p_i})) \\
&\iff \vDash_{\mathcal{M}} \exists\vec{y}'.\ \psi
\end{aligned}
$$

Thus, we have $\vDash_{\mathcal{M}} (\exists\vec{y}'.\ \psi')\sigma \Leftrightarrow (\exists\vec{y}'.\ \psi)$. Since we can eliminate bound variables which do not appear in the constraint, we conclude $\vDash_{\mathcal{M}} (\exists\vec{y}'.\ \psi')\sigma \Leftrightarrow (\exists\vec{y}.\ \psi)$. □

Finally, we are able to prove the main result.

Theorem 8.7 (Commutation of Rewrite Steps and Equivalence by Left-Value-Free Rules). *Let ρ be a left-value-free constrained rewrite rule, and $\Pi X.\ s\ [\exists\vec{x}.\ \varphi], \Pi Y.\ t\ [\exists\vec{y}.\ \psi]$ be satisfiable existentially constrained terms. If $\Pi X'.\ s'\ [\exists\vec{x}'.\ \varphi'] \leftarrow_\rho \Pi X.\ s\ [\exists\vec{x}.\ \varphi] \sim \Pi Y.\ t\ [\exists\vec{y}.\ \psi]$, then we have $\Pi X'.\ s'\ [\exists\vec{x}'.\ \varphi'] \sim \Pi Y'.\ t'\ [\exists\vec{y}'.\ \psi'] \leftarrow_\rho \Pi Y.\ t\ [\exists\vec{y}.\ \psi]$ for some $\Pi Y'.\ t'\ [\exists\vec{y}'.\ \psi']$.*

Proof. Suppose that $\Pi X'.\ s'\ [\exists\vec{x}'.\ \varphi'] \leftarrow_\rho \Pi X.\ s\ [\exists\vec{x}.\ \varphi] \sim \Pi Y.\ t\ [\exists\vec{y}.\ \psi]$. By Theorem 8.6, there exists an existentially constrained term $\Pi X''.\ s''\ [\exists\vec{x}''.\ \varphi'']$ such that $\Pi X'.\ s'\ [\exists\vec{x}'.\ \varphi'] \sim \Pi X''.\ s''\ [\exists\vec{x}''.\ \varphi''] \leftarrow_\rho \mathrm{PG}(\Pi X.\ s\ [\exists\vec{x}.\ \varphi])$. By $\Pi X.\ s\ [\exists\vec{x}.\ \varphi] \sim \Pi Y.\ t\ [\exists\vec{y}.\ \psi]$, $\Pi X''.\ s''\ [\exists\vec{x}''.\ \varphi''] \leftarrow_\rho \mathrm{PG}(\Pi X.\ s\ [\exists\vec{x}.\ \varphi]) \sim \mathrm{PG}(\Pi Y.\ t\ [\exists\vec{y}.\ \psi])$. Hence, by Theorem 6.2, $\Pi X''.\ s''\ [\exists\vec{x}''.\ \varphi''] \sim \Pi Y'.\ t'\ [\exists\vec{y}'.\ \psi'] \leftarrow_\rho \mathrm{PG}(\Pi Y.\ t\ [\exists\vec{y}.\ \psi])$ for some existentially constrained term $\Pi Y'.\ t'\ [\exists\vec{y}'.\ \psi']$. Hence, $\Pi X'.\ s'\ [\exists\vec{x}'.\ \varphi'] \sim \Pi Y'.\ t'\ [\exists\vec{y}'.\ \psi'] \leftarrow_\rho \mathrm{PG}(\Pi Y.\ t\ [\exists\vec{y}.\ \psi])$. □

Theorem 8.7 enables us to defer the equivalence transformations until after the application of rewrite rules.

Corollary 8.8. *Let $\mathcal{R}$ be an LCTRS consisting of left-value-free constrained rewrite rules. If $\Pi X.\ s\ [\exists\vec{x}.\ \varphi] \tilde{\to}_{\mathcal{R}} \Pi Y.\ t\ [\exists\vec{y}.\ \psi]$, then $\Pi X.\ s\ [\exists\vec{x}.\ \varphi] \to^*_{\mathcal{R}} \Pi X'.\ s'\ [\exists\vec{x}'.\ \varphi'] \sim \Pi Y.\ t\ [\exists\vec{y}.\ \psi]$ for some $\Pi X'.\ s'\ [\exists\vec{x}'.\ \varphi']$.*

Proof Sketch. The claim can be proved by induction on n to show that if we have $\Pi X.\ s\ [\exists\vec{x}.\ \varphi](\sim \cdot \to_{\mathcal{R}})^n \cdot \sim \Pi Y.\ t\ [\exists\vec{y}.\ \psi]$, then there exists $\Pi X'.\ s'\ [\exists\vec{x}'.\ \varphi']$ such that $\Pi X.\ s\ [\exists\vec{x}.\ \varphi] \to^*_{\mathcal{R}} \Pi X'.\ s'\ [\exists\vec{x}'.\ \varphi'] \sim \Pi Y.\ t\ [\exists\vec{y}.\ \psi]$. □

9 Related Work

As mentioned earlier, tools working with LCTRSs, like Ctrl or crest, do already implement (mitigated) versions of the results in this paper. Implementation details w.r.t. crest are discussed in [13, 14]. This includes transformations like [13, Definition 12] that produces left-value-free rules, or to check joinability of $s \approx t\ [\varphi]$ crest does not compute any intermediate equivalences but checks triviality by using [13, Lemma 3] at the end of the rewrite sequence. Ctrl [9] makes constrained rewrite rules left-value-free in advance [3, 6], and thus, equivalence transformations are not used in applying constrained rewrite rules to constrained terms. Crisys [4] does not modify the given LCTRSs but uses an extension of matching during rewriting constrained terms, which can be seen as an alternative to equivalence transformations before rewrite steps: When matching a term ℓ with s in a constrained term $s\ [\varphi]$, a value v can match a logical variable x if $x = v$ is guaranteed by φ, i.e., $\vDash_{\mathcal{M}} \varphi \Rightarrow (x = v)$. However, the results of this paper do show promising evidence that tools incorporating these approximations do not loose much power.

Regarding rewriting of constrained terms, the *symbolic rewriting module SMT* has been proposed in [12]. The rewriting formalism there is different from constrained rewriting in this paper. We use the original definition of LCTRSs and constrained rewriting on them with the only difference on the presentation to ease its analysis. In particular, constrained rewriting reduces a constrained term to another one such that every instance in the former has its reduct in the latter. On the other hand, symbolic rewriting in [12] does not guarantee this, hence this is similar to narrowing of constrained terms.

10 Conclusion

In this paper, we have revisited the formalism of constrained rewriting in LCTRSs. We have introduced the new notion of most general rewriting on existentially constrained terms for left-linear LCTRSs. It was discussed in which way our new formalism of most general rewriting extracts the so called "most general" part of the corresponding original formalism. We have shown the uniqueness of reducts for our formalism of constrained rewriting, and commutation between rewrite steps and equivalence for pattern-general constrained terms. Then, by using the notion of non-left-value-free rewrite rules we showed that left-value-free rewrite rules can simulate non-left-value-free rules. Finally, we did recover general commutation between rewrite steps with left-value-free rules and equivalent transformations.

Because of the expected complications, our focus in this paper was on rewriting with left-linear rules. To deal with non-left-linear rules, we need to extend the matching mechanism underlying rewrite steps. However, this complicates the definition of rewrite steps drastically. This extension remains as future work.

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