Matrix Interpretations for Polynomial Derivational Complexity of Rewrite Systems

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joint work with

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LPAR-18

Outline

- Introduction
- History
- Matrix Interpretations
- Algebraic Methods
- Automata-Based Methods
- Concluding Remarks

Example

```
let rec reverse =
function
| [] -> []
| x :: xs -> (reverse xs) @ (x :: []) ;;


let rec shuffle =
function
| [] -> []
| x :: xs -> x :: shuffle (reverse xs) ;;
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Example

rewrite rules

$$
\begin{aligned}
\mathsf{reverse}(\mathsf{nil}) &\longrightarrow \mathsf{nil} \\
\mathsf{reverse}(x :: xs) &\longrightarrow \mathsf{append}(\mathsf{reverse}(xs), x :: \mathsf{nil}) \\
\mathsf{shuffle}(\mathsf{nil}) &\longrightarrow \mathsf{nil} \\
\mathsf{shuffle}(x :: xs) &\longrightarrow x :: \mathsf{shuffle}(\mathsf{reverse}(xs)) \\
\mathsf{append}(\mathsf{nil}, ys) &\longrightarrow ys \\
\mathsf{append}(x :: xs, ys) &\longrightarrow x :: \mathsf{append}(xs, ys)
\end{aligned}
$$

Example

signature nil 0 (constants) reverse shuffle s (unary) append :: (binary)

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\mathsf{reverse}(\mathsf{nil}) &\rightarrow \mathsf{nil} \\
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\end{aligned}
$$

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terms $\mathsf{s}(\mathsf{s}(0))$ $\mathsf{shuffle}(0 :: \mathsf{nil})$

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\mathsf{reverse}(\mathsf{nil}) &\to \mathsf{nil} \\
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rewriting $\quad$ $\mathsf{shuffle}(\mathsf{0} :: \mathsf{s}(\mathsf{0}) :: \mathsf{s}(\mathsf{s}(\mathsf{0})) :: \mathsf{s}(\mathsf{s}(\mathsf{s}(\mathsf{0}))) :: \mathsf{s}(\mathsf{s}(\mathsf{s}(\mathsf{s}(\mathsf{0})))) :: \mathsf{nil})$

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rewriting $\quad$ $\mathsf{shuffle}(0 :: 1 :: 2 :: 3 :: 4 :: \mathsf{nil})$

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Definition

TRS is terminating if $\to_{\mathcal{R}}^{+}$ is well-founded

Definitions

- well-founded monotone $\mathcal{F}$-algebra $(\mathcal{A}, >)$ consists of nonempty algebra $\mathcal{A} = (A, \{f_{\mathcal{A}}\}_{f \in \mathcal{F}})$ together with well-founded order $>$ on A such that every $f_{\mathcal{A}}$ is strictly monotone in all coordinates:

$$f_{\mathcal{A}}(a_1, \ldots, a_i, \ldots, a_n) > f_{\mathcal{A}}(a_1, \ldots, b, \ldots, a_n)$$

for all $a_1, \ldots, a_n, b \in A$ and $i \in \{1, \ldots, n\}$ with $a_i > b$

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- relation $>_\mathcal{A}$ on terms: $s >_\mathcal{A} t$ if $[\alpha]_\mathcal{A}(s) > [\alpha]_\mathcal{A}(t)$ for all assignments α

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Definitions

- derivation height $\quad \mathrm{dh}_\mathcal{R}(t) = \max\{\, n \mid t \to_\mathcal{R}^n u \text{ for some term } u \,\}$
- derivational complexity $\quad \mathrm{dc}_\mathcal{R}(k) = \max\{\, \mathrm{dh}(t) \mid |t| \leqslant k \,\}$

Example

TRS $\mathcal{R}$

$$0 + y \to y \qquad \mathsf{s}(x) + y \to \mathsf{s}(x + y)$$

Example

TRS $\mathcal{R}$ is terminating

$$0 + y \to y \qquad \mathsf{s}(x) + y \to \mathsf{s}(x + y)$$

polynomial interpretation

$$0_{\mathbb{N}} = 0 \qquad \mathsf{s}_{\mathbb{N}}(x) = x + 1 \qquad +_{\mathbb{N}}(x, y) = 2x + y + 1$$

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$$\mathsf{dh}_{\mathcal{R}}(\mathsf{s}^m(0) + \mathsf{s}^n(0)) = m + 1$$

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derivational complexity

$$t_i = \underbrace{\mathsf{s}(0) + \cdots + \mathsf{s}(0)}_{i+1} \qquad t_i = \begin{cases} \mathsf{s}(0) & \text{if } i = 0 \\ t_{i-1} + \mathsf{s}(0) & \text{if } i > 0 \end{cases}$$

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$$|t_i| = 3i + 2 \qquad \mathsf{dh}_{\mathcal{R}}(t_i) = \begin{cases} 0 & \text{if } i = 0 \\ \mathsf{dh}_{\mathcal{R}}(t_{i-1}) + i + 1 & \text{if } i > 0 \end{cases}$$

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derivation height

$$\mathsf{dh}_{\mathcal{R}}(\mathsf{s}^m(0) + \mathsf{s}^n(0)) = m + 1$$

derivational complexity

$$t_i = \underbrace{\mathsf{s}(0) + \cdots + \mathsf{s}(0)}_{i+1} \qquad t_i = \begin{cases} \mathsf{s}(0) & \text{if } i = 0 \\ t_{i-1} + \mathsf{s}(0) & \text{if } i > 0 \end{cases} \qquad \mathsf{dc}_{\mathcal{R}}(k) \in \Theta(k^2)$$

$$|t_i| = 3i + 2 \qquad \mathsf{dh}_{\mathcal{R}}(t_i) = \begin{cases} 0 & \text{if } i = 0 \\ \mathsf{dh}_{\mathcal{R}}(t_{i-1}) + i + 1 & \text{if } i > 0 \end{cases}$$

Examples

1. TRS $\mathcal{R}$ $\quad \mathsf{a}(\mathsf{a}(x)) \rightarrow \mathsf{a}(\mathsf{b}(\mathsf{a}(x)))$

Examples

1. TRS $\mathcal{R}$ $\quad \mathsf{aa} \longrightarrow \mathsf{aba}$

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Examples

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Examples

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3. TRS $\mathcal{R}$ $\quad \mathsf{ab} \to \mathsf{bba}$

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Inferring Complexity Bounds

how to (automatically) establish

- upper bounds on derivational complexity of TRSs ?

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Hofbauer and Lautemann 1989

adapt termination techniques

> *proving termination with one of these specific techniques in general proves more than just the absence of infinite derivations. It turns out that in many cases such a proof implies an upper bound on the maximal length of derivations*

Outline

	termination	derivational complexity	1967
1967	Knuth-Bendix order		

	termination	derivational complexity	1968
1967	Knuth-Bendix order		

	termination	**derivational complexity**	1969
1967	Knuth-Bendix order		

	termination	**derivational complexity**	1970
1967	Knuth-Bendix order		

	termination	derivational complexity	1971
1967	Knuth-Bendix order		

	termination	**derivational complexity**	1972
1967	Knuth-Bendix order		

	termination	derivational complexity	1973
1967	Knuth-Bendix order		

1967	**termination**	**derivational complexity**	1974
1967	Knuth-Bendix order		

	termination	derivational complexity	1975
1967	Knuth-Bendix order		
1975	polynomial interpretations		

	termination	**derivational complexity**	1979
1967	Knuth-Bendix order		
1975	polynomial interpretations		
1979	simple path order		

	termination	**derivational complexity**	1980
1967	Knuth-Bendix order		
1975	polynomial interpretations		
1979	simple path order		
1980	lexicographic path order semantic path order		

	termination	**derivational complexity**	1981
1967	Knuth-Bendix order		
1975	polynomial interpretations		
1979	simple path order		
1980	lexicographic path order semantic path order		
1981	recursive decomposition order		

	termination	**derivational complexity**	1982
1967	Knuth-Bendix order		
1975	polynomial interpretations		
1979	simple path order		
1980	lexicographic path order semantic path order		
1981	recursive decomposition order		
1982	multiset path order		

	termination	**derivational complexity**	1983
1967	Knuth-Bendix order		
1975	polynomial interpretations		
1979	simple path order		
1980	lexicographic path order semantic path order		
1981	recursive decomposition order		
1982	multiset path order		
1983	recursive path order		

	termination		derivational complexity 1989
1967	Knuth-Bendix order		
1975	polynomial interpretations	1989	Hofbauer and Lautemann
1979	simple path order		
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Theorem (Hofbauer and Lautemann 1989)

interpretation in $\mathbb{N}$	bound on derivational complexity
polynomial	double exponential

Example

rewrite system

$$x + 0 \to x \qquad \mathsf{d}(0) \to 0 \qquad \mathsf{q}(0) \to 0$$

$$x + \mathsf{s}(y) \to \mathsf{s}(x + y) \qquad \mathsf{d}(\mathsf{s}(x)) \to \mathsf{s}(\mathsf{s}(\mathsf{d}(x))) \qquad \mathsf{q}(\mathsf{s}(x)) \to \mathsf{q}(x) + \mathsf{s}(\mathsf{d}(x))$$

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interpretations

$$0_{\mathbb{N}} = 2 \quad \mathsf{s}_{\mathbb{N}}(x) = x + 1 \quad +_{\mathbb{N}}(x, y) = x + 2y \quad \mathsf{d}_{\mathbb{N}}(x) = 3x \quad \mathsf{q}_{\mathbb{N}}(x) = x^3$$

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Theorem (Hofbauer and Lautemann 1989)

	interpretation in $\mathbb{N}$	bound on derivational complexity
	polynomial	double exponential
$a_1 x_1 + \cdots + a_n x_n + b$	linear	exponential

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Theorem (Hofbauer and Lautemann 1989)

	interpretation in $\mathbb{N}$	bound on derivational complexity
	polynomial	double exponential
$a_1 x_1 + \cdots + a_n x_n + b$	linear	exponential
$x_1 + \cdots + \quad x_n + b$	strongly linear	linear

Example

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1967	Knuth-Bendix order			
1975	polynomial interpretations	1989	Hofbauer and Lautemann	
1979	simple path order			
1980	lexicographic path order semantic path order			
1981	recursive decomposition order			
1982	multiset path order	1990	Hofbauer	
1983	recursive path order			
1990	transformation order			

Theorem (Hofbauer 1990)

termination proof by multiset path order implies primitive recursive upper bound on derivational complexity

	termination		derivational complexity	1992
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1975	polynomial interpretations	1989	Hofbauer and Lautemann	
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1982	multiset path order	1990	Hofbauer	
1983	recursive path order			
1990	transformation order			
1992	elementary interpretations type introduction			

	termination		derivational complexity
			1995
1967	Knuth-Bendix order		
1975	polynomial interpretations	1989	Hofbauer and Lautemann
1979	simple path order		
1980	lexicographic path order semantic path order	1995	Weiermann
1981	recursive decomposition order		
1982	multiset path order	1990	Hofbauer
1983	recursive path order		
1990	transformation order		
1992	elementary interpretations type introduction		
1995	general path order semantic labeling dummy elimination		

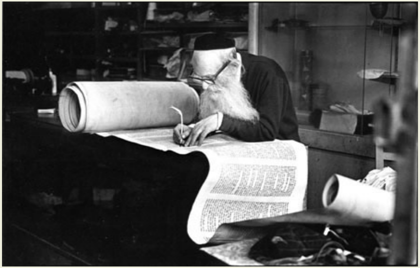
Theorem (Hofbauer 1990)

termination proof by multiset path order implies primitive recursive upper bound on derivational complexity

Theorem (Weiermann 1995)

termination proof by lexicographic path order implies multiple recursive upper bound on derivational complexity

Termination and Complexity Research



	termination	**derivational complexity**	1997
1997	dependency pairs		

	termination	**derivational complexity**	2000
1997	dependency pairs		
2000	monotonic semantic path order		

	termination		derivational complexity 2001
1967	Knuth-Bendix order	2001	Lepper
1975	polynomial interpretations	1989	Hofbauer and Lautemann
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Theorem (Lepper 2001)

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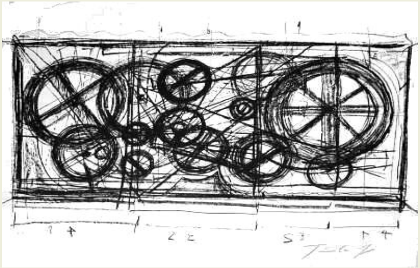
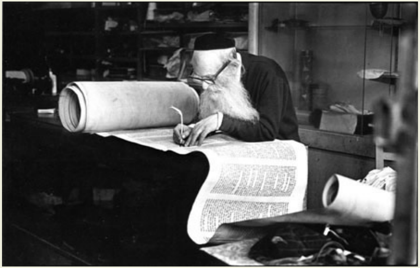
	termination		**derivational complexity**	2003
1997	dependency pairs			
2000	monotonic semantic path order			
2001	context-dependent interpretations	2001		
2003	match-bounds size-change principle			

	termination		**derivational complexity**	2003
1997	dependency pairs			
2000	monotonic semantic path order			
2001	context-dependent interpretations	2001		
2003	match-bounds size-change principle			
2003	**termination competition**			

Termination Research / Complexity Research





Termination Tools

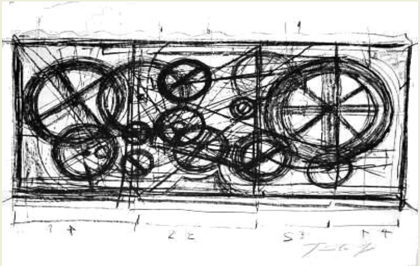
CiME, $\mathsf{T_TT_2}$, AProVE, Termptation, Cariboo, Torpa, TPA, Matchbox, Jambox, MuTerm, NTI, ...

	termination		derivational complexity	2004
1997	dependency pairs			
2000	monotonic semantic path order			
2001	context-dependent interpretations	2001		
2003	match-bounds size-change principle	2004		
2003	**termination competition**			

	termination		derivational complexity	2006
1997	dependency pairs			
2000	monotonic semantic path order			
2001	context-dependent interpretations	2001		
2003	match-bounds size-change principle	2004		
2003	**termination competition**			
2006	matrix interpretations predictive labeling uncurrying			

	termination		**derivational complexity**	2007
1997	dependency pairs			
2000	monotonic semantic path order			
2001	context-dependent interpretations	2001		
2003	match-bounds size-change principle	2004		
2003	**termination competition**			
2006	matrix interpretations predictive labeling uncurrying			
2007	bounded increase quasi-periodic interpretations			

Termination and Complexity Research



Termination Tools

CiME, $\mathsf{T_TT_2}$, AProVE, Matchbox, Jambox, MuTerm, . . .

Complexity Tools

$\mathsf{T_CT}$, Matchbox, CaT

	termination		**derivational complexity**	2008
1997	dependency pairs			
2000	monotonic semantic path order			
2001	context-dependent interpretations	2001		
2003	match-bounds size-change principle	2004		
2003	**termination competition**			
2006	matrix interpretations predictive labeling uncurrying	2008		
2007	bounded increase quasi-periodic interpretations			
2008	arctic interpretations root-labeling	2008	**complexity competition**	

	termination		derivational complexity	2009
1997	dependency pairs	2009		
2000	monotonic semantic path order			
2001	context-dependent interpretations	2001		
2003	match-bounds size-change principle	2004		
2003	**termination competition**			
2006	matrix interpretations predictive labeling uncurrying	2008		
2007	bounded increase quasi-periodic interpretations			
2008	arctic interpretations root-labeling	2008	**complexity competition**	

	termination		**derivational complexity** 2010
1997	dependency pairs	2009	
2000	monotonic semantic path order		
2001	context-dependent interpretations	2001	
2003	match-bounds size-change principle	2004	
2003	**termination competition**		
2006	matrix interpretations predictive labeling uncurrying	2008, 2010	
2007	bounded increase quasi-periodic interpretations		
2008	arctic interpretations root-labeling	2008	**complexity competition**

	termination		derivational complexity 2011
1997	dependency pairs	2009, 2011	
2000	monotonic semantic path order		
2001	context-dependent interpretations	2001, 2008	
2003	match-bounds size-change principle	2004	
2003	**termination competition**		
2006	matrix interpretations predictive labeling uncurrying	2008, 2010, 2011	
2007	bounded increase quasi-periodic interpretations		
2008	arctic interpretations root-labeling	2008	**complexity competition**

	termination		derivational complexity	2012
1997	dependency pairs	2009, 2011		
2000	monotonic semantic path order			
2001	context-dependent interpretations	2001, 2008		
2003	match-bounds size-change principle	2004		
2003	**termination competition**			
2006	matrix interpretations predictive labeling uncurrying	2008, 2010, 2011		
2007	bounded increase quasi-periodic interpretations			
2008	arctic interpretations root-labeling	2008	**complexity competition**	
2012	binomial interpretations ordinal interpretations			

	termination		derivational complexity	2012
1997	dependency pairs	2009, 2011		
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2003	**termination competition**			
2006	matrix interpretations predictive labeling uncurrying	2008, 2010, 2011		
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Outline

- Introduction
- History
- Matrix Interpretations
- Algebraic Methods
- Automata-Based Methods
- Concluding Remarks

Definition (Endrullis, Waldmann, Zantema 2006)

algebra $\mathcal{M}$ with well-founded order $>$

- carrier of $\mathcal{M}$ is $\mathbb{N}^d$ with $d > 0$

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- $(x_1, \ldots, x_d)^\mathsf{T} > (y_1, \ldots, y_d)^\mathsf{T} \quad \Longleftrightarrow \quad x_1 > y_1 \ \wedge \ \bigwedge_{i=2}^{d} x_i \geqslant y_i$

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- interpretations (for every n-ary f)

$$f_\mathcal{M}(\vec{x}_1, \dots, \vec{x}_n) = F_1\, \vec{x}_1 + \cdots + F_n\, \vec{x}_n + f$$

  with

  - matrices $F_1, \dots, F_n \in \mathbb{N}^{d \times d}$ with $(F_i)_{1,1} \geqslant 1$ for all $1 \leqslant i \leqslant n$

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  - vector $\vec{f} \in \mathbb{N}^d$

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- $(x_1, \ldots, x_d)^\mathsf{T} > (y_1, \ldots, y_d)^\mathsf{T} \quad \iff \quad x_1 > y_1 \ \wedge \ \bigwedge_{i=2}^{d} x_i \geqslant y_i$
- interpretations (for every n-ary f)

$$f_\mathcal{M}(\vec{x}_1, \ldots, \vec{x}_n) = F_1\,\vec{x}_1 + \cdots + F_n\,\vec{x}_n + f$$

  with

  - matrices $F_1, \ldots, F_n \in \mathbb{N}^{d \times d}$ with $(F_i)_{1,1} \geqslant 1$ for all $1 \leqslant i \leqslant n$
  - vector $f \in \mathbb{N}^d$

Lemma

$(\mathcal{M}, >)$ is well-founded monotone algebra

Theorem

termination proof by matrix interpretation implies exponential upper bound on derivational complexity

Example

rewrite rule

$$\mathsf{ab} \longrightarrow \mathsf{bba}$$

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Aim

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allow only special upper triangular matrices in interpretations

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Theorem (Moser, Schnabl, Waldmann 2008)

if $\mathcal{R}$ has compatible triangular matrix interpretation of dimension d then

$$\mathsf{dc}_{\mathcal{R}}(k) \in \mathcal{O}(k^d)$$

Example

rewrite system $\mathcal{R}$

$$\mathsf{aba} \longrightarrow \mathsf{abba}$$
$$\mathsf{bbb} \longrightarrow \mathsf{bb}$$

compatible triangular matrix interpretation $\mathcal{M}$

$$\mathsf{a}_{\mathcal{M}}(\vec{x}) = \begin{pmatrix} 1 & 2 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \vec{x} + \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} \quad \mathsf{b}_{\mathcal{M}}(\vec{x}) = \begin{pmatrix} 1 & 0 & 2 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \vec{x} + \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

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- $\mathsf{dc}_{\mathcal{R}}(k) \in \mathcal{O}(k^3)$

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- $\mathsf{dc}_{\mathcal{R}}(k) \in \mathcal{O}(k^3)$ but $\mathsf{dc}_{\mathcal{R}}(k)$ is linear
- no compatible triangular matrix interpretation of dimension 1 or 2 exists

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Extensions

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Extensions

1. using weighted automata techniques (Waldmann 2010)
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3. joint spectral radius theory to unify and strengthen earlier extensions (Middeldorp, Moser, Neurauter, Waldmann, Zankl 2011)

Example

TRS

$$\mathsf{aa} \longrightarrow \mathsf{aba}$$

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Observation

given TRS $\mathcal{R}$ and compatible matrix interpretation $\mathcal{M}$ over $\mathbb{N}$

- every derivation
$$t \to t_1 \to t_2 \to t_3 \to t_4 \to \cdots$$
maps to decreasing sequence of vectors of natural numbers
$$[t] \ > [t_1] \ > [t_2] \ > [t_3] \ > [t_4] \ > \cdots$$
where $[t] = [\alpha_0]_{\mathcal{M}}(t)$

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$$\mathsf{growth}_{\mathcal{M}}(k) = \max \{ [t]_1 \mid |t| \leqslant k \}$$

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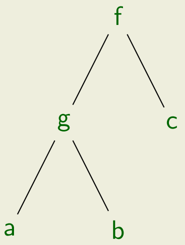
Example

term $t = \mathsf{f}(\mathsf{g}(\mathsf{a}, \mathsf{b}), \mathsf{c})$



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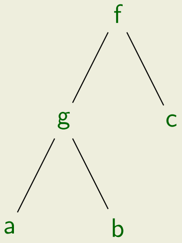

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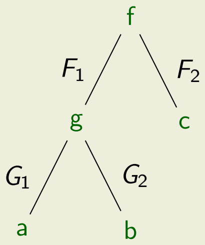

interpretation of t

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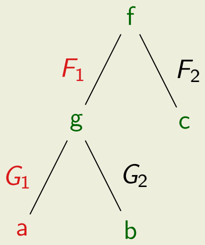

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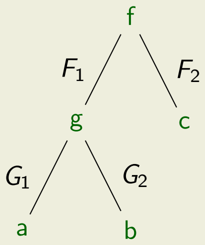

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Remark

term t of size at most k

- *. . . has at most k subterms*
- *. . . each subterm corresponds to product of at most k matrices*

Definition

$S_{\mathcal{M}}$ is set of matrices occurring in matrix interpretation $\mathcal{M}$:

$$S_{\mathcal{M}} = \bigcup_{n\text{-ary } f} \{ F_1, \ldots, F_n \mid f_{\mathcal{M}}(\vec{x}_1, \ldots, \vec{x}_n) = F_1 \vec{x}_1 + \cdots + F_n \vec{x}_n + f \}$$

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Observation

if growth of entries of matrix products

$$A_1 \cdots A_k$$

with $A_1, \dots, A_k \in S_{\mathcal{M}}$ is bounded by a function $f(k)$ then $[t]_1 \in \mathcal{O}(f(|t|) \cdot |t|)$

Definition

matrix interpretation $\mathcal{M}$ is polynomially bounded (with degree d) if growth of entries of matrix products

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if $\mathcal{R}$ has compatible matrix interpretation $\mathcal{M}$ that is polynomially bounded with degree d then $\mathit{growth}_{\mathcal{M}}(k) \in \mathcal{O}(k^{d+1})$

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if $\mathcal{R}$ has compatible matrix interpretation $\mathcal{M}$ that is polynomially bounded with degree d then $\mathrm{growth}_{\mathcal{M}}(k) \in \mathcal{O}(k^{d+1})$ and thus $\mathrm{dc}_{\mathcal{R}}(k) \in \mathcal{O}(k^{d+1})$

Outline

- Introduction
- History
- Matrix Interpretations
- Algebraic Methods
  - Spectral Radius
  - Joint Spectral Radius
- Automata-Based Methods
- Concluding Remarks

Approximation

over-approximate growth of entries of matrix products

$$A_1 \cdots A_k \in S_{\mathcal{M}}^k \quad \text{by} \quad M^k$$

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square matrix $A \in R^{n \times n}$ over ring R ($\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$)

- characteristic polynomial $\chi_A(\lambda)$ of A is $\det(\lambda I_n - A)$

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- polynomial $p \in R[x]$ annihilates A if $p(A) = 0$

Theorem (Cayley-Hamilton)

characteristic polynomial χ_A *annihilates* A

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Example

matrix

$$A = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{pmatrix}$$

characteristic polynomial

$$\chi_A(\lambda) = \lambda^4 - 3\lambda^3 + 3\lambda^2 - \lambda$$

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annihilation

$$A^4 - 3A^3 + 3A^2 - A = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

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minimal polynomial $\mathrm{m}_A(\lambda)$ of square matrix $A \in R^{n \times n}$ is unique monic polynomial of minimum degree that annihilates A

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minimal polynomial

$$\mathrm{m}_A(\lambda) = \lambda^3 - 2\lambda^2 + \lambda = \lambda(\lambda - 1)^2$$

Lemma

given matrix $A \in \mathbb{R}_0^{n \times n}$

$\rho(A) \leqslant 1 \quad \Longleftrightarrow$

entries of A^k *are asymptotically bounded by polynomial in* k

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given matrix $A \in \mathbb{R}_0^{n \times n}$ *and monic polynomial* $p \in \mathbb{R}[x]$ *that annihilates* A

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Corollary

if $\mathcal{R}$ *has compatible matrix interpretation* $\mathcal{M}$ *such that* $\rho(M) \leqslant 1$ *then*

$$\mathsf{dc}_{\mathcal{R}}(k) \in \mathcal{O}(k^{d+1})$$

where $d = \max_{\lambda}(0, \#\mathrm{m}_M(\lambda) - 1)$

Example

TRS

$$\mathsf{aa} \longrightarrow \mathsf{aba}$$

triangular matrix interpretation

$$\mathsf{a}_{\mathcal{M}}(\vec{x}) = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} \vec{x} + \begin{pmatrix} 0 \\ 1 \end{pmatrix} \qquad \mathsf{b}_{\mathcal{M}}(\vec{x}) = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \vec{x}$$

component-wise maximum matrix

$$M = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} \qquad \rho(M) = 1$$

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Theorem (Moser, Schnabl, Waldmann 2008)

if $\mathcal{R}$ has compatible triangular matrix interpretation of dimension n then

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$\rho(M) = 1$ *for every upper triangular complexity matrix* M

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if $\mathcal{R}$ has compatible triangular matrix interpretation $\mathcal{M}$ then

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$$\mathsf{aa} \to \mathsf{aba} \qquad \mathsf{bb} \to \epsilon$$

matrix interpretation

$$\mathsf{a}_{\mathcal{M}}(\vec{x}) = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \vec{x} + \begin{pmatrix} 0 \\ 4 \\ 0 \end{pmatrix} \quad \mathsf{b}_{\mathcal{M}}(\vec{x}) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \vec{x} + \begin{pmatrix} 1 \\ 0 \\ 3 \end{pmatrix}$$

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no compatible triangular matrix interpretations

Example (limitation of spectral radius)

TRS $\mathcal{R}$

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compatible matrix interpretation $\mathcal{M}$

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derivational complexity is linear but

$$M^k = \begin{pmatrix} 1 & 2^{k-1} & 2^{k-1} - 1 \\ 0 & 2^{k-1} & 2^{k-1} \\ 0 & 2^{k-1} & 2^{k-1} \end{pmatrix}$$

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  hence $(M^k)_{ii}$ grows exponentially

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derivational complexity is linear: joint spectral radius

$$\rho\left(\left\{\begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\right\}\right) = 1$$

Outline

Definition

matrix norm is function $\|\cdot\| \colon \mathbb{R}^{n \times n} \to \mathbb{R}$ such that for all $A, B \in \mathbb{R}^{n \times n}$

1. $\|A\| \geqslant 0$ with $\|A\| = 0 \iff A = 0_n$
2. $\|cA\| = |c| \cdot \|A\|$ for all $c \in \mathbb{R}$
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Example

ℓ_1 norm $\|\cdot\|_1$

$$\|A\|_1 = \sum_{1 \leqslant i,j \leqslant n} |A_{ij}|$$

Definitions

finite set $S \subseteq \mathbb{R}^{n \times n}$ of real square matrices and matrix norm $\|\cdot\|$

- growth function

$$\mathrm{growth}_S(k, \|\cdot\|) = \max \{ \|A_1 \cdots A_k\| \mid A_1, \ldots, A_k \in S \}$$

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Lemma

if $S = \{A\}$ then

$$\rho(S) = \lim_{k\to\infty} \|A^k\|^{1/k} = \max\{\, |\lambda| \mid \lambda \textit{ is eigenvalue of } A \,\} = \rho(A)$$

Theorem

problem

instance:	*finite set* $S \subseteq \mathbb{R}^{n \times n}$
question:	$\rho(S) \leqslant 1$ *?*

is undecidable in general

Theorem

problem

> *instance:* finite set $S \subseteq \mathbb{R}^{n \times n}$
> *question:* $\rho(S) \leqslant 1$?

is undecidable in general and decidable (in polynomial time) if $S \subseteq \mathbb{N}^{n \times n}$

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Theorem (based on Jungers, Protasov, Blondel 2008)

if $\rho(S) \leqslant 1$ *for finite set* $S \subseteq \mathbb{N}^{n \times n}$ *then*

$$growth_S(k) \in \begin{cases} \Theta(k^d) & \text{if } d \geqslant 1 \\ \mathcal{O}(k^d) & \text{if } d = 1 \end{cases}$$

where d *is largest integer such that* $\exists$ d *different pairs of indices* $(i_1, j_1), \dots, (i_d, j_d)$

- $\forall\ 1 \leqslant s \leqslant d \quad i_s \neq j_s$ *and* $\exists$ *product* $A \in S^*$ *such that* $A_{i_s\, i_s}, A_{i_s\, j_s}, A_{j_s\, j_s} \geqslant 1$
- $\forall\ 1 \leqslant s < d \quad \exists$ *product* $B \in S^*$ *such that* $B_{j_s\, i_{s+1}} \geqslant 1$

Theorem

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Corollary

if $\mathcal{R}$ has compatible matrix interpretation $\mathcal{M}$ such that

$$\rho(S_{\mathcal{M}}) \leqslant 1$$

then $\mathrm{dc}_{\mathcal{R}}(k) \in \mathcal{O}(k^{d+1})$ where d is largest integer such that ...

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Remark

degree $d + 1$ can be computed in polynomial time

Outline

- Introduction
- History
- Matrix Interpretations
- Algebraic Methods
- **Automata-Based Methods**
- Concluding Remarks

Example

matrix interpretation $\mathcal{M}$ of dimension 3

$$\mathsf{a}_{\mathcal{M}}(\vec{x}) = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \vec{x} \qquad \mathsf{f}_{\mathcal{M}}(\vec{x},\vec{y}) = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 0 & 1 \\ 0 & 0 & 1 \end{pmatrix} \vec{x} + \begin{pmatrix} 1 & 0 & 2 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{pmatrix} \vec{y} + \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$$

weighted automaton $\mathcal{A}$

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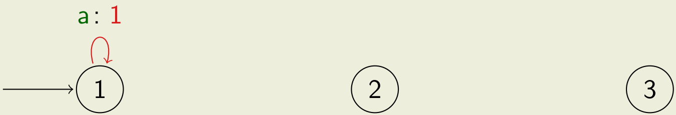

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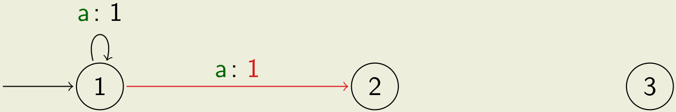

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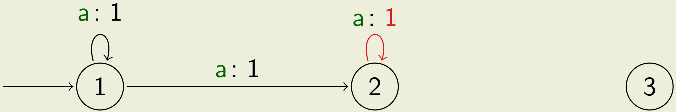

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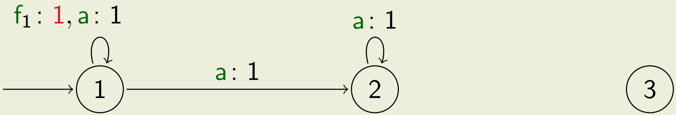

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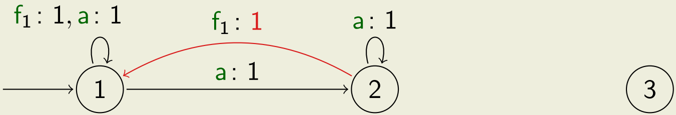

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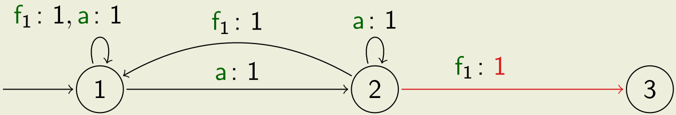

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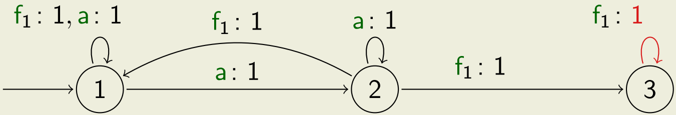

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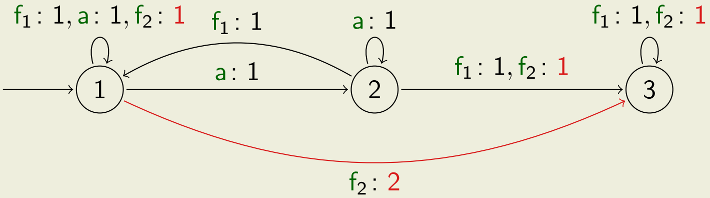

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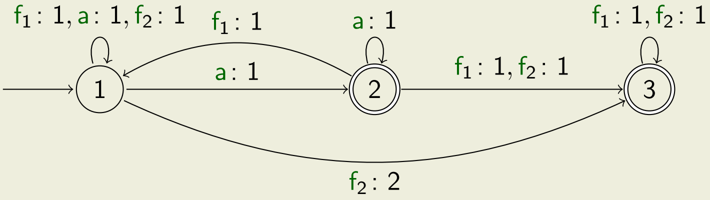

Definition

weighted automaton is quintuple $\mathcal{A} = (Q, \Sigma, \lambda, \mu, \gamma)$ with

1. Q: finite set of states
2. Σ: finite alphabet
3. $\lambda \in Q$ initial state
4. $\mu \colon \Sigma \to \mathbb{N}^{|Q| \times |Q|}$ transition matrix
5. $\gamma \subseteq Q$ final states

$\mu(a)_{pq}$ denotes weight of transition $p \xrightarrow{a} q$

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Definition

weight of string $x \in \Sigma^*$

$$\mathrm{weight}_{\mathcal{A}}(x) = \sum_{q \in \gamma} \mu(x)_{\lambda q}$$

Definition

growth function of weighted automaton $\mathcal{A} = (Q, \Sigma, \lambda, \mu, \gamma)$

$$\text{growth}_{\mathcal{A}}(k) = \max \{ \text{weight}_{\mathcal{A}}(x) \mid x \in \Sigma^k \}$$

skip

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▸ skip

Definition

given matrix interpretation $\mathcal{M}$ of dimension n for signature $\mathcal{F}$
define weighted automaton $\mathcal{A} = (Q, \Sigma, \lambda, \mu, \gamma)$ as follows:

- $Q = \{1, \dots, n\}$
- $\Sigma = \{\, f_i \mid f \in \mathcal{F}$ has arity m and $1 \leqslant i \leqslant m \,\}$
- $\lambda = 1$
- $\mu(f_i) = F_i$ where F_i denotes i-th matrix of $f_{\mathcal{M}}$
- $\gamma = \{i \mid c_i > 0$ for some vector c in $\mathcal{M}\}$

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Lemma

$\forall$ *weighted automaton* $\mathcal{A}$ $\exists$ *trim automaton* $\mathcal{B}$ *such that*

$$\mathit{growth}_{\mathcal{A}}(k) = \mathit{growth}_{\mathcal{B}}(k)$$

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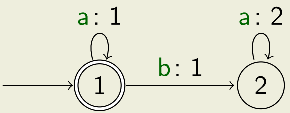
weighted automaton $\mathcal{A}$ is not trim: state 2 is not useful



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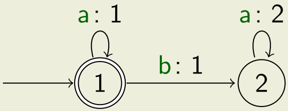
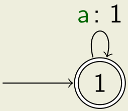


weighted automaton $\mathcal{B}$



Theorem

matrix interpretation $\mathcal{M}$ and corresponding weighted automaton $\mathcal{A}$

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Definitions (based on Weber and Seidl 1991)

weighted automaton $\mathcal{A} = (Q, \Sigma, \lambda, \mu, \gamma)$

EDA $\quad \exists\, q \in Q \;\exists\, x \in \Sigma^*$ such that q is useful and $\mu(x)_{qq} \geqslant 2$

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$\forall\, i \geqslant 1$ p_i and q_i are useful, $p_i \neq q_i$ and $p_i \xrightarrow{v_i} p_i \xrightarrow{v_i} q_i \xrightarrow{v_i} q_i$
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$$\begin{tikzcd}
{} \arrow[r] & p_1 \arrow[loop above, "v_1"] \arrow[r, "v_1"] & q_1 \arrow[loop above, "v_1"] \arrow[r, "u_2"] & \cdots \arrow[r, "u_d"] & p_d \arrow[loop above, "v_d"] \arrow[r, "v_d"] & q_d \arrow[loop above, "v_d"] \arrow[r] & {}
\end{tikzcd}$$

Theorem

weighted automaton $\mathcal{A}$

$$\mathit{growth}_{\mathcal{A}}(k) \in \mathcal{O}(k^{d+1}) \quad \iff \quad \mathcal{A} \not\models \mathsf{EDA},\ \mathcal{A} \not\models \mathsf{IDA}_{d+1}$$

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Remark

conditions are decidable in time $\mathcal{O}(|Q|^6 \cdot |\Sigma|)$ *for* $\mathcal{A} = (Q, \Sigma, \lambda, \mu, \gamma)$

Example

rewrite rule

$$\mathsf{f}(x) \to x$$

compatible matrix interpretation $\mathcal{M}$

$$\mathsf{f}_{\mathcal{M}}(\vec{x}) = \begin{pmatrix} 1 & 1 \\ 0 & 2 \end{pmatrix} \vec{x} + \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

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Lemma

for every TRS $\mathcal{R}$

$\forall$ compatible matrix interpretation $\mathcal{M}$ $\exists$ compatible matrix interpretation $\mathcal{N}$

such that corresponding automaton is trim and $\mathrm{growth}_{\mathcal{M}}(k) = \mathrm{growth}_{\mathcal{N}}(k)$

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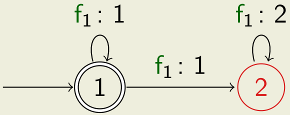

is not trim because state 2 is not useful

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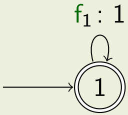

is trim and $\mathcal{M}$ is polynomially bounded

Theorem

for every TRS $\mathcal{R}$

$$\mathrm{dc}_{\mathcal{R}}(k) \in \mathcal{O}(k^d) \text{ can be shown using automata-based approach}$$

$$\iff$$

$$\mathrm{dc}_{\mathcal{R}}(k) \in \mathcal{O}(k^d) \text{ can be shown using algebraic approach}$$

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Lemma

matrix interpretations are incomplete for polynomial derivational complexity

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Example

rewrite system $\mathcal{R}$ with linear derivational complexity

$$\mathsf{aa} \to \mathsf{aba} \qquad \mathsf{bb} \to x \qquad \mathsf{c} \to \epsilon$$

compatible matrix interpretation $\mathcal{M}$

$$\mathsf{a}_{\mathcal{M}}(\vec{x}) = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \vec{x} + \begin{pmatrix} 0 \\ 4 \\ 0 \end{pmatrix} \qquad \mathsf{b}_{\mathcal{M}}(\vec{x}) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \vec{x} + \begin{pmatrix} 1 \\ 0 \\ 3 \end{pmatrix}$$

$$\mathsf{c}_{\mathcal{M}}(\vec{x}) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \vec{x} + \begin{pmatrix} 4 \\ 4 \\ 3 \end{pmatrix} \qquad \rho(\{\dots\}) = 1$$

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- compatible matrix interpretation $\mathcal{M}$ of dimension n:

$$\mathsf{a}_{\mathcal{M}}(\vec{x}) = A\,\vec{x} + a \qquad \mathsf{b}_{\mathcal{M}}(\vec{x}) = B\,\vec{x} + b \qquad \mathsf{c}_{\mathcal{M}}(\vec{x}) = C\,\vec{x} + c$$

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rewrite system $\mathcal{R}$ with linear derivational complexity

$$\mathsf{aa} \to \mathsf{aba} \qquad \mathsf{bb} \to x \qquad \mathsf{c} \to \epsilon \qquad \mathsf{c} \to \mathsf{b}$$

no polynomially bounded compatible matrix interpretation

- compatible matrix interpretation $\mathcal{M}$ of dimension n:

$$\mathsf{a}_{\mathcal{M}}(\vec{x}) = A\,\vec{x} + a \qquad \mathsf{b}_{\mathcal{M}}(\vec{x}) = B\,\vec{x} + b \qquad \mathsf{c}_{\mathcal{M}}(\vec{x}) = C\,\vec{x} + c$$

- $C \geqslant \max(I_n, B)$

Lemma

matrix interpretations are incomplete for polynomial derivational complexity

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- $C \geqslant \max(I_n, B)$ ... hence entries in C^k grows exponentially

Remarks

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Definitions (Hirokawa and Moser 2008)

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Definitions (Hirokawa and Moser 2008)

- runtime complexity $\mathsf{rc}_{\mathcal{R}}(k) = \max\{\, \mathsf{dh}(t) \mid t$ is basic term and $|t| \leqslant k \,\}$
- term $f(t_1, \ldots, t_n)$ is basic if
  1. f is defined symbol
  2. $t_1, \ldots, t_n$ are constructor terms

Example

rewrite system $\mathcal{R}$

$$\begin{aligned}
\mathsf{reverse}(\mathsf{nil}) &\to \mathsf{nil} \\
\mathsf{reverse}(x :: xs) &\to \mathsf{append}(\mathsf{reverse}(xs), x :: \mathsf{nil}) \\
\mathsf{shuffle}(\mathsf{nil}) &\to \mathsf{nil} \\
\mathsf{shuffle}(x :: xs) &\to x :: \mathsf{shuffle}(\mathsf{reverse}(xs)) \\
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derivational complexity

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runtime complexity

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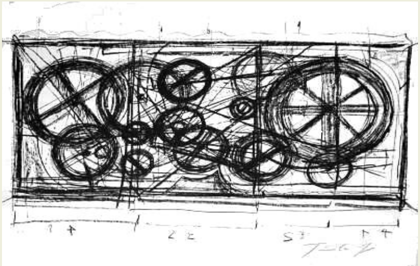
... beyond reach of current complexity tools

Termination and Complexity Research



Termination Tools

CiME, $\mathsf{T_TT_2}$, AProVE, Matchbox, MuTerm, VMTL, WANDA, THOR, ...

Complexity Tools

$\mathsf{T_CT}$, Matchbox, CaT, AProVE

Confluence Research

Confluence Tools

ACP, CSI, Saigawa