Layer Systems for Proving Confluence

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We introduce layer systems for proving generalizations of the modularity of confluence for first-order rewrite systems. Layer systems specify how terms can be divided into layers. We establish structural conditions on those systems that imply confluence. Our abstract framework covers known results like modularity, many-sorted persistence, layer-preservation, and currying. We present a counterexample to an extension of persistence to order-sorted rewriting and derive new sufficient conditions for the extension to hold. All our proofs are constructive.

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1. INTRODUCTION

We revisit the celebrated modularity result of confluence, due to Toyama [1987]. It states that the union of two confluent rewrite systems is confluent, provided the participating rewrite systems do not share function symbols. This result has been reproved several times, using category theory [Lüth 1996], ordered completion [Jouannaud and Toyama 2008], and decreasing diagrams [van Oostrom 2008]. While confluence is also modular for rewriting modulo [Jouannaud and Toyama 2008; Jouannaud and Liu 2012], the situation is different for higher-order rewriting [Appel et al. 2010]. In practice, modularity is of limited use. More useful techniques, in the sense that rewrite systems can be decomposed into smaller systems that share function symbols and rules, are based on type introduction [Aoto and Toyama 1997], layer-preservation [Ohlebusch 1994a], and commutativity [Rosen 1973].

Type introduction [Zantema 1994] restricts the set of terms that have to be considered to the well-typed terms according to some many-sorted type discipline that is compatible with the rewrite system under consideration. A property of (many-sorted) rewrite systems that is preserved and reflected under type removal is called persistent, and Aoto and Toyama [1997] showed that confluence is persistent. Aoto and Toyama [1996] extended the latter result by considering an order-sorted type discipline. However, we show that the conditions imposed in [Aoto and Toyama 1996] are not sufficient for confluence.

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The proofs in Ohlebusch [1994a] and Aoto and Toyama [1996, 1997] are adaptations of the proof of Toyama's modularity result by Klop et al. [1994]. A more complicated proof using concepts from Klop et al. [1994] has been given by Kahrs, who showed in [Kahrs 1995] that confluence is preserved under currying [Kennaway et al. 1996]. In this article, we introduce *layer systems* as a common framework to capture the results of Aoto and Toyama [1997], Kahrs [1995], Ohlebusch [1994a], and Toyama [1987] and to identify appropriate conditions to restore the persistence of confluence for order-sorted rewriting [Aoto and Toyama 1996]. Layer systems identify the parts that are available when decomposing terms. The key proof idea remains the same. We treat each such layer independently from the others where possible and deal with interactions between layers separately. The main advantage of and motivation for our proof is that the result becomes reusable; rather than checking every detail of a complex proof, we have to check a couple of comparatively simple, structural conditions on layer systems instead. Such a common framework also facilitates a formalization of these results in a theorem prover like Isabelle or Coq.

Besides the theoretical results of this article, we stress practical implications: due to an implementation of Theorem 6.3 in our confluence tool CSI [Zankl et al. 2011b], it supports a decomposition result based on ordered sorts, exceeding the criteria available in other tools. A second result of practical importance is preservation and reflection of confluence under currying [Kahrs 1995], which is used as a preprocessing step when deciding confluence of ground term rewrite systems (TRSs) [Felgenhauer 2012].

The remainder of this article is organized as follows. In the next section, we recall preliminaries. Section 3 introduces layer systems and establishes results on how rewriting interacts with layers. The main (abstract) results for confluence via layer systems are presented in Section 4 and instantiated in Section 5 to obtain various known results. The new result on order-sorted persistence is covered in Section 6. Differences to related work are discussed in Section 7, which might be consulted in advance by readers familiar with the literature. We conclude in Section 8.

This article is an extended and significantly revised version of [Felgenhauer et al. 2011]. Since here we build upon [van Oostrom 2008], all our proofs are constructive. Furthermore, this work is based on a more intuitive definition of layer systems. The result for nonduplicating systems has been generalized to the strictly larger class of *bounded duplicating* systems. The application of quasi-ground systems (Section 5.3) is new. Moreover, all important concepts are demonstrated by examples, and detailed proofs are provided.

2. PRELIMINARIES

We assume familiarity with rewriting [Baader and Nipkow 1998; Terese 2003] and the decreasing diagrams technique [van Oostrom 1994].

Let $\mathcal{V}$ be a countably infinite set of variables and $\mathcal{F}$ a signature, that is, a set of function symbols $f \in \mathcal{F}$, each associated with a fixed arity, denoted by $\mathsf{arity}(f)$. The set of terms over $\mathcal{F}$ and $\mathcal{V}$ is denoted by $\mathcal{T}(\mathcal{F}, \mathcal{V})$. The sets of variables and function symbols occurring in a term t are referred to by $\mathcal{V}\mathsf{ar}(t)$ and $\mathcal{F}\mathsf{un}(t)$, respectively. A term is ground if it does not contain variables. The set of ground terms over $\mathcal{F}$ is denoted by $\mathcal{T}(\mathcal{F})$. A term is linear if every variable occurs at most once.

Let $\square \notin \mathcal{F} \cup \mathcal{V}$ be a constant (i.e., a function symbol of arity 0) called hole and abbreviate $\mathcal{T}(\mathcal{F} \cup \{\square\}, \mathcal{V})$ by $\mathcal{C}(\mathcal{F}, \mathcal{V})$. We write $\mathcal{V}_\square$ for the set of symbols $\mathcal{V} \cup \{\square\}$. Contexts are terms from $\mathcal{C}(\mathcal{F}, \mathcal{V})$ containing an arbitrary number of holes. They are partially ordered by $\sqsubseteq$, defined as the smallest reflexive and transitive relation that is monotone and satisfies $\square \sqsubseteq C$ for all $C \in \mathcal{C}(\mathcal{F}, \mathcal{V})$. There is a corresponding partial supremum operation, $\sqcup$, which merges contexts. The strict order $\sqsubset$ is defined by $C \sqsubset D$ if $C \sqsubseteq D$ and $C \neq D$. The minimum context with respect to $\sqsubseteq$ is the empty context $\square$. By

$C[t_1, \ldots, t_n]$, we denote the result of replacing holes in C by the terms $t_1, \ldots, t_n$ from left to right.

The size of a term t is denoted by $|t|$, and $|t|_W$ for a subset $W \subseteq \mathcal{F} \cup \mathcal{V}_\square$ denotes the number of occurrences of function symbols and variables from W in t. We write $|t|_w$ for $|t|_{\{w\}}$. Positions of a term t are strings of positive natural numbers, ϵ for the root, and ip if $t = f(t_1, \ldots, t_i, \ldots, t_n)$ and p is a position of t_i. Then $\mathcal{P}\mathsf{os}(t)$ is the set of all positions of t. Two positions p, q are parallel if neither p is a prefix of q nor q is a prefix of p. Given terms t and s, $t|_p$ is the subterm at position p of t and $t[s]_p$ denotes the result of replacing $t|_p$ by s in t. This operation is extended to sets of pairwise parallel positions P, resulting in the notation $t[s_p]_{p \in P}$. By $\mathrm{root}(t)$ we denote the root symbol of t. For $W \subseteq \mathcal{F} \cup \mathcal{V}_\square$ and $w \in \mathcal{F} \cup \mathcal{V}_\square$, we let $\mathcal{P}\mathsf{os}_W(t) = \{p \in \mathcal{P}\mathsf{os}(t) \mid \mathrm{root}(t|_p) \in W\}$ and $\mathcal{P}\mathsf{os}_w(t) = \mathcal{P}\mathsf{os}_{\{w\}}(t)$. A substitution is a map $\sigma : \mathcal{V} \to \mathcal{T}(\mathcal{F}, \mathcal{V})$ which extends homomorphically to terms. For terms s and t, we write $s \leqslant t$ if there exists a substitution σ such that $s\sigma = t$.

A rewrite rule is a pair of terms $(\ell, r) \in \mathcal{T}(\mathcal{F}, \mathcal{V})^2$, written $\ell \to r$, such that $\ell \notin \mathcal{V}$ and $\mathcal{V}\mathsf{ar}(\ell) \supseteq \mathcal{V}\mathsf{ar}(r)$. A rewrite rule $\ell \to r$ is left-linear if ℓ is linear, duplicating if there is a variable $x \in \mathcal{V}$ with $|\ell|_x < |r|_x$, and collapsing if its right-hand side is a variable. A TRS consists of a signature and a set of rewrite rules. If we do not specify differently, a TRS will always be over the signature $\mathcal{F}$ and variables $\mathcal{V}$. The rewrite relation induced by a TRS $\mathcal{R}$ is denoted $\to_\mathcal{R}$. We write $s \to_{p,\ell\to r} t$ if $s \to_\mathcal{R} t$ using a rule $\ell \to r \in \mathcal{R}$ at position p. Two rewrite steps $s \to_\mathcal{R} t$ and $s' \to_\mathcal{R} t'$ *mirror* each other if both steps use the same rule at the same position. This notion is extended to rewrite sequences. We write $\leftarrow$, $\to^=$, $\to^+$, and $\to^*$ to denote the inverse, the reflexive closure, the transitive closure, and the reflexive and transitive closure of a relation $\to$, respectively. A relation $\to$ is terminating if $\to^+$ is well-founded and confluent if ${}^*\!\leftarrow \cdot \to^* \subseteq \to^* \cdot {}^*\!\leftarrow$. We say that $\to$ is *confluent on* a set S of terms if S is closed under $\to$ and $\to \cap (S \times S)$ is confluent. A TRS $\mathcal{R}$ inherits these properties from $\to_\mathcal{R}$. A relative TRS $\mathcal{R}/\mathcal{S}$ is a pair of TRSs $\mathcal{R}$ and $\mathcal{S}$ with the induced rewrite relation $\to_{\mathcal{R}/\mathcal{S}} = \to_\mathcal{S}^* \cdot \to_\mathcal{R} \cdot \to_\mathcal{S}^*$. It is terminating if $\to_{\mathcal{R}/\mathcal{S}}^+$ is well-founded.

Let $>$ be a well-founded order on an index set I and $\to$ the union of $\to_\alpha$ for all $\alpha \in I$. We write $\to_{\vee\alpha_1\ldots\alpha_n}$ for the union of $\to_\beta$, where $\alpha_i > \beta$ for some $1 \leqslant i \leqslant n$. A local peak $t \;{}_\alpha\!\leftarrow s \to_\beta u$ is said to be *decreasing* if

$$t \to_{\vee\alpha}^* \cdot \to_\beta^= \cdot \to_{\vee\alpha\beta}^* \cdot {}_{\vee\alpha\beta}^{\;\;*}\!\leftarrow \cdot {}_{\alpha}^{=}\!\leftarrow \cdot {}_{\vee\beta}^{\;*}\!\leftarrow u.$$

Furthermore, $\to$ is *locally decreasing* if for all $\alpha, \beta \in I$ every local peak ${}_\alpha\!\leftarrow \cdot \to_\beta$ is decreasing. Van Oostrom [1994] established the following result.

THEOREM 2.1. *Every locally decreasing relation is confluent.*

3. LAYER SYSTEMS

In this section, we introduce layer systems, which are sets of contexts satisfying special properties. The top-down decomposition of a term into maximal layers admits the notion of the rank of a term. Since for suitable layer systems rewriting does not increase the rank, this is a valid measure for proofs by induction.

Definition 3.1. Let $\mathbb{L} \subseteq \mathcal{C}(\mathcal{F}, \mathcal{V})$ be a set of contexts. Then $L \in \mathbb{L}$ is called a *top* of a context $C \in \mathcal{C}(\mathcal{F}, \mathcal{V})$ (according to $\mathbb{L}$) if $L \sqsubseteq C$. A top is a *max-top* of C if it is maximal with respect to $\sqsubseteq$ among the tops of C.

Note that terms are contexts without holes, so they have tops and max-tops as well. In the sequel, we use subsets $\mathbb{L} \subseteq \mathcal{C}(\mathcal{F}, \mathcal{V})$ to layer terms. The process is top-down, taking the max-top of a term as layer and proceeding recursively.

Example 3.2. Let $\mathcal{F}$ consist of a binary function symbol f, a unary function symbol g, and constants a, b, and c. We consider the following candidates for $\mathbb{L}$:

$$\begin{aligned}
\mathbb{L}_0 &= \varnothing \\
\mathbb{L}_1 &= \{\mathsf{f}(v, w), \mathsf{g}(v), \mathsf{a}, \mathsf{b}, \mathsf{c}, v \mid v, w \in \mathcal{V}_\square\} \\
\mathbb{L}_2 &= \{\mathsf{f}(\mathsf{g}^n(v), \mathsf{g}^m(w)), \mathsf{g}^n(v), \mathsf{g}^n(\mathsf{c}), \mathsf{a}, \mathsf{b} \mid v, w \in \mathcal{V}_\square,\ n, m \in \mathbb{N}\} \\
\mathbb{L}_3 &= \{\mathsf{f}(\mathsf{g}^n(v), \mathsf{g}^m(w)), \mathsf{g}^n(v), \mathsf{a}, \mathsf{b} \mid v, w \in \mathcal{V}_\square \cup \{\mathsf{c}\},\ n, m \in \mathbb{N}\}.
\end{aligned}$$

Regard the terms $s = \mathsf{f}(\mathsf{c}, \mathsf{c})$ and $t = \mathsf{f}(\mathsf{c}, \mathsf{g}(\mathsf{c}))$. According to $\mathbb{L}_0$, neither s nor t has any tops. According to $\mathbb{L}_1$, the tops of both s and t are $\square$ and $\mathsf{f}(\square, \square)$, and the latter is the max-top of s and t. According to $\mathbb{L}_2$, $\square$ and $\mathsf{f}(\square, \square)$ are the tops of s and t, and $\mathsf{f}(\square, \mathsf{g}(\square))$ is a top of t but not of s. The max-tops of s and t are $\mathsf{f}(\square, \square)$ and $\mathsf{f}(\square, \mathsf{g}(\square))$, respectively. Finally, the max-tops of s and t according to $\mathbb{L}_3$ are s and t themselves.

Our goal is to deduce confluence of $\mathcal{R}$ when rewriting is restricted to $\mathbb{L} \cap \mathcal{T}(\mathcal{F}, \mathcal{V})$. To this end, we need to impose restrictions on $\mathbb{L}$. This leads to the central definition of the article.

Definition 3.3. Let $\mathcal{F}$ be a signature. A set $\mathbb{L} \subseteq \mathcal{C}(\mathcal{F}, \mathcal{V})$ of contexts is called a *layer system* if it satisfies properties (L_1), (L_2), and (L_3). The elements of $\mathbb{L}$ are called *layers*. A TRS $\mathcal{R}$ over $\mathcal{F}$ is *weakly layered (according to a layer system* $\mathbb{L}$*)* if condition (W) is satisfied for each $\ell \to r \in \mathcal{R}$. It is *layered (according to a layer system* $\mathbb{L}$*)* if conditions (W), (C_1), and (C_2) are satisfied. The conditions are as follows:

(L_1) Each term in $\mathcal{T}(\mathcal{F}, \mathcal{V})$ has a nonempty top.
(L_2) If $x \in \mathcal{V}$ and $C \in \mathcal{C}(\mathcal{F}, \mathcal{V})$, then $C[x]_p \in \mathbb{L}$ if and only if $C[\square]_p \in \mathbb{L}$.
(L_3) If $L, N \in \mathbb{L}$, $p \in \mathcal{P}\mathsf{os}_{\mathcal{F}}(L)$, and $L|_p \sqcup N$ is defined, then $L[L|_p \sqcup N]_p \in \mathbb{L}$.
(W) If M is a max-top of s, $p \in \mathcal{P}\mathsf{os}_{\mathcal{F}}(M)$, and $s \to_{p,\ell\to r} t$, then $M \to_{p,\ell\to r} L$ for some $L \in \mathbb{L}$.
(C_1) In (W), either L is a max-top of t or $L = \square$.
(C_2) If $L, N \in \mathbb{L}$ and $L \sqsubseteq N$, then $L[N|_p]_p \in \mathbb{L}$ for any $p \in \mathcal{P}\mathsf{os}_\square(L)$.

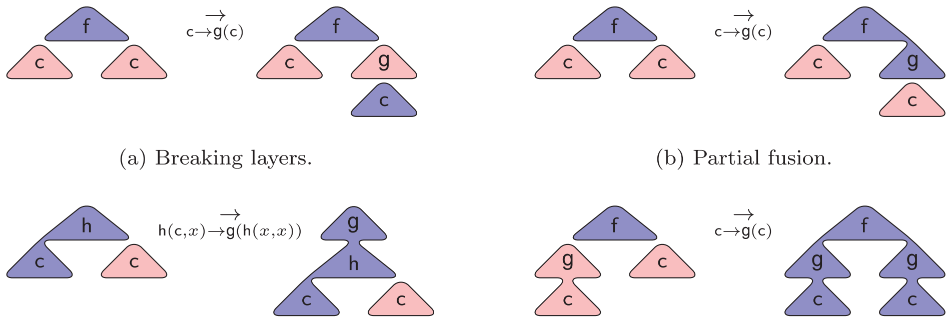
Example 3.4 (*Example 3.2 revisited*). Consider the TRS $\mathcal{R}_1$ consisting of the rewrite rules

$$\mathsf{f}(x, x) \to \mathsf{a} \qquad\qquad \mathsf{f}(x, \mathsf{g}(x)) \to \mathsf{b} \qquad\qquad \mathsf{c} \to \mathsf{g}(\mathsf{c})$$

from Huet [1980]. It is nonconfluent because $\mathsf{a} \ {}_{\mathcal{R}_1}\!\leftarrow \mathsf{f}(\mathsf{c}, \mathsf{c}) \to_{\mathcal{R}_1} \mathsf{f}(\mathsf{c}, \mathsf{g}(\mathsf{c})) \to_{\mathcal{R}_1} \mathsf{b}$, and a, b are in normal form. However, $\mathcal{R}_1$ is confluent on $\mathbb{L}_0 \cap \mathcal{T}(\mathcal{F}, \mathcal{V})$ and $\mathbb{L}_2 \cap \mathcal{T}(\mathcal{F}, \mathcal{V})$, and $\mathcal{R}_1$ is confluent if rewriting is restricted to terms of $\mathbb{L}_1 \cap \mathcal{T}(\mathcal{F}, \mathcal{V})$ (which rules out the rewrite step $\mathsf{c} \to \mathsf{g}(\mathsf{c})$), but $\mathcal{R}_1$ is not confluent on $\mathbb{L}_3 \cap \mathcal{T}(\mathcal{F}, \mathcal{V})$, because $\mathsf{a}, \mathsf{f}(\mathsf{c}, \mathsf{c}), \mathsf{f}(\mathsf{c}, \mathsf{g}(\mathsf{c})), \mathsf{b} \in \mathbb{L}_3 \cap \mathcal{T}(\mathcal{F}, \mathcal{V})$. Clearly $\mathbb{L}_0$ violates (L_1), and therefore any attempt of proving confluence of $\mathcal{R}_1$ by decomposing terms into a max-top and remaining subterms is doomed to fail. Our basic idea for establishing confluence of a (weakly) layered TRS is to perform rewrite steps on arbitrary terms on the corresponding elements of a layer system in the terms' decomposition, with subterms replaced by variables (this replacement is enabled by (L_2)).

Figure 1(a) depicts the rewrite step $\mathsf{f}(\mathsf{c}, \mathsf{c}) \to_{\mathcal{R}_1} \mathsf{f}(\mathsf{c}, \mathsf{g}(\mathsf{c}))$ with both terms decomposed according to $\mathbb{L}_1$. Note that the subterm c rewrites to $\mathsf{g}(\mathsf{c})$, but the resulting subterm is split into two layers. Note furthermore that $\mathsf{f}(\mathsf{c}, \mathsf{g}(\mathsf{c})) \to_{\mathcal{R}_1} \mathsf{b}$, but the corresponding left-hand side $\mathsf{f}(x, \mathsf{g}(x))$ does not match any part of the decomposition of $\mathsf{f}(\mathsf{c}, \mathsf{g}(\mathsf{c}))$. Condition (W) (which is violated by $\mathbb{L}_1$) helps ensure that rewrite steps on terms can be adequately simulated on layers.

Next, consider Figure 1(b), depicting the rewrite step $\mathsf{f}(\mathsf{c}, \mathsf{c}) \to_{\mathcal{R}_1} \mathsf{f}(\mathsf{c}, \mathsf{g}(\mathsf{c}))$ with terms decomposed according to $\mathbb{L}_2$. Note that $\mathbb{L}_2$ satisfies (L_1), (L_2), and (W). Nevertheless,



(a) Breaking layers. (b) Partial fusion.

(c) Fusion from above. (d) Conspiring aliens.

Fig. 1. Undesired behavior on layers.

the result of the rewrite step $\mathsf{c} \to_{\mathcal{R}_1} \mathsf{g}(\mathsf{c})$ is broken apart: only a part of $\mathsf{g}(\mathsf{c})$ is merged with the max-top of $\mathsf{f}(\mathsf{c},\mathsf{g}(\mathsf{c}))$. Condition ($\mathrm{L}_3$) prevents such partial fusion. We can see that it is violated by $\mathbb{L}_2$: we have $\mathsf{f}(\square,\mathsf{g}(\square)) \in \mathbb{L}_2$ and $\mathsf{g}(\mathsf{c}) \in \mathbb{L}_2$, but $\mathsf{f}(\square,\mathsf{g}(\square) \sqcup \mathsf{g}(\mathsf{c})) = \mathsf{f}(\square,\mathsf{g}(\mathsf{c})) \notin \mathbb{L}_2$. Finally, $\mathbb{L}_3$ weakly layers $\mathcal{R}$.

In order to motivate (C_1), we consider the TRS $\mathcal{R}_2$ consisting of the rewrite rules

$$\mathsf{f}(x,x) \to \mathsf{a} \qquad \mathsf{f}(x,\mathsf{g}(x)) \to \mathsf{b} \qquad \mathsf{h}(\mathsf{c},x) \to \mathsf{g}(\mathsf{h}(x,x)),$$

which is closely related to $\mathcal{R}_1$; instead of the rewrite step $\mathsf{c} \to_{\mathcal{R}_1} \mathsf{g}(\mathsf{c})$, we have $t_c \to_{\mathcal{R}_2} \mathsf{g}(t_c)$ for $t_c = \mathsf{h}(\mathsf{c},\mathsf{c})$, and therefore $\mathsf{a}\ {}_{\mathcal{R}_2}\!\leftarrow \mathsf{f}(t_c,t_c) \to_{\mathcal{R}_2} \mathsf{f}(t_c,\mathsf{g}(t_c)) \to_{\mathcal{R}_2} \mathsf{b}$. We define a layer system $\mathbb{L}_4$ by

$$\mathbb{L}_c = \{v, \mathsf{h}(v,w), \mathsf{h}(\mathsf{c},v) \mid v,w \in \mathcal{V}_\square\}$$
$$\mathbb{L}_4 = \{\mathsf{f}(\mathsf{g}^n(s),\mathsf{g}^m(t)), \mathsf{g}^n(t), \mathsf{a}, \mathsf{b}, \mathsf{c}, s \mid s,t \in \mathbb{L}_c,\ n,m \in \mathbb{N}\}.$$

It is straightforward to verify that $\mathbb{L}_4$ weakly layers $\mathcal{R}_2$ and that $\mathcal{R}_2$ is confluent on $\mathbb{L}_4 \cap \mathcal{T}(\mathcal{F},\mathcal{V})$. Figure 1(c) depicts the rewrite step $t_c \to_{\mathcal{R}_2} \mathsf{g}(t_c)$. It affects the max-top of t_c, but the max-top of the result, $\mathsf{g}(\mathsf{h}(\mathsf{c},\square))$, is larger than the result of rewriting the max-top $\mathsf{h}(\mathsf{c},\square)$ of t_c: $\mathsf{h}(\mathsf{c},\square) \to \mathsf{g}(\mathsf{h}(\square,\square))$. In the case of $\mathcal{R}_2$, there are rewrite sequences in which such fusion from above happens infinitely often, and that presents another obstacle to confluence. Condition (C_1) is designed to rule out such fusion from above completely, and indeed the rewrite step $t_c \to_{\mathcal{R}_2} \mathsf{g}(t_c)$ shows that (C_1) is violated by $\mathbb{L}_4$ and $\mathcal{R}_2$.

Finally, consider the layer system

$$\mathbb{L}_5 = \{\mathsf{f}(v,w), \mathsf{f}(\mathsf{g}^{n+1}(\mathsf{c}),\mathsf{g}^{m+1}(\mathsf{c})), \mathsf{a}, \mathsf{g}^n(\mathsf{c}), \mathsf{g}^n(v), v \mid v,w \in \mathcal{V}_\square,\ n,m \in \mathbb{N}\},$$

which weakly layers the TRS $\mathcal{R}_3$ consisting of the rewrite rules

$$\mathsf{f}(x,x) \to \mathsf{a} \qquad \mathsf{c} \to \mathsf{g}(\mathsf{c})$$

and satisfies (C_1). Figure 1(d) depicts the rewrite step $\mathsf{f}(\mathsf{g}(\mathsf{c}),\mathsf{c}) \to_{\mathcal{R}_3} \mathsf{f}(\mathsf{g}(\mathsf{c}),\mathsf{g}(\mathsf{c}))$. What happens here is that the result of rewriting the subterm $\mathsf{c} \to \mathsf{g}(\mathsf{c})$ fuses with the previous top, $\mathsf{f}(\square,\square)$, but only if the unrelated first subterm $\mathsf{g}(\mathsf{c})$ fuses at the same time. This phenomenon causes problems in our proof, and (C_2) prevents that. To wit, we have $\mathsf{f}(\square,\square) \in \mathbb{L}_5$ and $\mathsf{f}(\mathsf{g}(\mathsf{c}),\mathsf{g}(\mathsf{c})) \in \mathbb{L}_5$, so by ($\mathrm{C}_2$) with $p = 1$, there should be $\mathsf{f}(\square \sqcup \mathsf{g}(\mathsf{c}),\square) \in \mathbb{L}_5$, but this is not the case.

The following convention helps to differentiate various contexts.

Convention 3.1. *We use C and D to denote contexts, B to denote* base contexts *(to be introduced in Section 4), L and N to denote arbitrary layers, and M to denote a max-top of a term or context.*

In the sequel, we implicitly assume a given layer system $\mathbb{L}$. In light of the next lemma, we speak of *the* max-top of a term or context.

Lemma 3.5. *Any nonempty context has a unique and nonempty max-top.*

Proof. Let C be a nonempty context. To show that C has a nonempty top, let x be a variable not occurring in C and consider $C[x,\ldots,x]$, which has a nonempty top L_x by (L_1). Then $L := L_x\sigma$ with $\mathrm{dom}(\sigma) = \{x\}$ and $\sigma(x) = \Box$ is a top of C since $L \in \mathbb{L}$ by (L_2) and $L \sqsubseteq C$ by construction. It is nonempty since $L = \Box$ implies $L_x = x$; hence, $C[x,\ldots,x] = x$ and consequently $C = \Box$ because x is fresh, contradicting the premises. Hence, the set S of nonempty tops of C is nonempty. Since it also is finite, it has a (nonempty) maximal element.

To show uniqueness, let M and M' be max-tops of C. Then $M \sqsubseteq C$ and $M' \sqsubseteq C$ ensures that $M \sqcup M'$ is defined, and a layer by (L_3) (if $\Box \in \{M, M'\}$, then (L_3) is not needed). If $M \neq M'$, then $M \sqsubset M \sqcup M'$ or $M' \sqsubset M \sqcup M'$. Since $M \sqcup M' \sqsubseteq C$, this gives the desired contradiction. □

Next, we introduce the key notion of the rank of a term.

Definition 3.6. Let $t = M[t_1,\ldots,t_n]$, with M being the max-top of t. Then $t_1,\ldots,t_n$ are the *aliens* of t. We define $\mathrm{rank}(t) = 1 + \max\{\mathrm{rank}(t_i) \mid 1 \leqslant i \leqslant n\}$, where $\max(\varnothing) = 0$ by convention.

Since the max-top of a term is uniquely defined (Lemma 3.5), it follows that also its aliens are uniquely defined. The next example shows that rewriting might increase the rank of a term. In Lemma 3.12, we show that this cannot happen in weakly layered TRSs.

Example 3.7. Consider the layer system

$$\mathbb{L}_6 = \{v, \mathsf{f}(v), \mathsf{g}(v), \mathsf{h}(v), \mathsf{f}(\mathsf{g}(\mathsf{h}(v))), \mathsf{g}(\mathsf{g}(v)), \mathsf{a} \mid v \in \mathcal{V}_\Box\}.$$

Note that (in contrast to modularity) subterms can have larger rank. For example, if $s = \mathsf{f}(\mathsf{g}(\mathsf{h}(x)))$ and $t = \mathsf{g}(\mathsf{h}(x))$, then $\mathrm{rank}(s) = 1 < 2 = \mathrm{rank}(t)$. Furthermore, $s \to_{\mathcal{R}} t$ in the TRS $\mathcal{R}$ containing the rule $\mathsf{f}(\mathsf{g}(x)) \to \mathsf{g}(x)$. Note that $\mathcal{R}$ is not weakly layered according to $\mathbb{L}_6$.

The next lemma states a useful decomposition result.

Lemma 3.8. *Let $t = L[t_1,\ldots,t_n]$, L be a top of t, and k be the maximum of* $\mathrm{rank}(t_i)$ *for $1 \leqslant i \leqslant n$. Then* $\mathrm{rank}(t) \leqslant k+1$ *and aliens of t that are not rooted at hole positions of L have rank less than k.*

Proof. Let M be the max-top of t. We show the (stronger) property for any context C with $C \sqsubseteq M$ (instead of a top L of t). Note that $L \sqsubseteq M$. The proof is by induction on $|t| - |C|_{\mathcal{F}\cup\mathcal{V}}$, which is a natural number because $C \sqsubseteq t$. If $C = M$, then $\mathrm{rank}(t) = 1 + \max\{\mathrm{rank}(t_i) \mid 1 \leqslant i \leqslant n\} = 1 + k$ and all aliens of t are rooted at hole positions of C, so we are done. Otherwise, let M_i be the max-top of t_i. There is a unique maximal context C' such that $C' \sqsubseteq C[M_1,\ldots,M_n]$ and $C' \sqsubseteq M$. Furthermore, we have $C \sqsubset C'$ because the M_i are nonempty by Lemma 3.5. Because $C' \sqsubseteq M \sqsubseteq t$, $t = C'[t'_1,\ldots,t'_m]$, where t'_j is the subterm of t at the position of the jth hole in C'. For each $p \in \mathcal{P}\mathsf{os}_\Box(C)$, there are three possibilities. Let $C[t_1,\ldots,t_n]|_p = t_i$.

(1) If $p \in \mathcal{P}\mathsf{os}_\square(C')$, then $C'[t'_1, \ldots, t'_m]|_p = t'_j$ and $t_i = t'_j$ for some j.
(2) If $p \in \mathcal{P}\mathsf{os}_\mathcal{V}(C')$, then there are no holes below p in C'.
(3) If $p \in \mathcal{P}\mathsf{os}_\mathcal{F}(C')$, then $p \in \mathcal{P}\mathsf{os}_\mathcal{F}(M)$ and $M[M|_p \sqcup M_i]_p \in \mathbb{L}$ by (L_3). Because M is the max-top of t, this implies $M_i \sqsubseteq M|_p$ and therefore $C'|_p = M_i$ by construction of C'. Hence, all t'_j corresponding to holes of C' below p are aliens of t_i having rank less than $\mathrm{rank}(t_i)$.

We can now apply the induction hypothesis to $C'[t'_1, \ldots, t'_m]$ since $C \sqsubset C'$ implies $|t| - |C|_{\mathcal{F}\cup\mathcal{V}} > |t| - |C'|_{\mathcal{F}\cup\mathcal{V}}$. To conclude, note that any alien rooted at a hole position of C' but not at a hole position of C equals a t'_j from Case (3) and therefore has rank less than k. $\square$

LEMMA 3.9. *Let $\mathcal{R}$ be a TRS that is weakly layered according to $\mathbb{L}$. Then $\mathbb{L}$ is closed under rewriting by $\mathcal{R}$.*

PROOF. Let $L \in \mathbb{L}$ and $L \to_\mathcal{R} N$. Obviously, $L[x, \ldots, x] \to_\mathcal{R} N[x, \ldots, x]$ for a fresh variable x. Since $L[x, \ldots, x] \in \mathbb{L}$ by (L_2), it is its own max-top. We conclude since $N[x, \ldots, x] \in \mathbb{L}$ by (W) and hence $N \in \mathbb{L}$ by (L_2). $\square$

We now present technical results about rewriting contexts. In the sequel, we often want to replace variables affected by some substitution σ by holes. We therefore denote by $\sigma_\square(x)$ the substitution obtained by letting $\sigma_\square(x) = \square$ for $x \in \mathrm{dom}(\sigma)$ and $\sigma_\square(x) = x$ otherwise. For a context C, we denote by $C_\square$ the context obtained from C by replacing all variables by holes.

LEMMA 3.10. *Let C be a context and ℓ a nonvariable term. If $\ell \leqslant C|_p$, then there is a term c such that*

(1) $\ell \leqslant c|_p$ and $C = c\sigma_\square$ for some substitution σ, and
(2) if $C \sqsubseteq D$ for a context D and $\ell \leqslant D|_p$, then $c \leqslant D$.

PROOF. Assume that C has $n \geqslant 0$ holes. We may assume without loss of generality that C and ℓ have no variables in common. Let $c_0 := C[x_1, \ldots, x_n]$ with fresh variables $x_1, \ldots, x_n$. The context C witnesses the fact that c_0 and $c_1 := c_0[\ell]_p$ are unifiable. Let c be a most general instance of c_0 and c_1. Note that variables in c can be renamed freely. If $C \sqsubseteq D$, then D is an instance of c_0. Furthermore, if $\ell \leqslant D|_p$, then D must be an instance of c_1 as well and therefore $c \leqslant D$. In particular, $c \leqslant C$ and thus $C = c\sigma$ for some substitution σ. Let τ be a substitution such that $c = c_0\tau$. For $x \in \mathcal{V}\mathsf{ar}(C)$, $\sigma(\tau(x)) = x$, which implies that $\tau(x)$ is a variable. We can rename each $\tau(x)$ to x in c. Therefore, we may assume without loss of generality that $\sigma(x) = \tau(x) = x$ for $x \in \mathcal{V}\mathsf{ar}(C)$. For the variables x_i, we have $\sigma(\tau(x_i)) = \square$ for all $1 \leqslant i \leqslant n$, which is only possible if σ maps those variables to $\square$. Consequently, $\sigma_\square = \sigma$. $\square$

If a rewrite rule is applied to a context, then each hole may be erased, copied, or duplicated. The same holds for the terms used to fill the holes in a context, as formalized by the next lemma.

LEMMA 3.11. *If $C \to_{p,\ell\to r} C'$ and $\ell \leqslant C[s_1, \ldots, s_n]|_p$, then $C[s_1, \ldots, s_n] \to_{p,\ell\to r} C'[t_1, \ldots, t_m]$ and $\{t_1, \ldots, t_m\} \subseteq \{s_1, \ldots, s_n\}$.*

PROOF. Since $\ell \leqslant C|_p$, Lemma 3.10(1) yields a term c and a substitution $\sigma_\square$ such that $\ell \leqslant c|_p$ and $C = c\sigma_\square$. Furthermore, due to $C \sqsubseteq C[s_1, \ldots, s_n]$ and $\ell \leqslant C[s_1, \ldots, s_n]|_p$, there is a substitution σ with $c\sigma = C[s_1, \ldots, s_n]$ by Lemma 3.10(2). Hence, $C \to_{p,\ell\to r} C'$ mirrors a rewrite step $c \to_{p,\ell\to r} c'$ with $C' = c'\sigma_\square$ and $C'[t_1, \ldots, t_m] = c'\sigma$. Since $t_1, \ldots, t_m$ can only come from σ, we conclude. $\square$

This section ends with a key lemma that enables the use of induction on the rank of terms for proving confluence of $\mathcal{R}$.

LEMMA 3.12. *Let $\mathcal{R}$ be a weakly layered TRS. If $s \to_{\mathcal{R}} t$, then* $\text{rank}(s) \geqslant \text{rank}(t)$.

PROOF. By induction on the rank of s. Let $s \to_p t$ and $s = M[s_1, \ldots, s_n]$ be the decomposition of s into max-top and aliens. We distinguish two cases.

If $p \in \mathcal{P}\mathsf{os}_{\mathcal{F}}(M)$, then condition (W) yields $M \to_p L$ and L is a top of t. Let $t = L[t_1, \ldots, t_m]$. By Lemma 3.11, $\{t_1, \ldots, t_m\} \subseteq \{s_1, \ldots, s_n\}$ since $M \to_p L$. Hence, $\text{rank}(t) \leqslant 1 + \max\{\text{rank}(t_i) \mid 1 \leqslant i \leqslant m\} \leqslant 1 + \max\{\text{rank}(s_i) \mid 1 \leqslant i \leqslant n\} = \text{rank}(s)$ using Lemma 3.8.

If $p \notin \mathcal{P}\mathsf{os}_{\mathcal{F}}(M)$, then $s_j \to s'_j$ and $t = M[s_1, \ldots, s'_j, \ldots, s_n]$ for some $1 \leqslant j \leqslant n$. The induction hypothesis yields $\text{rank}(s_j) \geqslant \text{rank}(s'_j)$. Since M is a top of t, Lemma 3.8 yields $\text{rank}(t) \leqslant 1 + \max\{\text{rank}(s'_j), \text{rank}(s_i) \mid 1 \leqslant i \leqslant n, i \neq j\} \leqslant 1 + \max\{\text{rank}(s_i) \mid 1 \leqslant i \leqslant n\} = \text{rank}(s)$. □

4. CONFLUENCE BY LAYER SYSTEMS

We start this long section by stating our main results. All results reduce the task of proving confluence of a TRS to the easier task of proving confluence of the terms in a suitable layer system, that is, the terms in $\mathbb{L} \cap \mathcal{T}(\mathcal{F}, \mathcal{V})$, which are precisely the terms of rank one. The first result imposes left-linearity.

THEOREM 4.1. *Let $\mathcal{R}$ be a weakly layered TRS that is confluent on terms of rank one. If $\mathcal{R}$ is left-linear, then $\mathcal{R}$ is confluent.*

The second result exchanges left-linearity for a condition that is weaker than nonduplication.

Definition 4.2. Let $\mathcal{R}$ be a TRS and $\diamond$ a fresh unary function symbol. Then $\mathcal{R}$ is *bounded duplicating* if the relative rewrite system $\{\diamond(x) \to x\}/\mathcal{R}$ is terminating.

THEOREM 4.3. *Let $\mathcal{R}$ be a weakly layered TRS that is confluent on terms of rank one. If $\mathcal{R}$ is bounded duplicating, then $\mathcal{R}$ is confluent.*

LEMMA 4.4. *Nonduplicating TRSs are bounded duplicating.*

PROOF. Let $\mathcal{R}$ be a nonduplicating TRS. In order to show termination of $\{\diamond(x) \to x\}/\mathcal{R}$, we measure terms by counting the number of occurrences of the $\diamond$ symbol. Clearly, each application of the $\diamond(x) \to x$ rule decreases that number and rules of $\mathcal{R}$ do not increase it because they do not duplicate $\diamond$ symbols and cannot introduce any new ones. □

Bounded duplication strictly extends nonduplication; the TRS consisting of the rewrite rule $\mathsf{f}(x, x) \to \mathsf{g}(x, x, x)$ is duplicating but still bounded duplicating. This can be shown by the polynomial interpretation [Lankford 1979] given by

$$\mathsf{f}_{\mathbb{N}}(x, y) = 2x + 2y \qquad \mathsf{g}_{\mathbb{N}}(x, y, z) = x + y + z \qquad \diamond_{\mathbb{N}}(x) = x + 1.$$

By combining Theorem 4.3 with Lemma 4.4, we obtain the following corollary.

COROLLARY 4.5. *Let $\mathcal{R}$ be a weakly layered TRS that is confluent on terms of rank one. If $\mathcal{R}$ is nonduplicating, then $\mathcal{R}$ is confluent.* □

The third result does not impose any conditions on $\mathcal{R}$ but further limits the layer systems that can be employed to derive confluence.

THEOREM 4.6. *Let $\mathcal{R}$ be a layered TRS that is confluent on terms of rank one. Then $\mathcal{R}$ is confluent.*

Table I. Incomparability of the Main Results

Rewrite Rule	Layer System	Theorem 4.1	Theorem 4.3	Theorem 4.6
$\mathsf{f}(\mathsf{g}(\mathsf{h}(x))) \to \mathsf{g}(x)$	$\mathbb{L}_6$	✓	✓	×
$\mathsf{k}(\mathsf{b}, x) \to \mathsf{k}(x, x)$	$\mathbb{L}_7$	✓	×	✓
$\mathsf{k}(x, x) \to \mathsf{k}(x, x)$	$\mathbb{L}_7$	×	✓	✓

Hence, for duplicating TRSs, there are three possibilities to prove confluence, either by weakly layering a left-linear rewrite system (Theorem 4.1), by establishing bounded duplication for a weakly layered rewrite system (Theorem 4.3), or by layering the rewrite system (Theorem 4.6). Table I shows that the three results are pairwise incomparable where $\mathbb{L}_7 = \{v, \mathsf{k}(v, w), \mathsf{b} \mid v, w \in \mathcal{V}_\Box\}$ and $\mathbb{L}_6$ is as in Example 3.7.

In the following subsections, we develop proofs for Theorems 4.1, 4.3, and 4.6. In Section 4.1, we describe the proof setup and introduce auxiliary rewrite relations. In Sections 4.2 and 4.3, we show that the auxiliary relations are locally decreasing. Finally, we wrap up the proofs in Section 4.4.

4.1. Proof Setup

Assume we are given a weakly layered TRS $\mathcal{R}$ such that $\mathcal{R}$ is confluent on terms of rank one. We will show confluence of $\mathcal{R}$ on all terms by induction on the rank of terms. In the sequel, we prepare for the induction step, hence:

We fix r and assume terms with rank at most r to be confluent.

Next, we generalize the crucial concepts of van Oostrom [2008] from the modularity setting to layer systems. We have renamed *nonnative* to *foreign* because nonnative is not the complement of native.

Definition 4.7. Terms with rank at most $r + 1$ are called *native*. An alien of a native term is *tall* if its rank equals r and *short* otherwise. *Foreign* terms have rank less than or equal to r.

Note that by definition, foreign terms are also native. However, we will only call terms foreign if they are descendants of aliens of a native term.

Definition 4.8. Let t be a native term. Its *base context* B is obtained by replacing all tall aliens in t with holes. The tall aliens form the *base sequence* $\mathbf{t}$, which satisfies $t = B[\mathbf{t}]$.

Definition 4.9. Sequences of foreign terms are called *foreign sequences*. The *imbalance* of a foreign sequence $\mathbf{t}$ is the number of distinct terms in $\mathbf{t}$. The imbalance of a native term t is the imbalance of its base sequence. If $\mathbf{s}$ and $\mathbf{t}$ are sequences of length n, then we write $\mathbf{s} \propto \mathbf{t}$ if $s_i = s_j$ implies $t_i = t_j$ for all $1 \leqslant i, j \leqslant n$.

Note that the relation $\propto$ is transitive. It is useful for analyzing the imbalance of foreign sequences. If $\mathbf{s} \propto \mathbf{t}$, then the imbalance of $\mathbf{t}$ is no larger than that of $\mathbf{s}$.

Definition 4.10. Let s and t be native terms. A *short step* $s \blacktriangleright\!\!\blacktriangleright_{s_0} t$ is a sequence of $\mathcal{R}$-steps $s \to_{\mathcal{R}}^* t$ that is mirrored by a rewrite sequence $B \to_{\mathcal{R}}^* C$ from the base context B of s. Short steps are labeled by terms s_0 that are predecessors of the source: $s_0 \to_{\mathcal{R}}^* s$. We omit the label when it is irrelevant.

There are two ways in which short steps arise: either by rewriting short aliens (hence the name) or by rewriting the max-top of a term. In the sequel, we will sometimes use the fact that in Definition 4.10, $C \sqsubseteq t$ by Lemma 3.11, and when writing $s = B[\mathbf{s}]$ and $t = C[\mathbf{t}]$, each element of $\mathbf{t}$ is an element of $\mathbf{s}$.

Table II. Properties for $r = 2$

Term	Foreign	Native	Max-Top	Base Context	Base Sequence	Imbalance
$s = \mathsf{f}(\mathsf{G}(\mathsf{a}), \mathsf{G}(\mathsf{a}))$	×	✓	$\mathsf{f}(\Box, \Box)$	$\mathsf{f}(\Box, \Box)$	$(\mathsf{G}(\mathsf{a}), \mathsf{G}(\mathsf{a}))$	1
$t = \mathsf{f}(\mathsf{H}(\mathsf{a}), \mathsf{G}(\mathsf{a}))$	×	✓	$\mathsf{f}(\Box, \Box)$	$\mathsf{f}(\Box, \Box)$	$(\mathsf{H}(\mathsf{a}), \mathsf{G}(\mathsf{a}))$	2
$u = \mathsf{f}(\mathsf{J}, \mathsf{G}(\mathsf{a}))$	×	✓	$\mathsf{f}(\Box, \Box)$	$\mathsf{f}(\mathsf{J}, \Box)$	$(\mathsf{G}(\mathsf{a}))$	1
$v = \mathsf{f}(\mathsf{K}, \mathsf{K})$	✓	✓	$\mathsf{f}(\Box, \Box)$	$\mathsf{f}(\mathsf{K}, \mathsf{K})$	$()$	0

Definition 4.11. Let B and $\mathbf{s}$ be the base context and base sequence of a native term s. If $\mathbf{s} \to_{\mathcal{R}}^* \mathbf{t}$, then $s = B[\mathbf{s}] \rhd\!\!\rhd_\iota B[\mathbf{t}] = t$ is a *tall step*. Here, the label ι is the imbalance of $\mathbf{t}$.

Note that $\mathbf{t}$ in Definition 4.11 is a foreign sequence because $\mathcal{R}$ is weakly layered. Further note that the imbalance of t may be smaller than ι (since B need not be the base context of t). The following example illustrates these concepts.

Example 4.12. Consider the TRSs $\mathcal{R}_1 = \{\mathsf{f}(x, x) \to x\}$ over $\mathcal{F}_1 = \{\mathsf{f}, \mathsf{a}\}$ and $\mathcal{R}_2 = \{\mathsf{G}(x) \to \mathsf{I}, \mathsf{I} \to \mathsf{K}, \mathsf{G}(x) \to \mathsf{H}(x), \mathsf{H}(x) \to \mathsf{J}, \mathsf{J} \to \mathsf{K}\}$ over $\mathcal{F}_2 = \{\mathsf{I}, \mathsf{J}, \mathsf{K}, \mathsf{G}, \mathsf{H}\}$ and let $\mathcal{R} = \mathcal{R}_1 \cup \mathcal{R}_2$. Then $\mathbb{L} = \mathcal{C}(\mathcal{F}_1, \mathcal{V}) \cup \mathcal{C}(\mathcal{F}_2, \mathcal{V})$ layers $\mathcal{R}_1$ and $\mathcal{R}_2$ (cf. the proof of Theorem 5.1). Assume that $r = 2$. Table II demonstrates some properties and notions. We have $\mathsf{f}(\underline{\mathsf{G}(\mathsf{a})}, \underline{\mathsf{G}(\mathsf{a})}) \blacktriangleright\!\!\blacktriangleright \mathsf{G}(\mathsf{a})$, but $\mathsf{f}(\underline{\mathsf{G}(\mathsf{a})}, \underline{\mathsf{G}(\mathsf{a})}) \blacktriangleright\!\!\blacktriangleright \mathsf{I}$ is not possible since the step $\mathsf{G}(\mathsf{a}) \to_{\mathcal{R}} \mathsf{I}$ is not in the base context of $\mathsf{f}(\underline{\mathsf{G}(\mathsf{a})}, \underline{\mathsf{G}(\mathsf{a})})$. (Here we have underlined the tall aliens.) We also have $\mathsf{f}(\underline{\mathsf{G}(\mathsf{a})}, \underline{\mathsf{G}(\mathsf{a})}) \rhd\!\!\rhd_2 \mathsf{f}(\mathsf{J}, \mathsf{G}(\mathsf{a})) = u$, despite the imbalance of u being 1 (note that $\mathsf{f}(\Box, \Box)$ is not the base context of u). Furthermore, $(\mathsf{G}(\mathsf{a}), \mathsf{G}(\mathsf{a})) \not\propto (\mathsf{J}, \mathsf{G}(\mathsf{a}))$, but as the latter can be further rewritten $(\mathsf{J}, \mathsf{G}(\mathsf{a})) \to_{\mathcal{R}}^* (\mathsf{J}, \mathsf{J})$, we obtain $(\mathsf{G}(\mathsf{a}), \mathsf{G}(\mathsf{a})) \propto (\mathsf{J}, \mathsf{J})$.

Remark 4.13. The constraint on short steps is subtle. It implies that the rewrite steps do not overlap with any descendants of the tall aliens of s, but furthermore it also has the effect of delaying fusion of those tall aliens with the base context until the end of the rewrite sequence, in the sense of Felgenhauer et al. [2011].

We prove confluence of $\mathcal{R}$ on native terms by showing that any local peak consisting of short steps and/or tall steps may be joined decreasingly. Steps are compared as follows. Tall steps are ordered by their imbalance, tall steps are ordered above short steps, and short steps are compared by a well-founded order introduced later (in the proof of Lemma 4.33).

In the remainder of this section, we use s, t, and u to denote native terms.

4.2. Local Decreasingness of Peaks Involving Tall Steps

Based on Lemma 3.11, we obtain the following result:

Lemma 4.14. *Let* $\mathbf{s}$ *and* $\mathbf{t}$ *be sequences of contexts with* $\mathbf{s} \propto \mathbf{t}$ *and* $C \to_{p,\ell\to r} C'$. *If* $\ell \leqslant C[\mathbf{s}]|_p$, *then* $C[\mathbf{t}] \to_{p,\ell\to r} C'[\mathbf{t}']$, *with each element of* $\mathbf{t}'$ *belonging to* $\mathbf{t}$.

Proof. We extend the proof of Lemma 3.11 as follows. Let τ be the substitution

$$\tau(x) = \begin{cases} t_i & \text{if } x \in \mathrm{dom}(\sigma_\Box) \text{ and } \sigma(x) = s_i \\ x & \text{otherwise.} \end{cases}$$

Note that $C[\mathbf{t}] = c\tau$ because $\mathbf{s} \propto \mathbf{t}$. We have $c\tau \to_{p,\ell\to r} c'\tau$. Comparing $c'\tau$ and $C' = c'\sigma_\Box$ establishes the claim that $c'\tau = C'[\mathbf{t}']$, with each element of $\mathbf{t}'$ equaling some element of $\mathbf{t}$. □

Lemma 4.15. *Let* $\mathbf{s}$, $\mathbf{t}$, $\mathbf{u}$ *be foreign sequences. If* $\mathbf{s} \to_{\mathcal{R}}^* \mathbf{t}$ *and* $\mathbf{s} \to_{\mathcal{R}}^* \mathbf{u}$, *then there is a foreign sequence* $\mathbf{v}$ *such that* $\mathbf{t} \to_{\mathcal{R}}^* \mathbf{v}$, $\mathbf{u} \to_{\mathcal{R}}^* \mathbf{v}$ *with* $\mathbf{t} \propto \mathbf{v}$ *and* $\mathbf{u} \propto \mathbf{v}$.

Proof. Let m be the length of $\mathbf{s}$. We use induction on the number of disequalities $t_i \neq u_i$ for $1 \leqslant i \leqslant m$. If this number is zero, then $\mathbf{t} = \mathbf{u}$ and we can take $\mathbf{v} = \mathbf{t}$. Otherwise,

$t_i \neq u_i$ for some $1 \leqslant i \leqslant m$. Both t_i and u_i are reducts of s_i and thus have a common reduct v since $\mathcal{R}$ is confluent on foreign terms. By replacing every occurrence of t_i and u_i in $\mathbf{t}$, $\mathbf{u}$ by v, we obtain new sequences $\mathbf{t}'$, $\mathbf{u}'$ that satisfy $\mathbf{s} \to_{\mathcal{R}}^* \mathbf{t} \to_{\mathcal{R}}^* \mathbf{t}'$, $\mathbf{s} \to_{\mathcal{R}}^* \mathbf{u} \to_{\mathcal{R}}^* \mathbf{u}'$, $\mathbf{t} \propto \mathbf{t}'$, and $\mathbf{u} \propto \mathbf{u}'$. Since the number of disequalities $t_i' \neq u_i'$ is decreased, we conclude by the induction hypothesis and the transitivity of $\propto$. □

A step in the base context is short.

Lemma 4.16. *Let p be a nonhole position of the base context of s. If $s \to_p t$, then $s \blacktriangleright\!\!\blacktriangleright t$.*

Proof. Let B be the base context of s and let $s \to_p t$. We show $B \to_p C$ for some context C. Because left-hand sides of rules are not variables, $p \in \mathcal{P}\mathsf{os}_{\mathcal{F}}(B)$. Let M be the max-top of s, which is also the max-top of B. We distinguish two cases. If $p \in \mathcal{P}\mathsf{os}_{\mathcal{F}}(M)$, then consider the decomposition $s = M[\mathbf{s}]$. According to (W), there is a layer L with $M \to_p L$. We have $B = M[\mathbf{s}']$, where $s_i' = s_i$ if s_i is a short alien and $s_i' = \Box$ if s_i is tall. Clearly, $\mathbf{s} \propto \mathbf{s}'$, and hence, we conclude by Lemma 4.14. If $p \notin \mathcal{P}\mathsf{os}_{\mathcal{F}}(M)$, then $s|_p$ is a subterm of a short alien of s and thus $B|_p = s|_p$. Hence, $B \to_p C$ for the context $C := B[t|_p]_p$. □

When doing a short step $s = B[\mathbf{s}] \blacktriangleright\!\!\blacktriangleright C[\mathbf{s}'] = t$, in general, the context C is not the base context of t (because of fusion from above or conspiring aliens). Similarly, for a tall step $s = B[\mathbf{s}] \rhd\!\!\rhd B[\mathbf{t}] = t$, in general, the context B is not the base context of t (because of fusion caused by steps in the aliens of t), but both contexts (B and C) satisfy the more general property defined next.

Definition 4.17. We call a context *shallow* if its rank is at most r and all its aliens are terms from $\mathcal{T}(\mathcal{F}, \mathcal{V})$.

Note that the base contexts of native terms are shallow. The same holds for the max-tops of native terms. Furthermore, shallow contexts are closed under rewriting, as shown by the next lemma.

Lemma 4.18. *If C is a shallow context and $C \to_{\mathcal{R}} D$, then D is a shallow context.*

Proof. Assume that $C \to_{p,\ell\to r} D$. Then $C[x,\ldots,x] \to_{p,\ell\to r} D[x,\ldots,x]$ for a fresh variable x. Let M_x be the max-top of $C[x,\ldots,x]$ and note that the max-top M of C is obtained by replacing each occurrence of x by a hole in M_x. If $p \in \mathcal{P}\mathsf{os}_{\mathcal{F}}(M) = \mathcal{P}\mathsf{os}_{\mathcal{F}}(M_x)$, then by (W), there is a rewrite step $M_x \to_{p,\ell\to r} L_x$, where L_x is a layer, and even a top of $D[x,\ldots,x]$ by Lemma 3.11. There is a mirroring rewrite step $M \to_{p,\ell\to r} L$, where L is a top of D. By Lemma 3.11, each hole of L corresponds to a hole or a term without holes in D. If $p \notin \mathcal{P}\mathsf{os}_{\mathcal{F}}(M)$, then we take $L = M$, which is a top of D. Again, each hole of L corresponds to a hole or a term in D. In both cases, we conclude by noting that any holes of D are holes of L and therefore also of the max-top of D and that the rank of D, which equals the rank of D_x, is at most r by Lemma 3.12. □

Let $s = B[\mathbf{s}]$ be the decomposition of s into base context and base sequence. From the previous result, we get that $B[\mathbf{s}] \blacktriangleright\!\!\blacktriangleright C[\mathbf{s}'] = t$ (with $B \to_{\mathcal{R}}^* C$) implies that C is shallow. The next result establishes that the shallow context C is never larger than the base context of t.

Lemma 4.19. *Let C be a shallow context and t a native term. If $C \sqsubseteq t$, then $C \sqsubseteq B$ for the base context B of t.*

Proof. Let $C = M[\mathbf{s}]$ be the decomposition of C into max-top and aliens. Since C is shallow, elements of $\mathbf{s}$ are either holes or terms of rank less than r. From $M \sqsubseteq C \sqsubseteq t$, we infer the existence of a sequence $\mathbf{t}'$ such that $t = M[\mathbf{t}']$ and $s_i = t_i'$ whenever $s_i \neq \Box$.

By Lemma 3.8, every tall alien in t is a subterm of a term of rank at least r in $\mathbf{t}'$. Hence, $C \sqsubseteq B$ as desired. □

Steps within shallow contexts are short steps.

LEMMA 4.20. *Let p be a nonhole position in a shallow context C with $s = C[\mathbf{s}]$. If $s \to_p t$, then $s \blacktriangleright\!\!\blacktriangleright t$.*

PROOF. By Lemmata 4.19 and 4.16. □

Steps below a shallow context can be decomposed into tall and short steps.

LEMMA 4.21 (TALL–SHORT FACTORIZATION). *Let $s = C[\mathbf{s}]$ with a shallow context C and a foreign sequence $\mathbf{s}$. If $\mathbf{s} \to_{\mathcal{R}}^{*} \mathbf{t}$ and ι is the imbalance of $\mathbf{t}$, then $C[\mathbf{s}] \rhd\!\!\rhd_{\leqslant \iota} \cdot \blacktriangleright\!\!\blacktriangleright^{*} C[\mathbf{t}]$.*

PROOF. Let B and $\mathbf{s}'$ be the base context and base sequence of s. Note that by Lemma 3.8 (with L equal to the max-top of C), the tall aliens $\mathbf{s}'$ of s are a subsequence of $\mathbf{s}$, because all aliens of C have rank less than r. For the corresponding subsequence $\mathbf{t}'$ of $\mathbf{t}$, we obtain $s = B[\mathbf{s}'] \rhd\!\!\rhd_{\leqslant \iota} B[\mathbf{t}']$, while the remaining elements of $\mathbf{s}$ and $\mathbf{t}$ give rise to a rewrite sequence $B = C[\mathbf{s}''] \to_{\mathcal{R}}^{*} C[\mathbf{t}'']$, where $\mathbf{s}''$ ($\mathbf{t}''$) is obtained by replacing the terms corresponding to the elements of $\mathbf{s}'$ ($\mathbf{t}'$) by holes. Consequently, $B[\mathbf{t}'] = C[\mathbf{s}''][\mathbf{t}'] \blacktriangleright\!\!\blacktriangleright^{*} C[\mathbf{t}''][\mathbf{t}'] = C[\mathbf{t}]$ by Lemma 4.20. □

Example 4.22. Continuing Example 4.12. Let $s = \mathsf{f}(\mathsf{J}, \mathsf{G}(\mathsf{a}))$. Then $s = C[\mathbf{s}]$ for the shallow context $C = \mathsf{f}(\Box, \Box)$ with $\mathbf{s} = (\mathsf{J}, \mathsf{G}(\mathsf{a}))$. Let $\mathbf{t} = (\mathsf{K}, \mathsf{I})$. Since $\mathbf{s} \to_{\mathcal{R}}^{*} \mathbf{t}$, the conditions of Lemma 4.21 hold and we have $C[\mathbf{s}] \rhd\!\!\rhd_{\leqslant 2} \cdot \blacktriangleright\!\!\blacktriangleright^{*} C[\mathbf{t}]$. The tall step arises as $s = \mathsf{f}(\mathsf{J}, \Box)[\mathsf{G}(\mathsf{a})] \rhd\!\!\rhd_{1} \mathsf{f}(\mathsf{J}, \Box)[\mathsf{I}] = \mathsf{f}(\mathsf{J}, \mathsf{I})$, while $\mathsf{f}(\mathsf{J}, \mathsf{I}) \blacktriangleright\!\!\blacktriangleright \mathsf{f}(\mathsf{K}, \mathsf{I})$ is a short step since $\mathsf{f}(\mathsf{J}, \mathsf{I})$ is its own base context.

LEMMA 4.23. *Local peaks of tall steps are decreasing:*

$$ {}_{\iota}\!\lhd\!\!\lhd \cdot \rhd\!\!\rhd_{\kappa} \;\subseteq\; \rhd\!\!\rhd_{\leqslant \kappa} \cdot \blacktriangleright\!\!\blacktriangleright^{*} \cdot {}^{*}\!\blacktriangleleft\!\!\blacktriangleleft \cdot {}_{\leqslant \iota}\!\lhd\!\!\lhd \,. $$

PROOF. Let $t \;{}_{\iota}\!\lhd\!\!\lhd\; s \rhd\!\!\rhd_{\kappa} u$ and let the base context and base sequence of s be B and $\mathbf{s}$. There are foreign sequences $\mathbf{t}$ and $\mathbf{u}$ such that $\mathbf{t} \;{}_{\mathcal{R}}^{*}\!\!\leftarrow \mathbf{s} \to_{\mathcal{R}}^{*} \mathbf{u}$ and $t = B[\mathbf{t}]$, $u = B[\mathbf{u}]$. By Lemma 4.15, we can find a foreign sequence $\mathbf{v}$ such that $\mathbf{t} \to_{\mathcal{R}}^{*} \mathbf{v} \;{}_{\mathcal{R}}^{*}\!\!\leftarrow \mathbf{u}$, $\mathbf{t} \propto \mathbf{v}$, and $\mathbf{u} \propto \mathbf{v}$. Hence, the imbalance of $\mathbf{v}$ is less than or equal to both ι and κ and we conclude by Lemma 4.21. □

Example 4.24. To demonstrate Lemma 4.23, we extend Example 4.12. Let $s = \mathsf{f}(\mathsf{G}(\mathsf{a}), \mathsf{G}(\mathsf{a}))$. Then $t = \mathsf{f}(\mathsf{H}(\mathsf{a}), \mathsf{I}) \;{}_{2}\!\lhd\!\!\lhd\; s \rhd\!\!\rhd_{2} \mathsf{f}(\mathsf{I}, \mathsf{H}(\mathsf{a})) = u$. Note that $\mathsf{I} \to_{\mathcal{R}} \mathsf{K}$ and $\mathsf{H}(\mathsf{a}) \to_{\mathcal{R}} \mathsf{J} \to_{\mathcal{R}} \mathsf{K}$. The base contexts of t and u are $\mathsf{f}(\Box, \mathsf{I})$ and $\mathsf{f}(\mathsf{I}, \Box)$, respectively. Consequently, $t \rhd\!\!\rhd_{1} \mathsf{f}(\mathsf{K}, \mathsf{I}) \blacktriangleright\!\!\blacktriangleright \mathsf{f}(\mathsf{K}, \mathsf{K}) \blacktriangleleft\!\!\blacktriangleleft \mathsf{f}(\mathsf{I}, \mathsf{K}) \;{}_{1}\!\lhd\!\!\lhd\; u$.

LEMMA 4.25. *Local peaks involving a tall and a short step are decreasing:*

$$ {}_{\iota}\!\lhd\!\!\lhd \cdot \blacktriangleright\!\!\blacktriangleright \;\subseteq\; \rhd\!\!\rhd_{<\iota}^{=} \cdot \blacktriangleright\!\!\blacktriangleright^{*} \cdot {}^{*}\!\blacktriangleleft\!\!\blacktriangleleft \cdot {}_{\leqslant \iota}\!\lhd\!\!\lhd \,. $$

PROOF. Let $t \;{}_{\iota}\!\lhd\!\!\lhd\; s \blacktriangleright\!\!\blacktriangleright u$ and let the base context and base sequence of s be B and $\mathbf{s}$. We have $t = B[\mathbf{t}]$ with $\mathbf{s} \to_{\mathcal{R}}^{*} \mathbf{t}$ for some foreign sequence $\mathbf{t}$ and $u = C[\mathbf{u}]$. We construct $\mathbf{v}$ and $\mathbf{w}$ such that $B[\mathbf{t}] \rhd\!\!\rhd_{<\iota}^{=} \cdot \blacktriangleright\!\!\blacktriangleright^{*} B[\mathbf{v}] \blacktriangleright\!\!\blacktriangleright^{*} C[\mathbf{w}] \;{}^{*}\!\blacktriangleleft\!\!\blacktriangleleft \cdot {}_{\leqslant \iota}\!\lhd\!\!\lhd\; C[\mathbf{u}]$. We distinguish two cases:

(1) If $\mathbf{s} \propto \mathbf{t}$, then we let $\mathbf{v} = \mathbf{t}$. Hence, $B[\mathbf{t}] = B[\mathbf{v}]$ and thus $B[\mathbf{t}] \rhd\!\!\rhd_{<\iota}^{=} \cdot \blacktriangleright\!\!\blacktriangleright^{*} B[\mathbf{v}]$.
(2) Otherwise, using Lemma 4.15 with $\mathbf{s} \to_{\mathcal{R}}^{*} \mathbf{t}$ and $\mathbf{s} \to_{\mathcal{R}}^{*} \mathbf{s}$, we can find a foreign sequence $\mathbf{v}$ such that $\mathbf{t} \to_{\mathcal{R}}^{*} \mathbf{v}$, $\mathbf{t} \propto \mathbf{v}$, and $\mathbf{s} \propto \mathbf{v}$. Since the imbalance of $\mathbf{v}$ is less than ι ($\mathbf{s} \not\propto \mathbf{t}$ means that there are i, j with $s_i = s_j$ and $t_i \neq t_j$. By $\mathbf{s} \propto \mathbf{v}$, we have $v_i = v_j$, and $\mathbf{t} \propto \mathbf{v}$ ensures that all other equalities between elements of $\mathbf{t}$ carry

over to $\mathbf{v}$, so the imbalance becomes smaller) we obtain $B[\mathbf{t}] \rhd\!\rhd^{=}_{<\iota} \cdot \blacktriangleright\!\blacktriangleright^* B[\mathbf{v}]$ from Lemma 4.21.

By the definition of $\blacktriangleright\!\blacktriangleright$, we get $B \to^*_{\mathcal{R}} C$ mirroring $s = B[\mathbf{s}] \to^*_{\mathcal{R}} C[\mathbf{u}] = u$. Hence, $\mathbf{u}$ is a sequence of foreign terms such that all elements of $\mathbf{u}$ are elements of $\mathbf{s}$, which follows by repeated application of Lemma 3.11. We define $w_i = v_j$ if $u_i = s_j$. Then $\mathbf{u} \to^*_{\mathcal{R}} \mathbf{w}$ and the imbalance of $\mathbf{w}$ is at most ι. Hence, $C[\mathbf{u}] \rhd\!\rhd_{\leqslant\iota} \cdot \blacktriangleright\!\blacktriangleright^* C[\mathbf{w}]$ by Lemma 4.21. We also have $B[\mathbf{v}] \to^*_{\mathcal{R}} C[\mathbf{w}]$ with no rewrite step affecting a tall alien and thus $B[\mathbf{v}] \blacktriangleright\!\blacktriangleright^* C[\mathbf{w}]$ by Lemma 4.20. □

Example 4.26. We revisit Example 4.12. Let $s = \mathsf{f}(\mathsf{f}(\mathsf{G}(\mathsf{a}), \mathsf{G}(\mathsf{a})), \mathsf{I})$. The base context of s is $\mathsf{f}(\mathsf{f}(\Box, \Box), \mathsf{I})$. Then $t = \mathsf{f}(\mathsf{f}(\mathsf{I}, \mathsf{H}(\mathsf{a})), \mathsf{I})\ {}_2\!\lhd\!\lhd\ s \blacktriangleright\!\blacktriangleright \mathsf{f}(\mathsf{G}(\mathsf{a}), \mathsf{K}) = u$. The base context of t is $\mathsf{f}(\mathsf{f}(\mathsf{I}, \Box), \mathsf{I})$, and we have $t \rhd\!\rhd_1 \mathsf{f}(\mathsf{f}(\mathsf{I}, \mathsf{K}), \mathsf{I}) \blacktriangleright\!\blacktriangleright \mathsf{f}(\mathsf{f}(\mathsf{K}, \mathsf{K}), \mathsf{K}) \blacktriangleright\!\blacktriangleright \mathsf{f}(\mathsf{K}, \mathsf{K}) = v$, whereas the base context of u is $\mathsf{f}(\Box, \mathsf{K})$ and $u \rhd\!\rhd_1 v$.

LEMMA 4.27 (MAIN LEMMA). *If $\blacktriangleright\!\blacktriangleright$ is locally decreasing, then $\mathcal{R}$ is confluent on native terms.*

PROOF. Every rewrite step $s \to_{\mathcal{R}} t$ can be written as $s \blacktriangleright\!\blacktriangleright t$ by Lemma 4.16 or $s \rhd\!\rhd t$ if the rewrite rule is applied to a tall alien of s. Hence, $\to_{\mathcal{R}} \subseteq \rhd\!\rhd \cup \blacktriangleright\!\blacktriangleright \subseteq \to^*_{\mathcal{R}}$ and thus the claim follows from the confluence of $\rhd\!\rhd \cup \blacktriangleright\!\blacktriangleright$. The latter is a consequence of Theorem 2.1 in connection with the assumption and Lemmata 4.23 and 4.25. □

The various versions of the main theorem will follow from Lemma 4.27.

4.3. Local Decreasingness of Short Steps

In this section, we study conditions to make short steps locally decreasing. The following result allows one to represent a native term s by a foreign term s' and a substitution π such that $s = s'\pi$. This will be the key for joining the peak originating from s by the confluence assumption of s'.

LEMMA 4.28 (PEAK ANALYSIS). *For a local peak $t \blacktriangleleft\!\blacktriangleleft s \blacktriangleright\!\blacktriangleright u$, there are foreign terms s', t', u', v' and substitutions π, $\pi_\Box$ such that*

(1) *π is a bijection with $\mathrm{dom}(\pi) \cap \mathcal{V}\mathrm{ar}(s) = \varnothing$;*
(2) *$s'\pi = s$, $t'\pi = t$, $u'\pi = u$, $s'\pi_\Box$ is the base context of s, and $t'\pi_\Box$ and $u'\pi_\Box$ are shallow contexts of t and u; and*
(3) *$v'\ {}_{\mathcal{R}}^{*}\!\leftarrow t'\ {}_{\mathcal{R}}^{*}\!\leftarrow s' \to^*_{\mathcal{R}} u' \to^*_{\mathcal{R}} v'$ and $t \to^*_{\mathcal{R}} v\ {}_{\mathcal{R}}^{*}\!\leftarrow u$ with $v = v'\pi$.*

PROOF. Let $s = B[\mathbf{s}]$ be the decomposition of s into base context and base sequence, and recall that base contexts are shallow. According to the definition of $\blacktriangleright\!\blacktriangleright$, there are rewrite sequences $B \to^*_{\mathcal{R}} C_t$, $B \to^*_{\mathcal{R}} C_u$ mirroring $s \to^*_{\mathcal{R}} t$, $s \to^*_{\mathcal{R}} u$, respectively. Using Lemma 4.18 repeatedly, we find that C_t and C_u are shallow contexts. Let π be a bijection between the tall aliens of s and fresh variables, and define $s' = B[\pi^{-1}(\mathbf{s})]$. We have $\mathbf{s} \propto \pi^{-1}(\mathbf{s})$, and therefore repeated application of Lemma 4.14 yields rewrite sequences $s' \to^*_{\mathcal{R}} t'$ and $s' \to^*_{\mathcal{R}} u'$ mirroring $s'\pi = s \to^*_{\mathcal{R}} t = t'\pi$ and $s'\pi = s \to^*_{\mathcal{R}} u = u'\pi$. Since s' is a foreign term and therefore confluent, t' and u' have a common reduct: $t' \to^*_{\mathcal{R}} v'\ {}_{\mathcal{R}}^{*}\!\leftarrow u'$. By applying π to this valley, we obtain $t \to^*_{\mathcal{R}} v\ {}_{\mathcal{R}}^{*}\!\leftarrow u$. Note that $s'\pi_\Box = B$, $t'\pi_\Box = C_t$, and $u'\pi_\Box = C_u$ are shallow contexts as claimed. □

Example 4.29. Consider the layer system $\mathbb{L}$ given by

$$\mathbb{L}_0 = \{v, \mathsf{a}, \mathsf{b}, \mathsf{f}(v), \mathsf{g}(v), \mathsf{g}(\mathsf{b}) \mid v \in \mathcal{V}_\Box\}$$
$$\mathbb{L} = \mathbb{L}_0 \cup \{\mathsf{h}(C, C', C'') \mid C, C', C'' \in \mathbb{L}_0\},$$

which weakly layers the TRS $\mathcal{R} = \{\mathsf{h}(x, y, z) \to \mathsf{h}(y, x, z), \mathsf{f}(x) \to \mathsf{g}(x), \mathsf{a} \to \mathsf{b}\}$. Assume that $r = 1$ and let $s = \mathsf{h}(\mathsf{a}, \mathsf{f}(\mathsf{a}), \mathsf{f}(\mathsf{b}))$. The base context of s is $\mathsf{h}(\mathsf{a}, \mathsf{f}(\Box), \mathsf{f}(\Box))$. There is a peak of short steps:

$$t = \mathsf{h}(\mathsf{b}, \mathsf{g}(\mathsf{a}), \mathsf{f}(\mathsf{b})) \twoheadleftarrow s \twoheadrightarrow \mathsf{h}(\mathsf{f}(\mathsf{a}), \mathsf{a}, \mathsf{g}(\mathsf{b})) = u.$$

From Lemma 4.28, we may obtain $\pi = \{\mathsf{a}/x, \mathsf{b}/y\}$, $s' = \mathsf{h}(\mathsf{a}, \mathsf{f}(x), \mathsf{g}(y))$, $t' = \mathsf{h}(\mathsf{b}, \mathsf{g}(x), \mathsf{f}(y))$, $u' = \mathsf{h}(\mathsf{f}(x), \mathsf{a}, \mathsf{g}(y))$, and $v' = \mathsf{h}(\mathsf{g}(x), \mathsf{b}, \mathsf{g}(y))$. Note that $t'\pi_\Box = \mathsf{h}(\mathsf{b}, \mathsf{g}(\Box), \mathsf{f}(\Box))$ is the base context of t, but $u'\pi_\Box = \mathsf{h}(\mathsf{f}(\Box), \mathsf{a}, \mathsf{g}(\Box))$ does not equal $\mathsf{h}(\mathsf{f}(\Box), \mathsf{a}, \mathsf{g}(\mathsf{b}))$, the base context of u.

Lemma 4.30. *If $\mathcal{R}$ is left-linear, then $\twoheadrightarrow$ is locally decreasing.*

Proof. Consider a local peak $t\;{}_{s_0}\!\!\twoheadleftarrow s \twoheadrightarrow_{s_1} u$. First we apply Lemma 4.28. Let t'' be a linearization of t', which we obtain by replacing each variable in t' by a fresh variable. Because $\mathcal{R}$ is left-linear, $t' \to_\mathcal{R}^* v'$ can be mirrored as $t'' \to_\mathcal{R}^* v''$. Let B_t be the base context of t and $C_t = t'\pi_\Box$. We have $C_t \sqsubseteq B_t$ by Lemma 4.19, which implies $t'' \leqslant B_t$ and thus $B_t = t''\sigma$ for some substitution σ. We have $B_t \to_\mathcal{R}^* v''\sigma$. Together with $t \to_\mathcal{R}^* v$, which mirrors $B_t \to_\mathcal{R}^* v''\sigma$, we obtain $t \twoheadrightarrow v$. This step can be labeled with s_1 because $s_1 \to_\mathcal{R}^* s \to_\mathcal{R}^* t$. By symmetry, we obtain $u \twoheadrightarrow_{s_0} v$ and hence $\twoheadrightarrow$ is locally decreasing. $\Box$

Next, we deal with bounded duplicating TRSs. In order to exploit relative termination, we insert $\diamond$ symbols in front of tall aliens as follows.

Definition 4.31. Let s be a native term with base context B and base sequence $\mathbf{s}$. Then $s^\diamond = B[\diamond(\mathbf{s})]$, where $\diamond(\mathbf{s})$ denotes the result of replacing each element u of $\mathbf{s}$ by $\diamond(u)$.

Lemma 4.32. *If $s \to_\mathcal{R} t$, then $s^\diamond \to_\mathcal{R} \cdot \to_{\diamond(x)\to x}^* t^\diamond$.*

Proof. Let $s \to_{p,\ell\to r} t$ and let B be the base context of s. If $p \in \mathcal{P}\mathsf{os}_\mathcal{F}(B)$, then by Lemma 4.14, we obtain a term t' and a context C such that $s^\diamond \to_{p,\ell\to r} t'$ and $B \to_{p,\ell\to r} C$. Decomposing t as $t = C[\mathbf{t}]$, we find that $t' = C[\diamond(\mathbf{t})]$. If $p \notin \mathcal{P}\mathsf{os}_\mathcal{F}(B)$, then the rewrite step is within a tall alien of s. Hence, letting $C = B$ and decomposing t as $C[\mathbf{t}]$, we find that $s^\diamond = C[\diamond(\mathbf{s})] \to_\mathcal{R} C[\diamond(\mathbf{t})]$. In either case, Lemma 3.8 (with L equal to the max-top of C) shows that the tall aliens of t are a subsequence of $\mathbf{t}$, and therefore $C[\diamond(\mathbf{t})] \to_{\diamond(x)\to x}^* t^\diamond$, using that $\diamond(t_i) \to_{\diamond(x)\to x} t_i$ for those t_i that are not tall aliens. $\Box$

Lemma 4.33. *If $\mathcal{R}$ is bounded duplicating, then $\twoheadrightarrow$ is locally decreasing.*

Proof. Since $\mathcal{R}$ is bounded duplicating, we may assume a fresh function symbol $\diamond$ such that $\{\diamond(x) \to x\}/\mathcal{R}$ is terminating. In order to compare the labels, we define a well-founded order on terms by $s_0 > s_1$ if $s_0^\diamond \to_{\{\diamond(x)\to x\}/\mathcal{R}}^+ s_1^\diamond$. Consider a local peak $t\;{}_{s_0}\!\!\twoheadleftarrow s \twoheadrightarrow_{s_1} u$, which we first subject to Lemma 4.28. We analyze the sequence $t \to_\mathcal{R}^* v$ resulting from the peak analysis by distinguishing two cases:

(1) If $t'\pi_\Box$ is the base context of t, then the rewrite sequence $t'\pi_\Box \to_\mathcal{R}^* v'\pi_\Box$ mirrors $t \to_\mathcal{R}^* v$. Hence, we obtain $t \twoheadrightarrow_{s_1} v$, noting that the label s_1 satisfies $s_1 \to_\mathcal{R}^* s \to_\mathcal{R}^* t$.
(2) If $t'\pi_\Box$ is not the base context, then like in the proof of Lemma 4.32, we can decompose t as $t = t'\pi_\Box[\mathbf{t}']$ in order to obtain $s^\diamond \to_\mathcal{R}^* t'\pi_\Box[\diamond(\mathbf{t}')]$. Since $t'\pi_\Box$ is not the base context, the tall aliens of t are a proper subsequence of $\mathbf{t}'$, and therefore, $t'\pi_\Box[\diamond(\mathbf{t}')] \to_{\diamond(x)\to x}^+ t^\diamond$. We also have $s_1 \to_\mathcal{R}^* s$, which implies $s_1^\diamond \to_{\mathcal{R}\cup\{\diamond(x)\to x\}}^* s$ by Lemma 4.32. As a consequence, $s_1^\diamond \to_{\mathcal{R}/\{\diamond(x)\to x\}}^+ t^\diamond$ and $s_1 > t$ follow. By repeated application of Lemma 4.16, we obtain $t \twoheadrightarrow_t^* v$ and thus $t \twoheadrightarrow_{\vee s_1}^* v$.

The analogous analysis of $u \to_\mathcal{R}^* v$ yields $u \twoheadrightarrow_{s_0} v$ or $u \twoheadrightarrow_{\vee s_0}^* v$, and hence $\twoheadrightarrow$ is locally decreasing. $\Box$

Finally, we prepare for the main result about layered TRSs, where condition (C_1) of Definition 3.3 is crucial.

LEMMA 4.34. *Let $\mathcal{R}$ be a layered TRS and $t \to_{p,\ell\to r} t'$ for native terms t and t'. If $p \in \mathcal{P}\mathsf{os}_{\mathcal{F}}(B)$ for the base context B of t, then either $B \to_{p,\ell\to r} B'$ for the base context B' of t' or t' is its own base context.*

PROOF. Let M and M' be the max-tops of t and t'. We distinguish two cases:

(1) If $p \in \mathcal{P}\mathsf{os}_{\mathcal{F}}(M)$, then by ($C_1$), either $M \to_{p,\ell\to r} \Box$ or $M \to_{p,\ell\to r} M'$. In the former case, t' equals an alien of t. Since the rank of t' is at most r, t' is its own base context. So assume $M \to_{p,\ell\to r} M'$. By Lemma 3.10, there exist a term m and a substitution σ such that $m \to_{p,\ell\to r} m'$ for some m' (since $\ell \leqslant m|_p$), $t = m\sigma$, and $M = m\sigma_\Box$. Define a substitution τ as follows:

$$\tau(x) = \begin{cases} \Box & \text{if } x \in \mathrm{dom}(\sigma_\Box) \text{ and } \sigma(x) \text{ is a tall alien of } t \\ \sigma(x) & \text{otherwise.} \end{cases}$$

We have $B = m\tau$ by construction of τ. Let $B' = m'\tau$. Clearly, $B \to_{p,\ell\to r} B'$. By comparing $m'\tau$ to $M' = m'\sigma_\Box$, we see that B' is the base context of t'.

(2) If $p \notin \mathcal{P}\mathsf{os}_{\mathcal{F}}(M)$, then a short alien of t is rewritten. By letting B and $\mathbf{t}$ be the base context and base sequence of t, by Lemma 3.11, we obtain a rewrite step $t = B[\mathbf{t}] \to_{p,\ell\to r} B'[\mathbf{t}'] = t'$ with $\mathbf{t}' = \mathbf{t}$ because p is parallel to the hole positions of B. We claim that B' is the base context of t'. Suppose to the contrary that some t_i is not a tall alien of t'. Let q be its position in t, which is also its position in t'. Since $q \in \mathcal{P}\mathsf{os}_\Box(M)$ and $q \notin \mathcal{P}\mathsf{os}_\Box(M')$, $M \sqsubset M[M'|_q]_q$. Hence, $M[M'|_q]_q \in \mathbb{L}$ by (C_2) and thus $M[M'|_q]_q \sqsubseteq t$, contradicting the fact that M is a max-top of t. □

The following example shows that (C_2) is essential for Lemma 4.34.

Example 4.35. Recall Figure 1 and the underlying layer system $\mathbb{L}$, which satisfies (W) and (C_1). However, (C_2) is violated; for example, we have $L = \mathsf{k}(\Box,\Box) \in \mathbb{L}$ and $N = \mathsf{k}(\mathsf{h}(\Box),\mathsf{h}(\Box)) \in \mathbb{L}$ but $L[N|_2]_2 = \mathsf{k}(\Box,\mathsf{h}(\Box)) \notin \mathbb{L}$. Consider the term $t = \mathsf{k}(\mathsf{f}(\mathsf{a}),\mathsf{h}(\mathsf{a}))$ of rank 3. Its base context is $B = \mathsf{k}(\mathsf{f}(\mathsf{a}),\Box)$. We have $t \to \mathsf{k}(\mathsf{h}(\mathsf{a}),\mathsf{h}(\mathsf{a})) =: t'$. The base context of t' is $\mathsf{k}(\mathsf{h}(\Box),\mathsf{h}(\Box)) =: B'$ but $B \not\to_{\mathcal{R}} B'$.

LEMMA 4.36. *If $\mathcal{R}$ is layered, then $\blacktriangleright\!\!\blacktriangleright$ is locally decreasing.*

PROOF. Consider a local peak $t \;{}_{s_0}\!\blacktriangleleft\!\!\blacktriangleleft\; s \;\blacktriangleright\!\!\blacktriangleright_{s_1}\; u$. First, we analyze the peak by Lemma 4.28. The rewrite sequence $t'\pi_\Box \to_{\mathcal{R}}^* v'\pi_\Box$ mirrors $t = t'\pi \to_{\mathcal{R}}^* v'\pi = v$. We find by repeated application of Lemma 4.34 that the base context B_t of t equals $t'\pi_\Box$ or t. In both cases, we have $t \blacktriangleright\!\!\blacktriangleright_{s_1} v$, noting that $t \to_{\mathcal{R}}^* v$ mirrors itself, and that $s_1 \to_{\mathcal{R}}^* t$. We obtain $u \blacktriangleright\!\!\blacktriangleright_{s_0} v$ in the same way, and hence $\blacktriangleright\!\!\blacktriangleright$ is locally decreasing. □

4.4. Proof of Main Theorems

Because the proofs are similar, we prove all main results in one go.

PROOF OF THEOREMS 4.1, 4.3, AND 4.6. By assumption, the TRS $\mathcal{R}$ is weakly layered and confluent on terms of rank one. We have to show that

—if $\mathcal{R}$ is left-linear, then $\mathcal{R}$ is confluent (Theorem 4.1);
—if $\mathcal{R}$ is bounded duplicating, then $\mathcal{R}$ is confluent (Theorem 4.3); and
—if $\mathcal{R}$ is layered, then $\mathcal{R}$ is confluent (Theorem 4.6).

We show confluence of all terms by induction on the rank r of a term. In the base case, we consider terms of rank one, which are confluent by assumption. Assume as induction hypothesis that confluence of terms of rank r or less has been established.

We consider terms of rank $r + 1$, to which the analysis of Sections 4.1 to 4.3 applies. By Lemma 4.27 in conjunction with Lemma 4.30 (for weakly layered left-linear $\mathcal{R}$), Lemma 4.33 (for weakly layered bounded duplicating $\mathcal{R}$), or Lemma 4.36 (for layered $\mathcal{R}$), we obtain confluence of $\mathcal{R}$ on terms of rank up to $r + 1$, completing the induction step. $\square$

5. APPLICATIONS

In this section, the abstract confluence results via layer systems are instantiated by concrete applications. Section 5.1 treats the plain modularity case [Toyama 1987], and Section 5.2 covers layer-preservation [Ohlebusch 1994a]. The result for quasi-ground systems [Kitahara et al. 1995] is less known but also fits our framework, as outlined in Section 5.3. Currying [Kahrs 1995] is the topic of Section 5.4, before many-sorted persistence [Aoto and Toyama 1997] is discussed in Section 5.5.

For the results in this section, the reverse directions also hold. We do not give the (easy) proofs since they do not require layer systems.

In Sections 5.1, 5.2, and 5.3, we deal with two TRSs $\mathcal{R}_1$ and $\mathcal{R}_2$ that are defined over the respective signatures $\mathcal{F}_1$ and $\mathcal{F}_2$. We let $\mathcal{R} = \mathcal{R}_1 \cup \mathcal{R}_2$ and $\mathcal{F} = \mathcal{F}_1 \cup \mathcal{F}_2$.

5.1. Modularity

We recall the classical modularity result for confluence [Toyama 1987].

THEOREM 5.1. *Suppose $\mathcal{F}_1 \cap \mathcal{F}_2 = \varnothing$. If $\mathcal{R}_1$ and $\mathcal{R}_2$ are confluent, then $\mathcal{R}$ is confluent.*

PROOF. Define

$$\mathbb{L} := \mathcal{C}(\mathcal{F}_1, \mathcal{V}) \cup \mathcal{C}(\mathcal{F}_2, \mathcal{V}).$$

We show that $\mathcal{R}$ is layered. Since $\mathcal{V} \subseteq \mathbb{L}$ and $f(\square, \ldots, \square) \in \mathbb{L}$ for all function symbols $f \in \mathcal{F}_1 \cup \mathcal{F}_2$, every term in $\mathcal{T}(\mathcal{F}, \mathcal{V})$ has a nonempty top. Hence, condition (L_1) holds. Also, condition (L_2) holds because $\mathbb{L}$ is closed under the operation of interchanging variables and holes. For condition (L_3), we observe that if $L \in \mathcal{C}(\mathcal{F}_i, \mathcal{V})$, $p \in \mathcal{P}\mathsf{os}_{\mathcal{F}}(L)$, and $N \in \mathbb{L}$ such that $L|_p \sqcup N$ is defined, then $\mathrm{root}(L|_p) \in \mathcal{F}_i$ and thus $N \in \mathcal{C}(\mathcal{F}_i, \mathcal{V})$. Consequently, $L[L|_p \sqcup N]_p \in \mathcal{C}(\mathcal{F}_i, \mathcal{V}) \subseteq \mathbb{L}$. Since each rule is over a single signature, and layers are closed under rewriting, condition (W) follows easily. For condition (C_1), we consider a term s with max-top M, $p \in \mathcal{P}\mathsf{os}_{\mathcal{F}}(M)$, and rewrite step $s \to_{p,\ell\to r} t$, which is mirrored by $M \to_{p,\ell\to r} L$. Suppose $M \in \mathcal{C}(\mathcal{F}_i, \mathcal{V})$. We have $L \in \mathcal{C}(\mathcal{F}_i, \mathcal{V})$. The case $L = \square$ is obtained when t is an alien of s, which is only possible if the rule $\ell \to r$ is collapsing. Otherwise, L is the max-top of t since the root symbols of aliens of s belong to $\mathcal{F}_{3-i}$ and hence cannot fuse with L to form a larger top. Finally, condition (C_2) holds because if $N \in \mathcal{C}(\mathcal{F}_i, \mathcal{V})$, then $L \sqsubseteq N$ implies $L \in \mathcal{C}(\mathcal{F}_i, \mathcal{V})$ and thus also $L[N|_p]_p$ belongs to $\mathcal{C}(\mathcal{F}_i, \mathcal{V})$.

According to Theorem 4.6, $\mathcal{R}$ is confluent if we show that $\mathcal{R}$ is confluent on terms of rank one. The latter follows from the fact that rewriting does not increase the rank of a term (Lemma 3.12) together with the observation that nonvariable terms of rank one belong to either $\mathcal{T}(\mathcal{F}_1, \mathcal{V})$ or $\mathcal{T}(\mathcal{F}_2, \mathcal{V})$ and only rewrite rules of $\mathcal{R}_i$ apply to terms in $\mathcal{T}(\mathcal{F}_i, \mathcal{V})$, in connection with the confluence assumptions of $\mathcal{R}_1$ and $\mathcal{R}_2$. $\square$

5.2. Layer-Preservation

Layer-preserving TRSs are a special class of TRSs with shared function symbols for which confluence is modular as shown in Ohlebusch [1994a]. In this section, we reprove this result using layer systems. Let $\mathcal{T}_X(\mathcal{F}, \mathcal{V})$ denote the set of terms with root symbol from X. Let $\mathcal{B} := \mathcal{F}_1 \cap \mathcal{F}_2$, $\mathcal{D}_1 := \mathcal{F}_1 \backslash \mathcal{F}_2$ and $\mathcal{D}_2 := \mathcal{F}_2 \backslash \mathcal{F}_1$. The result on layer-preservation can be stated as follows.

THEOREM 5.2. *Let* $\mathcal{R}_1 \subseteq \mathcal{T}(\mathcal{B},\mathcal{V})^2 \cup \mathcal{T}_{\mathcal{D}_1}(\mathcal{F}_1,\mathcal{V})^2$, $\mathcal{R}_2 \subseteq \mathcal{T}(\mathcal{B},\mathcal{V})^2 \cup \mathcal{T}_{\mathcal{D}_2}(\mathcal{F}_2,\mathcal{V})^2$, *and* $\mathcal{R}_1 \cap \mathcal{T}(\mathcal{B},\mathcal{V})^2 = \mathcal{R}_2 \cap \mathcal{T}(\mathcal{B},\mathcal{V})^2$. *If* $\mathcal{R}_1$ *and* $\mathcal{R}_2$ *are confluent, then* $\mathcal{R}$ *is confluent.*

PROOF. We define

$$\mathbb{L} := \mathcal{C}(\mathcal{B},\mathcal{V}) \cup \mathcal{T}_{\mathcal{D}_1}(\mathcal{F}_1 \cup \{\Box\},\mathcal{V}) \cup \mathcal{T}_{\mathcal{D}_2}(\mathcal{F}_2 \cup \{\Box\},\mathcal{V}).$$

It is easy to verify that $\mathbb{L}$ layers $\mathcal{R} := \mathcal{R}_1 \cup \mathcal{R}_2$, much like in the modularity case. In particular, $\mathbb{L}$ is closed under rewriting. Consider a term s of rank one and a peak $t \;{}_{\mathcal{R}}^{*}\!\leftarrow s \rightarrow_{\mathcal{R}}^{*} u$. Let $i \in \{1,2\}$ be such that $s \in \mathcal{T}(\mathcal{F}_i,\mathcal{V})$. The only rules of $\mathcal{R}_{3-i}$ that can be used in the peak come from $\mathcal{T}(\mathcal{B},\mathcal{V})^2$ and hence also appear in $\mathcal{R}_i$. Since $\mathcal{R}_i$ is confluent on $\mathcal{T}(\mathcal{F}_i,\mathcal{V})$, we obtain joinability of t and u in $\mathcal{R}_i$ and thus also in $\mathcal{R}$. Hence, $\mathcal{R}$ is confluent on terms of rank one and we conclude by Theorem 4.6. $\Box$

Toyama's modularity result has been adapted by Ohlebusch [1994b] to constructor-sharing combinations in which the participating TRSs may share constructor symbols under the additional condition that neither collapsing nor constructor-lifting rules are present. This result is subsumed by Theorem 5.2 (cf. [Ohlebusch 2002, p. 249]). Still, layer preservation and modularity are incomparable (since layer-preservation places collapsing rules in both systems).

5.3. Quasi-Ground Systems

We show modularity of quasi-ground TRSs [Kitahara et al. 1995, Theorem 1] using layer systems.

Definition 5.3. We call a context C *quasi-ground* if for all $p \in \mathcal{P}\mathsf{os}(C)$ with $\mathrm{root}(C|_p) \in \mathcal{F}_1 \cap \mathcal{F}_2$, $C|_p$ is ground over $\mathcal{F}$, that is, $C|_p \in \mathcal{T}(\mathcal{F})$.

THEOREM 5.4. *Suppose* $\mathrm{root}(\ell) \notin \mathcal{F}_1 \cap \mathcal{F}_2$ *and* ℓ *and* r *are quasi-ground, for all* $\ell \to r \in \mathcal{R}$. *If* $\mathcal{R}_1$ *and* $\mathcal{R}_2$ *are confluent, then* $\mathcal{R}$ *is confluent.*

PROOF. We define a layer system $\mathbb{L} := \mathbb{L}_1 \cup \mathbb{L}_2 \cup \mathbb{L}_c$ with

$$\mathbb{L}_i = \{C \in \mathcal{C}(\mathcal{F}_i,\mathcal{V}) \mid C \text{ is quasi-ground}\} \quad \text{for } i = 1,2$$
$$\mathbb{L}_c = \{f(v_1,\ldots,v_n) \mid f \in \mathcal{F}_1 \cap \mathcal{F}_2 \text{ and } v_i \in \mathcal{V}_\Box \text{ for } 1 \leqslant i \leqslant n\}.$$

We readily check that (L_1) and (L_2) are satisfied. For (L_3), $\mathbb{L}_1$, $\mathbb{L}_2$, and $\mathbb{L}_c$ are individually closed under merging at function positions. Fix $i \in \{1,2\}$. If we merge $L \in \mathbb{L}_i$ with $N \in \mathbb{L}_{3-i} \cup \mathbb{L}_c$ at $p \in \mathcal{P}\mathsf{os}_{\mathcal{F}}(L)$, then either $N = \Box$ and $L[L|_p \sqcup N] = L \in \mathbb{L}_i$, or $\mathrm{root}(L|_p) \in \mathcal{F}_1 \cap \mathcal{F}_2$, which implies $L|_p \in \mathcal{T}(\mathcal{F})$ and hence $L[L|_p \sqcup N]_p = L[L|_p]_p = L \in \mathbb{L}_i$. Note that $L \in \mathbb{L}_c$ can be merged with $N \in \mathbb{L}_i$ only at position $p = \epsilon$. If $N = \Box$, then $L \sqcup \Box = L \in \mathbb{L}_c$, and otherwise $L \sqcup N = N \in \mathbb{L}_i$. For (W), we let M be the max-top of s, $p \in \mathcal{P}\mathsf{os}_{\mathcal{F}}(M)$, and consider a rewrite step $s \rightarrow_{p,\ell\to r} t$. We assume without loss of generality that $\ell \to r \in \mathcal{R}_1$. Hence, $M \in \mathbb{L}_1$ because $\mathrm{root}(\ell) \in \mathcal{F}_1 \setminus \mathcal{F}_2$. Note that $\mathbb{L}_1$ is closed under taking subterms and that for any substitution $\tau : \mathcal{V} \to \mathbb{L}_1$, we have $\ell\tau \in \mathbb{L}_1$. Let σ be a substitution such that $s|_p = \ell\sigma$, and let τ be the substitution that maps each variable $x \in \mathcal{V}\mathsf{ar}(\ell)$ to the $\mathbb{L}_1$-max-top of $\sigma(x)$. We have $M = M[\ell\tau]_p$ and thus $M \rightarrow_{p,\ell\to r} L$ with $L = M[r\tau]_p \in \mathbb{L}_1$. For ($C_1$), it is easy to see that L is the $\mathbb{L}_1$-max-top of t. Suppose $L \neq \Box$. We claim that L is the max-top (with respect to $\mathbb{L}$) of t. This follows from the observation that if there is a top of t that comes from $\mathbb{L}_2$ or $\mathbb{L}_c$, then $\mathrm{root}(L) \in \mathcal{F}_1 \cap \mathcal{F}_2$ and thus $L \in \mathcal{T}(\mathcal{F})$, which cannot be made larger. Condition (C_2) follows as in the proof of Theorem 5.1.

Now let $\mathcal{R}_1$ and $\mathcal{R}_2$ be confluent. We show that $\mathcal{R}$ is confluent on terms of rank one. Consider a term of rank one. Note that rules from $\mathcal{R}_1$ only apply to elements of $\mathbb{L}_1$. Furthermore, $\mathbb{L}_1$ is closed under rewriting by $\mathcal{R}_1$. Likewise, rules from $\mathcal{R}_2$ only apply to elements of $\mathbb{L}_2$, which is closed under rewriting by $\mathcal{R}_2$. We conclude that $\mathcal{R}$ is confluent on terms of rank one, and by Theorem 4.6, this implies that $\mathcal{R}$ is confluent. □

5.4. Currying

Currying is a transformation of TRSs such that the resulting TRS has only one nonconstant function symbol Ap that represents partial applications. It is useful in the construction of polynomial-time procedures for deciding properties of TRSs (e.g., [Comon et al. 2001]). Kahrs [1995] proved that confluence is preserved by currying.

Definition 5.5. Given a TRS $\mathcal{R}$ over a signature $\mathcal{F}$, let $\mathcal{F}_\mathcal{C} = \{\mathsf{Ap}\} \cup \{f_0 \mid f \in \mathcal{F}\}$, where Ap is a fresh binary function symbol and all function symbols in $\mathcal{F}$ become constants. The *curried version* $\mathsf{Cu}(\mathcal{R})$ of $\mathcal{R}$ is the TRS over the signature $\mathcal{F}_\mathcal{C}$ with rules $\{\mathsf{Cu}(\ell) \to \mathsf{Cu}(r) \mid \ell \to r \in \mathcal{R}\}$. Here, $\mathsf{Cu}(t) = t$ if t is a variable or a constant and $\mathsf{Cu}(f(t_1, \ldots, t_n)) = \mathsf{Ap}(\cdots \mathsf{Ap}(f_0, \mathsf{Cu}(t_1)) \cdots, \mathsf{Cu}(t_n))$ (with n occurrences of Ap). Let $\mathcal{F}_\mathcal{U} = \{\mathsf{Ap}\} \cup \{f_i \mid f \in \mathcal{F} \text{ and } 0 \leqslant i \leqslant \text{arity}(f)\}$, where each f_i has arity i and $f_{\text{arity}(f)}$ is identified with f. The *partial parameterization* $\mathsf{PP}(\mathcal{R})$ of $\mathcal{R}$ is the TRS $\mathcal{R} \cup \mathcal{U}$ over the signature $\mathcal{F}_\mathcal{U}$, where $\mathcal{U}$ consists of all *uncurrying* rules:

$$\mathsf{Ap}(f_i(x_1, \ldots, x_i), x_{i+1}) \to f_{i+1}(x_1, \ldots, x_{i+1})$$

for all $f \in \mathcal{F}$ and $0 \leqslant i < \text{arity}(f)$.

The next example familiarizes the reader with the previous concepts.

Example 5.6. For the TRS $\mathcal{R} = \{\mathsf{f}(x, x) \to \mathsf{f}(\mathsf{a}, \mathsf{b})\}$, we have

$$\begin{aligned} \mathsf{Cu}(\mathcal{R}) &= \{\mathsf{Ap}(\mathsf{Ap}(\mathsf{f}_0, x), x) \to \mathsf{Ap}(\mathsf{Ap}(\mathsf{f}_0, \mathsf{a}), \mathsf{b})\} \\ \mathcal{U} &= \{\mathsf{Ap}(\mathsf{f}_0, x) \to \mathsf{f}_1(x), \mathsf{Ap}(\mathsf{f}_1(x), y) \to \mathsf{f}(x, y)\} \\ \mathsf{PP}(\mathcal{R}) &= \mathcal{R} \cup \mathcal{U}. \end{aligned}$$

Note that for a term $s = \mathsf{Ap}(\mathsf{Ap}(\mathsf{Ap}(\mathsf{f}_0, x), x), x)$,we have

$$s \to_{\mathsf{Cu}(\mathcal{R})} \mathsf{Ap}(\mathsf{Ap}(\mathsf{Ap}(\mathsf{f}_0, \mathsf{a}), \mathsf{b}), x)$$

and

$$s \to_\mathcal{U} \mathsf{Ap}(\mathsf{Ap}(\mathsf{f}_1(x), x), x) \to_\mathcal{U} \mathsf{Ap}(\mathsf{f}(x, x), x) \to_\mathcal{R} \mathsf{Ap}(\mathsf{f}(\mathsf{a}, \mathsf{b}), x),$$

so the partial parameterization is closely related to currying.

Note that $\mathcal{U}$ is both terminating and orthogonal, hence confluent. By $s{\downarrow_\mathcal{U}}$, we denote the unique $\mathcal{U}$-normal form of a term s.

LEMMA 5.7 [KAHRS 1995, PROPOSITION 3.1]. *Let $\mathcal{R}$ be a TRS. If* $\mathsf{PP}(\mathcal{R})$ *is confluent, then* $\mathsf{Cu}(\mathcal{R})$ *is confluent.*

THEOREM 5.8 [KAHRS 1995, THEOREM 5.2]. *Let $\mathcal{R}$ be a TRS. If $\mathcal{R}$ is confluent, then* $\mathsf{Cu}(\mathcal{R})$ *is confluent.*

PROOF. According to Lemma 5.7, it suffices to show that $\mathsf{PP}(\mathcal{R})$ is confluent. To this end, we let $\mathbb{L} := \mathbb{L}_1 \cup \mathbb{L}_2$, where $\mathbb{L}_1$ is the smallest extension of $\mathcal{V}_\square$ such that

$$\mathsf{Ap}(\cdots \mathsf{Ap}(f_m(s_1, \ldots, s_m), s_{m+1}) \cdots, s_n) \in \mathbb{L}_1$$

for all $f_m \in \mathcal{F}_\mathcal{U} \setminus \{\mathsf{Ap}\}$, $s_1, \ldots, s_n \in \mathbb{L}_1$, with n less than or equal to the arity of f in the original TRS $\mathcal{R}$, and

$$\mathbb{L}_2 = \{\mathsf{Ap}(v, t) \mid v \in \mathcal{V}_\square \text{ and } t \in \mathbb{L}_1\}.$$

It is not difficult to see that $\mathbb{L}_1$ consists of those contexts in $\mathcal{C}(\mathcal{F}_\mathcal{U}, \mathcal{V})$ whose $\mathcal{U}$-normal form contains no occurrences of Ap. See Figure 2 for some layered terms.

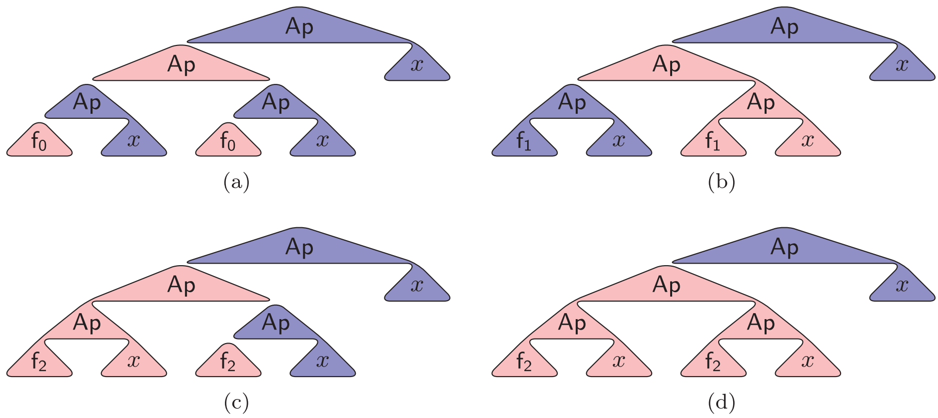


Fig. 2. Layering terms in $\mathsf{PP}(\mathcal{R})$ for the TRS $\mathcal{R}$ in Example 5.6.

We claim that $\mathsf{PP}(\mathcal{R})$ is layered. Conditions (L_1) and (L_2) are trivial and conditions (L_3) and (C_2) are easily shown by induction on the definition of $\mathbb{L}_1$. The interesting case for (L_3) is when $L \in \mathbb{L}_1$. Since merging cannot create new Ap symbols above any f_m, the result is in $\mathbb{L}_1$, whenever defined. For (W) and (C_1), we let M be the max-top of s, $p \in \mathcal{P}\mathsf{os}_{\mathcal{F}}(M)$, and consider a rewrite step $s \to_{p,\ell\to r} t$ with $\ell \to r \in \mathsf{PP}(\mathcal{R})$. Because $\mathbb{L}$ is closed under taking subterms, $M|_p$ is a top of $s|_p$. It is the max-top because otherwise we could merge the max-top of $s|_p$ with M at position p and obtain a larger top of s. Note that $\ell_\Box \in \mathbb{L}_1$ (recall that $\ell_\Box$ is obtained by replacing all variables in ℓ by $\Box$). We have $\ell_\Box \sqsubseteq s|_p$ and therefore $\ell_\Box \sqsubseteq M|_p$. As a matter of fact, $M|_p$ is obtained from $\ell_\Box$ by replacing each hole at position q by the max-top (in $\mathbb{L}_1$) of $s|_{pq}$. Because equal subterms have equal max-tops, $s \leqslant M|_p$, and hence there is a rewrite step $M \to_{p,\ell\to r} L$. We have $L \in \mathbb{L}_1$ because $\mathbb{L}_1$ is closed under rewriting by $\mathsf{PP}(\mathcal{R})$. Furthermore, the max-tops of the aliens of s do not belong to $\mathbb{L}_1$, and therefore the aliens of s are still aliens of L, unless $L = \Box$. It follows that both (W) and (C_1) hold.

To show confluence of $\mathsf{PP}(\mathcal{R})$ on terms of rank one, first note that elements of $\mathbb{L}_2$ allow no root steps, and therefore it suffices to show confluence on terms in $\mathbb{L}_1$. It is easy to see that $s \to_{\mathcal{R}\cup\mathcal{U}} t$ implies $s{\downarrow_\mathcal{U}} \to_{\mathcal{R}}^{=} t{\downarrow_\mathcal{U}}$. Hence, for a peak $t \;{}_{\mathcal{R}\cup\mathcal{U}}^{*}{\leftarrow}\; s \to_{\mathcal{R}\cup\mathcal{U}}^{*} u$, there is a corresponding peak $t{\downarrow_\mathcal{U}} \;{}_{\mathcal{R}}^{*}{\leftarrow}\; s{\downarrow_\mathcal{U}} \to_{\mathcal{R}}^{*} u{\downarrow_\mathcal{U}}$, which is joinable by the confluence of $\mathcal{R}$. Hence, t and u are joinable in $\mathsf{PP}(\mathcal{R})$. We conclude by Theorem 4.6. $\Box$

5.5. Many-Sorted Persistence

In this subsection, we prove persistence of confluence [Aoto and Toyama 1996]. We begin by recalling many-sorted terms and rewriting.

Definition 5.9. Let S be a set of *sorts*. A *sort attachment* $\mathcal{S}$ *associates with each function symbol* $f \in \mathcal{F}$ *of arity n a type* $f : \alpha_1 \times \cdots \times \alpha_n \to \alpha$ *with* $\alpha_i, \alpha \in S$ *for* $1 \leqslant i \leqslant n$, *and with each variable* $x \in \mathcal{V}$ *a sort from* S. Let $\mathcal{V}_\alpha$ denote the set of variables of sort α. We assume that each $\mathcal{V}_\alpha$ is countably infinite.

Note that $\mathcal{V}_\alpha \cap \mathcal{V}_\beta = \varnothing$ for all $\alpha, \beta \in S$ whenever $\alpha \neq \beta$.

Definition 5.10. Let $\mathcal{S}$ be a sort attachment. We define terms of sort α inductively by $\mathcal{T}_\alpha(\mathcal{F},\mathcal{V}) = \mathcal{V}_\alpha \cup \{ f(t_1,\ldots,t_n) \mid f : \alpha_1 \times \cdots \times \alpha_n \to \alpha \text{ and } t_i \in \mathcal{T}_{\alpha_i}(\mathcal{F},\mathcal{V}) \text{ for } 1 \leqslant i \leqslant n\}$. The set of many-sorted terms is defined as $\mathcal{T}_S(\mathcal{F},\mathcal{V}) = \bigcup_{\alpha\in S} \mathcal{T}_\alpha(\mathcal{F},\mathcal{V})$.

Definition 5.11. A TRS $\mathcal{R}$ is *compatible* with a sort attachment $\mathcal{S}$ if for each rule $\ell \to r \in \mathcal{R}$, there is a sort $\alpha \in S$ with $\ell, r \in \mathcal{T}_\alpha(\mathcal{F},\mathcal{V})$.

Remark 5.12. If a TRS $\mathcal{R}$ is compatible with a sort attachment $\mathcal{S}$, then $\mathcal{T}_\alpha(\mathcal{F}, \mathcal{V})$ is closed under rewriting by $\mathcal{R}$, for each $\alpha \in S$.

The following theorem states that confluence is a persistent property of TRSs.

THEOREM 5.13. *Let a TRS $\mathcal{R}$ be compatible with a sort attachment $\mathcal{S}$. If $\mathcal{R}$ is confluent on $\mathcal{T}_\mathcal{S}(\mathcal{F}, \mathcal{V})$, then $\mathcal{R}$ is confluent.*

PROOF. Assume that $\mathcal{R}$ is confluent on $\mathcal{T}_\mathcal{S}(\mathcal{F}, \mathcal{V})$. We let $\mathbb{L}$ be the smallest set such that $\mathcal{T}_\mathcal{S}(\mathcal{F}, \mathcal{V}) \subseteq \mathbb{L}$ and $\mathbb{L}$ is closed under replacing variables by holes and vice versa (cf. (L_2)). It is easy to see that $\mathcal{R}$ is layered according to $\mathbb{L}$. (W) and (C_1) follow from the compatibility assumption and Remark 5.12. Also, (C_2) is confirmed easily. We show that $\mathcal{R}$ is confluent on terms of rank one. To this end, consider a term $s \in \mathbb{L} \cap \mathcal{T}(\mathcal{F}, \mathcal{V})$. The confluence assumption on $\mathcal{T}_\mathcal{S}(\mathcal{F}, \mathcal{V})$ does not immediately apply to s since the variables need not match the type of their context. If s is a variable, then s is confluent. Otherwise, there is a term s' in $\mathcal{T}_\mathcal{S}(\mathcal{F}, \mathcal{V})$ that has s as an instance. Because subterms of sort α are interchangeable in many-sorted terms, we may choose s' in such a way that $s'|_p = s'|_q$ if $s'|_p, s'|_q \in \mathcal{V}_\alpha$ for some α and $s|_p = s|_q$. Note that for each p, the sort of $s'|_p$ is uniquely determined by s. Because the sets $\mathcal{T}_\alpha(\mathcal{F}, \mathcal{V})$ are pairwise disjoint, any rewrite sequence on $s \in \mathbb{L} \cap \mathcal{T}(\mathcal{F}, \mathcal{V})$ is mirrored by a rewrite sequence from $s' \in \mathcal{T}_\mathcal{S}(\mathcal{F}, \mathcal{V})$. By assumption, s' is confluent and hence s is confluent as well. We conclude that $\mathcal{R}$ is confluent on terms of rank one and hence confluent by Theorem 4.6. □

6. ORDER-SORTED PERSISTENCE

In this section, we establish order-sorted persistence. Section 6.1 introduces order-sorted rewriting, states the main result, and explains how to exploit it for establishing confluence. In Section 6.2, we prove the result for left-linear systems before Section 6.3 shows that layer systems cannot immediately cover arbitrary TRSs. We refine them such that they become suitable and give an alternative proof for many-sorted persistence (Section 6.4) before we finally prove order-sorted persistence in Section 6.5. We compare our result with the earlier result by Aoto and Toyama [1996] in Section 7.1.

6.1. Confluence via Order-Sorted Persistence

To obtain order-sorted terms, we equip a set of sorts S with a precedence $>$ and modify Definition 5.10 as follows.

Definition 6.1. Let $\mathcal{S}$ be a sort attachment. We define terms of sort α inductively by $\mathcal{T}_\alpha(\mathcal{F}, \mathcal{V}) = \mathcal{V}_\alpha \cup \{f(t_1, \ldots, t_n) \mid f : \alpha_1 \times \cdots \times \alpha_n \to \alpha,\ t_i \in \mathcal{T}_{\beta_i}(\mathcal{F}, \mathcal{V}),\ \alpha_i \geqslant \beta_i$, and $1 \leqslant i \leqslant n\}$. The set of order-sorted terms is $\mathcal{T}_\mathcal{S}(\mathcal{F}, \mathcal{V}) = \bigcup_{\alpha \in S} \mathcal{T}_\alpha(\mathcal{F}, \mathcal{V})$. A term t is *strictly order-sorted* if $\mathrm{root}(t|_p) : \alpha_1 \times \cdots \times \alpha_n \to \alpha$ and $t|_{pi} \in \mathcal{V}_\beta$ imply $\alpha_i = \beta$, for all $p \in \mathcal{P}\mathrm{os}_\mathcal{F}(t)$.

Note that we obtain many-sorted terms by letting $> \; = \varnothing$. Next, we define when a TRS is *compatible* with a sort attachment $\mathcal{S}$ in the order-sorted setting.

Definition 6.2. A TRS $\mathcal{R}$ is *compatible* with a sort attachment $\mathcal{S}$ if each rule $\ell \to r \in \mathcal{R}$ satisfies condition (1) and *strongly compatible* with $\mathcal{S}$ if condition (2) is satisfied as well.

(1) If $\ell \in \mathcal{T}_\alpha(\mathcal{F}, \mathcal{V})$ and $r \in \mathcal{T}_\beta(\mathcal{F}, \mathcal{V})$, then $\alpha \geqslant \beta$ and ℓ is strictly order-sorted.
(2) If $r \in \mathcal{V}_\beta$, then β is maximal in S. If $r \notin \mathcal{V}$, then r is strictly order-sorted.

Note that condition (1) ensures that well-typed terms are closed under rewriting. The main result on order-sorted persistence is stated later.

THEOREM 6.3. *Let $\mathcal{R}$ be compatible with a sort attachment $\mathcal{S}$. Furthermore, assume that $\mathcal{R}$ is left-linear, bounded duplicating, or strongly compatible with $\mathcal{S}$. If $\mathcal{R}$ is confluent on $\mathcal{T}_\mathcal{S}(\mathcal{F}, \mathcal{V})$, then it is confluent.*

Theorem 6.3 gives rise to a decomposition result (presented in [Aoto and Toyama 1996, 1997]) based on order-sorted persistence. The decomposition is based on the observation that the sort of a term restricts the rules that can be applied when rewriting it; therefore, we can decompose a TRS $\mathcal{R}$ that is compatible with a sort attachment $\mathcal{S}$ into several TRSs $\mathcal{R}_\alpha$ ($\alpha \in S$), each containing the rules applicable to terms of sort α or less. Formally, we define $\trianglerighteq$ on sorts as the smallest transitive relation such that $> \subseteq \trianglerighteq$ and $\alpha \trianglerighteq \alpha_i$ whenever $f : \alpha_1 \times \cdots \times \alpha_n \to \alpha$, and then define $\mathcal{R}_\alpha = \{\ell \to r \mid \ell \to r \in \mathcal{R}, \ell \in \mathcal{T}_\beta(\mathcal{F}, \mathcal{V})$, and $\alpha \trianglerighteq \beta\}$.

The next example shows that order-sorted persistence is more powerful than many-sorted persistence for decomposing TRSs.

Example 6.4 (*adapted from Aoto and Toyama [1996]*). Consider the TRS $\mathcal{R}$ consisting of the rewrite rules

$$(1)\ \mathsf{f}(x, \mathsf{a}) \to \mathsf{g}(x) \qquad (2)\ \mathsf{f}(x, \mathsf{f}(x, \mathsf{b})) \to \mathsf{b} \qquad (3)\ \mathsf{g}(\mathsf{c}) \to \mathsf{c} \qquad (4)\ \mathsf{h}(x) \to \mathsf{h}(\mathsf{g}(x))$$

and the set of sorts $S = \{0, 1, 2\}$ with $1 \geqslant 0$. Let the sort attachment be given by $\mathsf{a}, \mathsf{b} : 1$, $\mathsf{c} : 0$, $\mathsf{f} : 0 \times 1 \to 1$, $\mathsf{g} : 0 \to 0$, $\mathsf{h} : 0 \to 2$, and $x : 0$. It is straightforward to check that $\mathcal{R}$ is consistent with $\mathcal{S}$. In the order-sorted TRS, only rules (1), (2), and (3) can be applied to terms of sort 1 and their reducts; rules (3) and (4) can be applied to terms of sort 2; and only rule (3) can be applied to terms of sort 0. Hence, since $\mathcal{R}_1 = \{(1), (2), (3)\}$ (which is terminating and has no critical pairs), $\mathcal{R}_2 = \{(3), (4)\}$ (which is orthogonal), and $\mathcal{R}_0 = \{(3)\}$ (orthogonal) are confluent, $\mathcal{R}$ is confluent. No such decomposition can be obtained with many-sorted persistence. Consider a *most general* sort attachment making all rules many sorted: $\mathsf{a}, \mathsf{b}, \mathsf{c}, x : 0$, $\mathsf{f} : 0 \times 0 \to 0$, $\mathsf{g} : 0 \to 0$, and $\mathsf{h} : 0 \to 2$. Since terms of sort 2 can have subterms of sort 0, no decomposition is possible.

The weaker conditions in Definition 6.2 for left-linear TRSs are beneficial.

Example 6.5. Consider the TRS $\mathcal{R}$ consisting of the rewrite rules

$$\mathsf{f}(\mathsf{a}) \to \mathsf{f}(\mathsf{f}(\mathsf{h}(\mathsf{c}))) \qquad \mathsf{g}(\mathsf{b}) \to \mathsf{g}(\mathsf{g}(\mathsf{h}(\mathsf{c}))) \qquad \mathsf{h}(x) \to x$$

and the set of sorts $S = \{0, 1, 2\}$ with $1, 2 \geqslant 0$. Let the sort attachment be given by $\mathsf{a} : 1$, $\mathsf{b} : 2$, $\mathsf{c}, x : 0$, $\mathsf{f} : 1 \to 1$, $\mathsf{g} : 2 \to 2$, and $\mathsf{h} : 0 \to 0$. Note that $\mathcal{R}$ is compatible with $\mathcal{S}$. We can decompose $\mathcal{R}$ into the component induced by sort 1: $\mathcal{R}_1 = \{\mathsf{f}(\mathsf{a}) \to \mathsf{f}(\mathsf{f}(\mathsf{h}(\mathsf{c}))), \mathsf{h}(x) \to x\}$, sort 2: $\mathcal{R}_2 = \{\mathsf{g}(\mathsf{b}) \to \mathsf{g}(\mathsf{g}(\mathsf{h}(\mathsf{c}))), \mathsf{h}(x) \to x\}$, and sort 0: $\mathcal{R}_0 = \{\mathsf{h}(x) \to x\}$. If we add the restrictions for non-left-linear systems, the collapsing rule $\mathsf{h}(x) \to x$ enforces $\mathsf{h} : \alpha \to \alpha$ for a maximal sort α. Hence, also, the arguments of f and g have sort α, and α is greater than or equal to the sort of $\mathsf{a}, \mathsf{b}, \mathsf{c}, \mathsf{f}(x), \mathsf{g}(x)$. So the component induced by α contains all rules.

6.2. Order-Sorted Persistence for Left-linear Systems

In this section, we show that layer systems can establish order-sorted persistence for left-linear TRSs.

THEOREM 6.6. *Let $\mathcal{R}$ be compatible with a sort attachment $\mathcal{S}$. If $\mathcal{R}$ is left-linear and confluent on $\mathcal{T}_\mathcal{S}(\mathcal{F}, \mathcal{V})$, then it is confluent.*

PROOF. Let $\mathbb{L}$ be the smallest set such that $\mathcal{T}_\mathcal{S}(\mathcal{F}, \mathcal{V}) \subseteq \mathbb{L}$ and $\mathbb{L}$ is closed under (L_2). First, we show that $\mathcal{R}$ is weakly layered according to $\mathbb{L}$. In the sequel, we call contexts *weakly order-sorted* if they are order-sorted except that arbitrary variables may occur at any position. (These are exactly the elements of $\mathbb{L}$ and weakly order-sorted terms are those in $\mathbb{L} \cap \mathcal{T}(\mathcal{F}, \mathcal{V})$.)

Condition (L_1) holds trivially and condition (L_2) holds by assumption. For (L_3), we assume that $L|_p \sqcup N = N'$ with $p \in \mathcal{P}\mathsf{os}_\mathcal{F}(L)$ is defined. Since $L, N \in \mathbb{L}$, obviously N' is weakly order-sorted and so is $L[N']_p$ since $\mathrm{root}(L|_p) = \mathrm{root}(N')$ and hence $L[N']_p \in \mathbb{L}$. The final condition is (W). So let $s \to_{p,\ell\to r} t$ with $p \in \mathcal{P}\mathsf{os}_{\mathcal{F}}(M)$ for the max-top M of s. We have $\mathrm{root}(M|_p) = \mathrm{root}(\ell)$, and hence $M[\ell]_p$ is a layer. Since M is the max-top of s and ℓ is left-linear, there is a substitution σ such that $M[\ell\sigma]_p = M$. Hence, $M \to_{p,\ell\to r} M[r\sigma]_p$. By compatibility with the sort attachment $\mathcal{S}$, we have $r\sigma \in \mathbb{L}$. Furthermore, if α and β are the sorts of ℓ and r, then $\alpha \geqslant \beta$ ensures that $M[r\sigma]_p$ is weakly order-sorted and hence a member of $\mathbb{L}$.

Next, we show confluence of terms of rank one. To this end, let $s \in \mathbb{L} \cap \mathcal{T}(\mathcal{F}, \mathcal{V})$. Then there are a term $s' \in \mathcal{T}_{\mathcal{S}}(\mathcal{F}, \mathcal{V})$ and a variable substitution χ such that $s = s'\chi$. Let $t \;{}_{\mathcal{R}}^{*}\!\leftarrow s \to_{\mathcal{R}}^{*} u$. By left-linearity of $\mathcal{R}$, there are terms t' and u' with $t = t'\chi$ and $u = u'\chi$ such that $t' \;{}_{\mathcal{R}}^{*}\!\leftarrow s' \to_{\mathcal{R}}^{*} u'$. The confluence assumption on $\mathcal{T}_{\mathcal{S}}(\mathcal{F}, \mathcal{V})$ yields $t' \to_{\mathcal{R}}^{*} v' \;{}_{\mathcal{R}}^{*}\!\leftarrow u'$. Hence, $t = t'\chi \to_{\mathcal{R}}^{*} v'\chi \;{}_{\mathcal{R}}^{*}\!\leftarrow u'\chi = t$. We conclude by Theorem 4.1. $\square$

6.3. Variable-Restricted Layer Systems

The following example shows that Theorem 4.6 alone cannot establish Theorem 6.3 for TRSs that are neither left-linear nor bounded duplicating.

Example 6.7. Consider the set of sorts $S = \{0, 1, 2, 3, 4\}$, where $2 \geqslant 0$ and $2 \geqslant 1$. The sort attachment $\mathcal{S}$ is given by

$$u : 0 \qquad v : 1 \qquad \mathsf{f} : 3 \times 3 \to 4 \qquad \mathsf{h} : 2 \times 2 \times 0 \times 1 \to 3$$
$$x : 2 \qquad y : 3 \qquad \mathsf{g} : 3 \to 3 \qquad \mathsf{a}, \mathsf{b} : 4,$$

and the TRS $\mathcal{R}$ consists of the rules

$$\mathsf{f}(y, y) \to \mathsf{a} \qquad \mathsf{f}(y, \mathsf{g}(y)) \to \mathsf{b} \qquad \mathsf{h}(x, x, u, v) \to \mathsf{g}(\mathsf{h}(u, v, u, v)).$$

Then $\mathcal{R}$ is confluent on $\mathcal{T}_{\mathcal{S}}(\mathcal{F}, \mathcal{V})$ because it is locally confluent and terminating on order-sorted terms, noting that u and v never represent equal terms due to sort constraints. However, if we take $\mathbb{L}$ to be the closure of $\mathcal{T}_{\mathcal{S}}(\mathcal{F}, \mathcal{V})$ under (L_2), then the term $\mathsf{f}(t, t)$ with $t = \mathsf{h}(z, z, z, z)$ is not confluent because $\mathsf{a} \leftarrow \mathsf{f}(t, t) \to \mathsf{f}(t, \mathsf{g}(t)) \to \mathsf{b}$. Note that $\mathsf{f}(t, t)$ is not order-sorted but contained in $\mathbb{L}$. Furthermore, observe that $\mathcal{R}$ is layered according to $\mathbb{L}$. Finally, note that $\mathbb{L}$ is the smallest layer system with this property that contains $\mathcal{T}_{\mathcal{S}}(\mathcal{F}, \mathcal{V})$.

The previous example does not contradict Theorem 6.3 since $\mathcal{R}$ is not strongly compatible with $\mathcal{S}$; the right-hand sides of $\mathcal{R}$ are not strictly order-sorted, although $\mathcal{R}$ is neither left-linear nor bounded duplicating. In particular, we have an infinite reduction $\mathsf{h}(z, z, \Diamond(z), \Diamond(z)) \to_{\mathcal{R}} \mathsf{g}(\mathsf{h}(\Diamond(z), \Diamond(z), \Diamond(z), \Diamond(z))) \to_{\Diamond(x)\to x}^{+} \mathsf{g}(\mathsf{h}(z, z, \Diamond(z), \Diamond(z))) \to_{\mathcal{R}} \cdots$.

The problem is that layer systems allow one to replace variables by variables of a different sort and hence contain terms that are not order-sorted, enabling new rewrite steps (which never happens in the many-sorted case or for left-linear systems in the order-sorted setting). Since $\mathcal{T}_{\mathcal{S}}(\mathcal{F}, \mathcal{V}) \subsetneq \mathbb{L} \cap \mathcal{T}(\mathcal{F}, \mathcal{V})$, we have to study when confluence on $\mathcal{T}_{\mathcal{S}}(\mathcal{F}, \mathcal{V})$ implies confluence on $\mathbb{L} \cap \mathcal{T}(\mathcal{F}, \mathcal{V})$ in order to apply Theorem 4.6. Instead of proving the missing implication directly, we again pursue a general approach. To this end, we relax condition (L_2) such that variables need not be replaced by variables of different sort to enable the representation of $\mathcal{T}_{\mathcal{S}}(\mathcal{F}, \mathcal{V})$ as $\mathbb{L} \cap \mathcal{T}(\mathcal{F}, \mathcal{V})$, where $\mathbb{L}$ satisfies the following refined notion of layer systems.

Definition 6.8. Recall the conditions from Definition 3.3. We introduce the following condition:

(L_2') If $C[x]_p \in \mathbb{L}$, then $C[\Box]_p \in \mathbb{L}$. If $C[\Box]_p \in \mathbb{L}$, then $\{x \in \mathcal{V} \mid C[x]_p \in \mathbb{L}\}$ is an infinite set.

We call $\mathbb{L} \subseteq \mathcal{C}(\mathcal{F}, \mathcal{V})$ a *variable-restricted layer system* if it satisfies the conditions (L_1), (L_2'), and (L_3). Analogously, a variable-restricted layer system *weakly layers* $\mathcal{R}$ if (W) is satisfied and *layers* $\mathcal{R}$ if (W), (C_1), and (C_2) are satisfied.

To distinguish between variable-restricted and (unrestricted) layer systems, we denote the former by $\mathbb{V}$ in the future. Note that (L_2) implies (L_2'), and hence any layer system is also a variable-restricted layer system. Furthermore, for each variable-restricted layer system $\mathbb{V}$, there is a corresponding (unrestricted) layer system $\mathbb{L}_{\mathbb{V}} = \mathbb{V} \cup \{C[x]_p \mid C[\Box]_p \in \mathbb{V} \text{ and } x \in \mathcal{V}\}$. Obviously, $\mathbb{V} \subseteq \mathbb{L}_{\mathbb{V}}$.

With the new condition (L_2'), it is now possible to adequately represent $\mathcal{T}_{\mathcal{S}}(\mathcal{F}, \mathcal{V})$ by a variable-restricted layer system.

Example 6.9 (*Example 6.7 revisited*). To obtain a variable-restricted layer system, let $\mathbb{V}$ be the smallest set such that $\mathcal{T}_{\mathcal{S}}(\mathcal{F}, \mathcal{V}) \subseteq \mathbb{V}$ and $\mathbb{V}$ is closed under replacing variables by holes. Then it satisfies (L_2'). Note that $\mathbb{V} \cap \mathcal{T}(\mathcal{F}, \mathcal{V}) = \mathcal{T}_{\mathcal{S}}(\mathcal{F}, \mathcal{V})$, and hence $t = \mathsf{h}(z, z, z, z) \notin \mathbb{V}$ and thus $\mathsf{f}(t, t) \notin \mathbb{V}$.

For a weakly layered TRS, the reduct of a rank one term again is a rank one term.

LEMMA 6.10. *Let $\mathbb{V}$ be a variable-restricted layer system that weakly layers a TRS $\mathcal{R}$. Then $\mathbb{V} \cap \mathcal{T}(\mathcal{F}, \mathcal{V})$ is closed under rewriting by $\mathcal{R}$.*

PROOF. Let $t \in \mathbb{V} \cap \mathcal{T}(\mathcal{F}, \mathcal{V})$ and $t \to_{\mathcal{R}} u$. Note that t is its own max-top. By (W), its reduct u is a layer and hence $u \in \mathbb{V} \cap \mathcal{T}(\mathcal{F}, \mathcal{V})$. □

In the remainder of this section, we show the analogs of Theorems 4.1, 4.3, and 4.6 for variable-restricted layer systems (cf. Corollary 6.24).

The case of left-linear systems is straightforward.

LEMMA 6.11. *Let $\mathbb{V}$ be a variable-restricted layer system that weakly layers a left-linear TRS $\mathcal{R}$. If $\mathcal{R}$ is confluent on $\mathbb{V} \cap \mathcal{T}(\mathcal{F}, \mathcal{V})$, then $\mathcal{R}$ is confluent on $\mathbb{L}_{\mathbb{V}} \cap \mathcal{T}(\mathcal{F}, \mathcal{V})$.*

PROOF. Let $s \in \mathbb{L}_{\mathbb{V}} \cap \mathcal{T}(\mathcal{F}, \mathcal{V})$. By ($L_2$) and ($L_2'$), a term $s' \in \mathbb{V} \cap \mathcal{T}(\mathcal{F}, \mathcal{V})$ and a variable substitution χ exist such that $s'\chi = s$. Now consider rewrite sequences $t \; {}_{\mathcal{R}}^{*}\!\leftarrow s \to_{\mathcal{R}}^{*} u$. Thanks to left-linearity, there are terms t' and u' with $t'\chi = t$, $u'\chi = u$, and $t' \; {}_{\mathcal{R}}^{*}\!\leftarrow s' \to_{\mathcal{R}}^{*} u'$. By repeated application of Lemma 6.10, t', u', and all intermediate terms are elements of $\mathbb{V} \cap \mathcal{T}(\mathcal{F}, \mathcal{V})$. From the assumption, we obtain a valley $t' \to_{\mathcal{R}}^{*} v' \; {}_{\mathcal{R}}^{*}\!\leftarrow u'$, inducing a valley $t = t'\chi \to_{\mathcal{R}}^{*} v'\chi \; {}_{\mathcal{R}}^{*}\!\leftarrow u'\chi = u$. Note that $v'\chi \in \mathbb{L}_{\mathbb{V}}$ (by Lemma 3.9) and obviously $v'\chi \in \mathcal{T}(\mathcal{F}, \mathcal{V})$. □

To prepare for a result concerning bounded duplicating TRSs, we generalize bounded duplication to weakly bounded duplication, which turns out to be more suitable for the proof of Lemma 6.14.

Definition 6.12. We call $\mathcal{R}$ *weakly bounded duplicating* if $\{\top \to \bot\}/\mathcal{R}$ is terminating for fresh constants $\top$ and $\bot$.

LEMMA 6.13. *Any bounded duplicating TRS is weakly bounded duplicating.*

PROOF. Assume that $\mathcal{R}$ is not weakly bounded duplicating. So there exists an infinite rewrite sequence $t_0 \to t_1 \to \cdots$ in $\mathcal{R} \cup \{\top \to \bot\}$ that contains infinitely many applications of the rule $\top \to \bot$. Let t_i' be obtained from t_i by replacing all occurrences of $\top$ by $\diamond(\bot)$. Since $\top$ does not appear in the rules of $\mathcal{R}$, we obtain an infinite rewrite sequence $t_0' \to t_1' \to \cdots$ in $\mathcal{R} \cup \{\diamond(x) \to x\}$ with infinitely many applications of the instance $\diamond(\bot) \to \bot$ of $\diamond(x) \to x$. Hence, $\mathcal{R}$ is not bounded duplicating. □

To see that weakly bounded duplication generalizes bounded duplication, consider the TRS $\mathcal{R}$ consisting of the single rule $\mathsf{f}(\mathsf{a}, x) \to \mathsf{f}(x, x)$, which is not bounded duplicating since $\mathsf{f}(\mathsf{a}, \diamond(\mathsf{a})) \to_{\mathcal{R}} \mathsf{f}(\diamond(\mathsf{a}), \diamond(\mathsf{a})) \to_{\diamond(x)\to x} \mathsf{f}(\mathsf{a}, \diamond(\mathsf{a})) \to_{\mathcal{R}} \cdots$, but weakly bounded duplicating.

Next, we will establish the following two lemmata.

LEMMA 6.14. *Let $\mathbb{V}$ be a variable-restricted layer system that weakly layers a weakly bounded duplicating TRS $\mathcal{R}$. If $\mathcal{R}$ is confluent on $\mathbb{V} \cap \mathcal{T}(\mathcal{F}, \mathcal{V})$, then $\mathcal{R}$ is confluent on $\mathbb{L}_{\mathbb{V}} \cap \mathcal{T}(\mathcal{F}, \mathcal{V})$.*

LEMMA 6.15. *Let $\mathbb{V}$ be a variable-restricted layer system that layers a TRS $\mathcal{R}$. If $\mathcal{R}$ is confluent on $\mathbb{V} \cap \mathcal{T}(\mathcal{F}, \mathcal{V})$, then $\mathcal{R}$ is confluent on $\mathbb{L}_{\mathbb{V}} \cap \mathcal{T}(\mathcal{F}, \mathcal{V})$.*

For both proofs, we are given a variable-restricted layer system $\mathbb{V}$ that weakly layers a TRS $\mathcal{R}$. We fix an initial term $s \in \mathbb{L}_{\mathbb{V}} \cap \mathcal{T}(\mathcal{F}, \mathcal{V})$ and show that it is confluent. Since $\mathbb{V} \subseteq \mathbb{L}_{\mathbb{V}}$, the confluence assumption on $\mathbb{V} \cap \mathcal{T}(\mathcal{F}, \mathcal{V})$ may not apply to s. To overcome this problem, we use (L_2) and (L_2') to construct a term $s' \in \mathbb{V} \cap \mathcal{T}(\mathcal{F}, \mathcal{V})$ and a variable substitution χ such that $s = s'\chi$ and fix a well-order $\gg$ on $\mathcal{V}\mathsf{ar}(s')$. We extend $\gg$ to terms by closing it under contexts and transitivity.

Let $s \in \mathbb{L}_{\mathbb{V}} \cap \mathcal{T}(\mathcal{F}, \mathcal{V})$, $s' \in \mathbb{V} \cap \mathcal{T}(\mathcal{F}, \mathcal{V})$, and χ with $s = s'\chi$ be fixed.

Definition 6.16. A term $t' \in \mathbb{V} \cap \mathcal{T}(\mathcal{F}, \mathcal{V})$ is a *representative* of $t \in \mathbb{L}_{\mathbb{V}} \cap \mathcal{T}(\mathcal{F}, \mathcal{V})$ if $t = t'\chi$ and $\mathcal{V}\mathsf{ar}(t') \subseteq \mathcal{V}\mathsf{ar}(s')$. A representative t' of t is called *minimal* if it is minimal with respect to $\gg$.

Note that s' is a representative of s. Before proving key properties for representatives, we show how they help to avoid the situation of Example 6.7.

Example 6.17 (*Example 6.7 revisited*). Consider the variables with sorts

$$x_1, x_2, x_5, x_6 : 2 \qquad\qquad x_3, x_7 : 0 \qquad\qquad x_4, x_8 : 1$$

and order $x_8 \gg x_7 \gg \cdots \gg x_1$. The term $s = \mathsf{f}(t, t) \in \mathbb{L}_{\mathbb{V}} \cap \mathcal{T}(\mathcal{F}, \mathcal{V})$ has the representative $s' = \mathsf{f}(\mathsf{h}(x_1, x_2, x_3, x_4), \mathsf{h}(x_5, x_6, x_7, x_8)) \in \mathbb{V} \cap \mathcal{T}(\mathcal{F}, \mathcal{V})$ and the (unique) minimal representative $\hat{s} = \mathsf{f}(\hat{t}, \hat{t}) \in \mathbb{V} \cap \mathcal{T}(\mathcal{F}, \mathcal{V})$, where $\hat{t} = \mathsf{h}(x_1, x_1, x_3, x_4)$. The peak $\mathsf{a} \leftarrow \mathsf{f}(t, t) \to \mathsf{f}(t, \mathsf{g}(t)) \to \mathsf{b}$ in $\mathbb{L}_{\mathbb{V}} \cap \mathcal{T}(\mathcal{F}, \mathcal{V})$ is simulated by

$$\mathsf{a} \leftarrow \mathsf{f}(\hat{t}, \hat{t}) \to \mathsf{f}(\hat{t}, \mathsf{g}(\mathsf{h}(x_3, x_4, x_3, x_4))) \gg \mathsf{f}(\hat{t}, \mathsf{g}(\hat{t})) \to \mathsf{b}$$

in $\mathbb{V} \cap \mathcal{T}(\mathcal{F}, \mathcal{V})$. Note that the $\gg$ step replaces $\mathsf{f}(\hat{t}, \mathsf{g}(\mathsf{h}(x_3, x_4, x_3, x_4)))$ by the least representative $\mathsf{f}(\hat{t}, \mathsf{g}(\hat{t}))$ of $\mathsf{f}(t, \mathsf{g}(t))$.

The key operation on representatives and related terms is copying variables between them, as justified by the following lemma.

LEMMA 6.18. *Let $L, N \in \mathbb{V}$ be layers with $L_\square = N_\square$. If $p \in \mathcal{P}\mathsf{os}_{\mathcal{V}_\square}(L)$, then $L[N|_p]_p \in \mathbb{V}$.*

PROOF. If $p = \epsilon$, then the claim is trivial. Otherwise, let $L' = L[\square]_p$ and $N' = N[\square]_{q \in \mathcal{P}\mathsf{os}_{\mathcal{V}_\square}(L) \setminus \{p\}}$. We have $L', N' \in \mathbb{V}$ by applications of property (L_2') and $L[N|_p]_p = L' \sqcup N'$ by assumption. Property (L_3) yields the desired $L[N|_p]_p \in \mathbb{V}$. $\square$

The next lemma establishes that the minimal representative (if it exists) is unique, justifying the name *least* representative. The proof makes the construction in Example 6.17 explicit and is illustrated by Example 6.20.

LEMMA 6.19. *If $t \in \mathbb{L}_{\mathbb{V}} \cap \mathcal{T}(\mathcal{F}, \mathcal{V})$ has a representative, then it has a least representative.*

PROOF. We have to show the existence and uniqueness of a minimal representative of t. From a representative t' we obtain $t'_\square \in \mathbb{V}$ using (L_2') repeatedly. Consider $V_p = \{x \in \mathcal{V}\mathsf{ar}(s') \mid \chi(x) = t|_p \text{ and } t'_\square[x]_p \in \mathbb{V}\}$ for each $p \in \mathcal{P}\mathsf{os}_{\mathcal{V}}(t')$. Note that $t'|_p \in V_p$ because we can insert the variable $t'|_p$ into $t'_\square$ at position p by Lemma 6.18 to obtain a layer in $\mathbb{V}$. Hence, V_p is nonempty. Since it is also finite, it has a minimum element $\min(V_p)$ with respect to $\gg$. Let $\dot{t} = t'_\square[\min(V_p)]_{p \in \mathcal{P}\mathsf{os}_\mathcal{V}(t')}$. We have $\dot{t} \in \mathbb{V}$ by (L'_2) and the definition of V_p. Clearly, $\dot{t} \in \mathcal{T}(\mathcal{F}, \mathcal{V})$ and $\mathcal{V}\mathsf{ar}(\dot{t}) \subseteq \mathcal{V}\mathsf{ar}(s')$ because all holes are replaced by some variable from $\mathcal{V}\mathsf{ar}(s')$. Moreover, $\dot{t}\chi = t$ by construction, in particular the definition of V_p. It follows that $\dot{t}$ is a representative of t. Note that $\dot{t}$ does not depend on the choice of t' because $t'_\square = t_\square$. Therefore, $t' \gg^= \dot{t}$ for any representative t' of t, which makes $\dot{t}$ the least representative of t. $\square$

Example 6.20 (*Example 6.17 revisited*). Consider $s = \mathsf{f}(\mathsf{h}(z,z,z,z), \mathsf{h}(z,z,z,z))$ and $s' = \mathsf{f}(\mathsf{h}(x_1,x_2,x_3,x_4), \mathsf{h}(x_5,x_6,x_7,x_8))$ with $\chi(x_i) = z$ for all $1 \leqslant i \leqslant 8$. Then $s'_\square = \mathsf{f}(\mathsf{h}(\square,\square,\square,\square), \mathsf{h}(\square,\square,\square,\square))$. Since $V_{11} = V_{12} = V_{21} = V_{22} = \{x_1, \ldots, x_8\}$, $V_{13} = V_{23} = \{x_3, x_7\}$, and $V_{14} = V_{24} = \{x_4, x_8\}$, we obtain $\dot{s} = \mathsf{f}(\mathsf{h}(x_1,x_1,x_3,x_4), \mathsf{h}(x_1,x_1,x_3,x_4))$.

We denote the least representative term of a representable term $t \in \mathbb{L}_\mathbb{V} \cap \mathcal{T}(\mathcal{F},\mathcal{V})$ by $\hat{t} \in \mathbb{V} \cap \mathcal{T}(\mathcal{F},\mathcal{V})$. The following lemma states that a rewrite step performed on a term in $\mathbb{L}_\mathbb{V}$ can be mirrored on its least representative in $\mathbb{V}$. Recall that in Example 6.17, the representative s' is a normal form but the step from s can be mirrored on $\hat{s}$.

LEMMA 6.21. *Let $t, u \in \mathbb{L}_\mathbb{V} \cap \mathcal{T}(\mathcal{F},\mathcal{V})$ with $t \to_\mathcal{R} u$ such that $\hat{t}$ exists.*

(*1*) *If $\mathbb{V}$ weakly layers $\mathcal{R}$, then $\hat{t} \to_\mathcal{R} u'$ for some representative u' of u.*
(*2*) *If $\mathbb{V}$ layers $\mathcal{R}$, then $u' = \hat{u}$ or $u' \in \mathcal{V}$ in* (1).

PROOF.

(1) Assume that $\hat{t}$ is the least representative of t and let $t \to_{p,\ell\to r} u$. We obtain a context $C \in \mathbb{V}$ by replacing all variables in t by $\square$. By Lemma 3.10, there is a term c with $C = c\sigma_\square$ and $\ell \leqslant c|_p$. To ensure $c \leqslant \hat{t}$, we need to show $\hat{t}|_q = \hat{t}|_r$ for all $x \in \mathcal{V}\mathsf{ar}(c)$ and $q, r \in \mathcal{P}\mathsf{os}_x(c)$. To that end, fix x and let $P = \mathcal{P}\mathsf{os}_x(c)$. For each $q \in P$, $\hat{t}|_q$ is a variable. Let $y = \min\{\hat{t}|_q \mid q \in P\}$. We will show that $\hat{t}|_q = y$ for all $q \in P$. Consider the max-top $M \in \mathbb{V}$ of $C[y,\ldots,y]$. Note that $c \leqslant C[y,\ldots,y]$, so that $\ell \leqslant C[y,\ldots,y]|_p$. From condition (W), we obtain $\ell \leqslant M|_p$ and thus $c \leqslant M$ by Lemma 3.10(2) since $C \sqsubseteq M$. By construction, $\hat{t}|_q = y$ for some $q \in P$. Since $C[y]_q$ is a layer by Lemma 6.18, $M \sqcup C[y]_q$ is a layer according to (L_3). Because M is the max-top of $C[y,\ldots,y]$, $M \sqcup C[y]_q = M$ and thus $M|_q = y$. It follows that $M|_q = y$ for all $q \in P$, since otherwise M would fail to be an instance of c. Repeated applications of Lemma 6.18 yield $t' = \hat{t}[y]_{q \in P} \in \mathbb{V}$. We have $t' = \hat{t}$ by the choice of y and the minimality of $\hat{t}$. We conclude that $c \leqslant \hat{t}$ and hence $\ell \leqslant \hat{t}|_p$, which induces a rewrite step $\hat{t} \to_{p,\ell\to r} u'$ as claimed. The term u' is a representative of u because $u'\chi = u$, $u' \in \mathbb{V}$ by Lemma 6.10, and rewriting does not introduce variables.
(2) Assume that u' is not a least representative of u. We have $u' \gg \hat{u}$, so there is a position $q \in \mathcal{P}\mathsf{os}_\mathcal{V}(u)$ with $z = u'|_q \gg \hat{u}|_q = y$. Let $C = c\sigma_\square$ as in the proof of part (1). There is a rewrite step $c \to_{p,\ell\to r} d$ for some term d and $C \to_{p,\ell\to r} D = d\sigma_\square$. Let $M \in \mathbb{V}$ and $L \in \mathbb{V}$ be the max-tops of $C_y = C[y,\ldots,y]$ and $D_y = D[y,\ldots,y]$. Note that $C_y \to_{p,\ell\to r} D_y$, which implies $M \to_{p,\ell\to r} L$ by (C_1) except when $M \to_{p,\ell\to r} \square$. In the latter case, r and thus also u' is a variable, and we are done. So assume $M \to_{p,\ell\to r} L$. Consider the variable $x = d|_q$. We must have $L|_q = y$ because otherwise we could copy $\hat{u}|_q = y$ to L by Lemma 6.18. The term $\hat{t}$ and the context M are instances of c and so there are substitutions $\sigma_{\hat{t}}$ and σ_M such that $c\sigma_{\hat{t}} = \hat{t}$ and $c\sigma_M = M$. We have $\sigma_{\hat{t}}(x) = u'|_q = z$ and $\sigma_M(x) = y$ because $d\sigma_M = L$. Since $x \in \mathcal{V}\mathsf{ar}(d)$ and $c \to_\mathcal{R} d$, the set $\mathcal{P}\mathsf{os}_x(c)$ is nonempty. Let $q' \in \mathcal{P}\mathsf{os}_x(c)$. The layer $C[y]_{q'} \in \mathbb{V}$ can be obtained by copying $M|_{q'} = y$ to C using Lemma 6.18. Since $\hat{t}|_{q'} = \sigma_{\hat{t}}(x) = z$, we obtain

$\hat{t} \gg \hat{t}[y]_{q'} \in \mathbb{V}$. The term $\hat{t}[y]_{q'}$ is a representative of t because $\chi(y) = \chi(z)$ (recall that $u = u'\chi = \hat{u}\chi$). Hence, we obtained a contradiction with the minimality of $\hat{t}$. □

The following lemma shows that instead of adding a single rule $\top \to \bot$, we can extend a weakly bounded duplicating TRS with any terminating ARS, where the objects are regarded as fresh constants, and still obtain relative termination. The induced well-founded order will be used in the proof of Lemma 6.14.

LEMMA 6.22. *Let $\mathcal{R}$ be a weakly bounded duplicating TRS and $\mathcal{A}$ a terminating ARS. If $\mathcal{R}$ and $\mathcal{A}$ share no constants, then $\mathcal{A}$ is terminating relative to $\mathcal{R}$.*

PROOF. We use reduction pairs for this proof, which are pairs consisting of a quasi-order $\geqslant$ and a well-founded strict order $>$ that are compatible: $\geqslant \cdot > \cdot \geqslant \subseteq >$. Reduction pairs give rise to a multiset extension in a straightforward way (e.g., the definitions of $>_{\mathrm{gms}}$ and $\geqslant_{\mathrm{gms}}$ in [Thieman et al. 2012]). We denote the objects in $\mathcal{A}$ by $\mathcal{O}$. Let $\mathcal{F}$ be the signature of $\mathcal{R}$. From the termination of $\mathcal{A}$, we obtain a well-founded order $>$ on $\mathcal{O}$ such that $\mathcal{A} \subseteq >$. For each $\alpha \in \mathcal{O}$, define a map π_α from $\mathcal{T}(\mathcal{F} \cup \mathcal{O}, \mathcal{V})$ to $\mathcal{T}(\mathcal{F} \cup \{\top, \bot\}, \mathcal{V})$ as follows:

$$\pi_\alpha(t) = \begin{cases} \top & \text{if } t = \alpha \\ \bot & \text{if } t \in \mathcal{O} \setminus \{\alpha\} \\ f(\pi_\alpha(t_1), \ldots, \pi_\alpha(t_n)) & \text{if } t = f(t_1, \ldots, t_n) \text{ with } f \in \mathcal{F} \\ t & \text{if } t \in \mathcal{V}. \end{cases}$$

We measure terms by the set $\#t = \{(\alpha, \pi_\alpha(t)) \mid \alpha \in \mathcal{F}\mathsf{un}(t) \cap \mathcal{O}\}$. The measures of two terms are compared by the multiset extension of the lexicographic product of the precedence $>$ on $\mathcal{O}$ and the reduction pair consisting of the well-founded (by the weakly bounded termination assumption) order $\to^+_{\{\top \to \bot\}/\mathcal{R}}$ and the compatible quasi-order $\to^*_{\mathcal{R}}$. Each application of a rule $\alpha \to \beta$ from $\mathcal{A}$ decreases the component associated with α in $\#t$ and introduces or modifies a component associated with β in $\#t$, giving rise to a decrease in the strict part of the multiset extension. Moreover, if $t \to_{\mathcal{R}} u$, then $\pi_\alpha(t) \to_{\mathcal{R}} \pi_\alpha(u)$, for all $\alpha \in \mathcal{O}$. Hence, the terms are related by the nonstrict part of the multiset extension. It follows that $\mathcal{A}$ is terminating relative to $\mathcal{R}$. □

PROOF OF LEMMA 6.14. To show confluence of s, we introduce a relation $\blacktriangleright\!\!\rhd$ that allows one to map an $\mathcal{R}$-peak from s to a $\blacktriangleright\!\!\rhd$-peak. Afterward, we show confluence of $\blacktriangleright\!\!\rhd$ and conclude by $\blacktriangleright\!\!\rhd \subseteq \to^*_{\mathcal{R}}$.

We write $t \blacktriangleright\!\!\rhd_{t'_0} u$ if $t'_0 \to^*_{\mathcal{R}} \hat{t}$ and $s \to^*_{\mathcal{R}} t \to^*_{\mathcal{R}} u$ such that $t \to^*_{\mathcal{R}} u$ is mirrored by $\hat{t} \to^*_{\mathcal{R}} u'$ with $u = u'\chi$. Labels are compared using the order $> := \to^+_{\gg/\mathcal{R}}$, which is well-founded according to Lemma 6.22 applied to the ARS $(\mathcal{V}\mathsf{ar}(s'), \gg)$, where we regard the elements of $\mathcal{V}\mathsf{ar}(s')$ as constants for this purpose.

First, we show that a peak consisting of $\mathcal{R}$-steps can be represented as a peak of $\blacktriangleright\!\!\rhd$-steps. To this end, we claim that $t \blacktriangleright\!\!\rhd_{\hat{t}} u$ whenever $s \to^*_{\mathcal{R}} t \to_{\mathcal{R}} u$. To show the claim, note that s has a least representative by Lemma 6.19, and that by Lemmata 6.21(1) and Lemma 6.19, each immediate successor of a term with a least representative also has a least representative. Therefore, t has a least representative, and we conclude by another application of Lemma 6.21(1). Next, we establish that $\blacktriangleright\!\!\rhd$ is locally decreasing and hence confluent by Theorem 2.1. Consider a local peak $u \;{}_{t'_0}\!\lhd\!\!\blacktriangleleft\; t \blacktriangleright\!\!\rhd_{t'_1} v$. By definition of $\blacktriangleright\!\!\rhd$, there are representatives u' and v' of u and v such that $u' \;{}^*_{\mathcal{R}}\!\!\leftarrow \hat{t} \to^*_{\mathcal{R}} v'$. We obtain $u' \to^*_{\mathcal{R}} w' \;{}^*_{\mathcal{R}}\!\!\leftarrow v'$ from the confluence assumption on $\mathbb{V} \cap \mathcal{T}(\mathcal{F}, \mathcal{V})$. Consider the sequence $u' \to^*_{\mathcal{R}} w'$. If $u' = \hat{u}$, then $u \blacktriangleright\!\!\rhd_{t'_1} w'\chi$, noting that $t'_1 \to^*_{\mathcal{R}} \hat{t} \to^*_{\mathcal{R}} u'$. Otherwise, there is a rewrite sequence $u = u'\chi = u_1 \to_{\mathcal{R}} \cdots \to_{\mathcal{R}} u_n = w'\chi = w$, such that $u' \gg \hat{u} \to^*_{\gg \cup \mathcal{R}} \hat{u}_i$ and thus $u' > \hat{u}_i$ for all $1 \leqslant i \leqslant n$. Hence, we obtain $u \blacktriangleright\!\!\rhd^*_{\vee t'_1} w$ by repeated use of the previous claim. Analogously, we obtain $v \blacktriangleright\!\!\triangleright_{t'_0} w$ or $v \blacktriangleright\!\!\triangleright^*_{\vee t'_0} w$. The proof is concluded by the obvious observation that $\blacktriangleright\!\!\triangleright \subseteq \to^*_{\mathcal{R}}$. $\square$

Proof of Lemma 6.15. Consider a peak $t \;{}_{\mathcal{R}}^{*}\!\!\leftarrow s \to^*_{\mathcal{R}} u$. Obviously s has a representative and hence also a least representative $\hat{s}$ by Lemma 6.19. Using Lemma 6.21 repeatedly, we obtain a peak $t' \;{}_{\mathcal{R}}^{*}\!\!\leftarrow \hat{s} \to^*_{\mathcal{R}} u'$, noting that all reducts of $\hat{s}$ are least representatives of the corresponding reducts of s or variables, but since variables are normal forms, the latter can only happen in the last step. From the confluence assumption on $\mathbb{V} \cap \mathcal{T}(\mathcal{F}, \mathcal{V})$, we obtain $t' \to^*_{\mathcal{R}} v' \;{}_{\mathcal{R}}^{*}\!\!\leftarrow u'$. Applying the variable substitution χ yields $t = t'\chi \to^*_{\mathcal{R}} v'\chi \;{}_{\mathcal{R}}^{*}\!\!\leftarrow u'\chi = u$ on $\mathbb{L}_{\mathbb{V}} \cap \mathcal{T}(\mathcal{F}, \mathcal{V})$. $\square$

Lemma 6.23. *If a TRS is (weakly) layered according to a variable-restricted layer system, then it is (weakly) layered according to the corresponding (unrestricted) layer system.*

Proof. The result for weakly layered TRSs is obvious. The result for layered TRSs follows from Lemma 6.21. $\square$

Corollary 6.24. *The statements of Theorems 4.1, 4.3, and 4.6 remain true when based on a variable-restricted layer system.*

Proof. In case of left-linear TRSs, we conclude by Theorem 4.1 and Lemmata 6.11 and 6.23. For bounded duplicating TRSs, we use Theorem 4.3 and Lemmata 6.13, 6.14, and 6.23. For TRSs that are layered according to a variable-restricted layer system, we use Theorem 4.6 and Lemmata 6.15 and 6.23. $\square$

6.4. Many-Sorted Persistence by Variable-Restricted Layer Systems

We demonstrate the usefulness of variable-restricted layer systems by the following alternative proof of Theorem 5.13, which avoids the complication of establishing confluence on $\mathbb{L} \cap \mathcal{T}(\mathcal{F}, \mathcal{V})$.

Proof of Theorem 5.13. Assume that $\mathcal{R}$ is confluent on $\mathcal{T}_{\mathcal{S}}(\mathcal{F}, \mathcal{V})$. We let $\mathbb{V}$ be the smallest set such that $\mathcal{T}_{\mathcal{S}}(\mathcal{F}, \mathcal{V}) \subseteq \mathbb{V}$ and $\mathbb{V}$ is closed under replacing variables by holes. So $\mathbb{V}$ trivially satisfies (L_2'). Hence, $\mathbb{V} \cap \mathcal{T}(\mathcal{F}, \mathcal{V}) = \mathcal{T}_{\mathcal{S}}(\mathcal{F}, \mathcal{V})$, and thus $\mathcal{R}$ is confluent on $\mathbb{V} \cap \mathcal{T}(\mathcal{F}, \mathcal{V})$ by the assumption. It is easy to see that $\mathbb{V}$ is a variable-restricted layer system layering $\mathcal{R}$; conditions (W) and (C_1) follow from the compatibility assumption. Therefore, $\mathcal{R}$ is confluent by Corollary 6.24. $\square$

6.5. Order-Sorted Persistence by Variable-Restricted Layer Systems

In this section, we prove the main result on order-sorted persistence.

Proof of Theorem 6.3. Assume that $\mathcal{R}$ is compatible with $\mathcal{S}$. To define layers as order-sorted terms, we add a fresh, minimum sort $\perp$ with $\square : \perp$ and require that no variable has sort $\perp$. The set $\mathbb{V} := \mathcal{T}_{\mathcal{S}\cup\{\perp\}}(\mathcal{F} \cup \{\square\}, \mathcal{V})$ is a variable-restricted layer system that satisfies (C_2).

We show that $\mathbb{V}$ satisfies condition (W). So let M be the max-top of s, $p \in \mathcal{P}\mathsf{os}_{\mathcal{F}}(M)$, and $s \to_{p,\ell\to r} t$. Because ℓ is order-sorted, $\mathcal{P}\mathsf{os}(\ell) \subseteq \mathcal{P}\mathsf{os}(M|_p)$. We claim that $\ell \leqslant M|_p$. If $\ell|_q = \ell|_{q'} \in \mathcal{V}_\alpha$, then $M|_{pq} = M|_{pq'} \in \mathcal{T}_{\alpha'}(\mathcal{F} \cup \{\square\}, \mathcal{V})$ for some α' with $\alpha \geqslant \alpha'$, due to the fact that ℓ is strictly order-sorted. Let σ be a substitution such that $\ell\sigma = M|_p$. Using the compatibility condition (of Definition 6.1), we readily obtain $L = M[r\sigma]_p \in \mathbb{V}$.

Next, we show that if $\mathcal{R}$ is strongly compatible with $\mathcal{S}$, then condition (C_1) holds. So assume that $\mathcal{R}$ is neither left-linear nor bounded duplicating and $L \neq \square$. We show that L is the max-top of t. Let L' be the max-top of t. First of all, if r is not a variable and $\ell|_q = r|_{q'} \in \mathcal{V}_\alpha$, then $L'|_{pq'} = M|_{pq} = L|_{pq'}$ because ℓ and r are strictly order-sorted.

This implies $L = L'$. Next, suppose that $r = x \in \mathcal{V}_\beta$. Let p' be the position directly above p and let $\mathrm{root}(L|_{p'}) : \beta_1 \times \cdots \times \beta_n \to \beta'$. We have $p = p'i$ for some $1 \leqslant i \leqslant n$. We claim that $\beta_i = \beta$. Let α be the sort of ℓ. We have $\alpha \geqslant \beta$ and $\beta_i \geqslant \alpha$. According to the second compatibility condition, β is maximal in S and thus $\beta = \alpha = \beta_i$. It follows that $L'|_p = M|_{pq} = L|_p$ for any $q \in \mathcal{P}\mathsf{os}_x(\ell)$.

Note that $\mathbb{V} \cap \mathcal{T}(\mathcal{F}, \mathcal{V}) = \mathcal{T}_S(\mathcal{F}, \mathcal{V}) = \mathbb{V} \cap \mathcal{T}_S(\mathcal{F}, \mathcal{V})$. The proof is concluded with an appeal to Corollary 6.24. □

7. RELATED WORK

As we already mentioned in the introduction, modularity of term rewrite systems has been re-proved several times. A number of related results have been proved by adapting the proof of Klop et al. [1994], and there have been several previous attempts to make the result more reusable. Ohlebusch [1994b] casts the modularity result in terms of a collapsing reduction $\to_c$ and shows that for composable TRSs, confluence is modular if $\to_c$ is normalizing. Toyama's theorem arises as a special case. Kahrs [1995] proposes an abstract framework, based on so-called *preconfluences* and *context selectors* constructed from preconfluences. The latter can be seen as a precursor of layer systems. In particular, the selection of max-tops gives rise to a (proper) context selector. However, the notion of preconfluences is geared toward the uncurrying application and too restrictive to encompass modularity of confluence [Kahrs 2011]. A third approach to abstraction is taken in Lüth [1996]. In this work, modularity of confluence is proved using category theory, exploiting the fact that terms can conveniently be modeled by a monad. Unfortunately, the development is flawed and only applies to TRSs over unary function symbols and constants.[1]

In the remainder of this section, we discuss specific issues, starting with a comparison of our result on order-sorted persistence to Aoto and Toyama [1996] in Section 7.1. In Section 7.2, we reflect on the differences between [Klop et al. 1994] and [van Oostrom 2008], which correspond to changes from the earlier conference paper [Felgenhauer et al. 2011] to the present article. In Section 7.3, we elaborate on the constructivity claim made in Section 1.

7.1. Order-Sorted Persistence

In this section, we compare our result from Section 6 to the main result of Aoto and Toyama [1996], which can be stated as follows.

Definition 7.1. A sort attachment $\mathcal{S}$ is *compatible** with a TRS $\mathcal{R}$ if condition ($\star$) is satisfied for each rewrite rule $\ell \to r \in \mathcal{R}$:

($\star$) If $\ell \in \mathcal{T}_\alpha(\mathcal{F}, \mathcal{V})$ and $r \in \mathcal{T}_\beta(\mathcal{F}, \mathcal{V})$, then $\alpha \geqslant \beta$ and ℓ, r are strictly order-sorted.

The main claim in [Aoto and Toyama 1996] is that Theorem 6.3 holds for compatible* systems. We show that this is incorrect. The counterexample presented here is simpler than our previous example in [Felgenhauer et al. 2011].

Example 7.2. We use $\{0, 1, 2, 3\}$ as sorts where $1 \geqslant 0$ and sort attachment $\mathcal{S}$

$$\begin{array}{lllll} x : 0 & \mathsf{f} : 0 \to 2 & \mathsf{h} : 1 \times 0 \to 2 & \mathsf{e} : 0 \to 1 & \mathsf{c} : 1 \\ y : 2 & \mathsf{g} : 2 \to 2 & \mathsf{i} : 2 \times 2 \to 3 & \mathsf{a}, \mathsf{b} : 3 & \end{array}$$

[1]The article claims that for any TRS Θ, the monad T_Θ is *strongly finitary*, which implies that it preserves coequalizers. This is not true in general. As an example, let $\star$ be the trivial category and consider the coequalizer $Q : \star + \star \to \star$ of the injections $\iota_1, \iota_2 : \star \to \star + \star$. Furthermore, let $\Theta = \{\mathsf{f}(x, x) \to x\}$. Then $T_\Theta(Q)$ equates $\mathsf{f}(\acute{}\iota_1\star, \acute{}\iota_1\star)$ and $\mathsf{f}(\acute{}\iota_1\star, \acute{}\iota_2\star)$, but the coequalizer of $T_\Theta(\iota_1)$ and $T_\Theta(\iota_2)$ does not, because $\mathsf{f}(\acute{}\iota_1\star, \acute{}\iota_1\star)$ is not in the image of either of these functors.

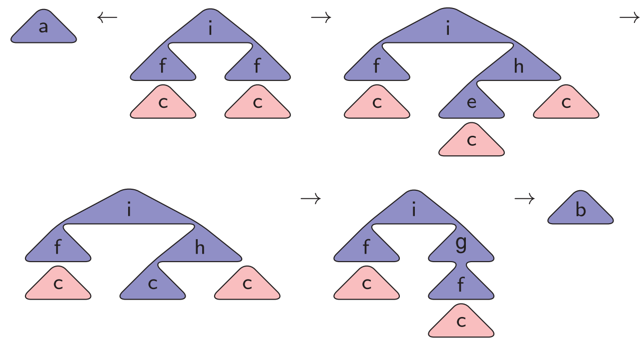


Fig. 3. Nonconfluence in Example 7.2.

Consider the TRS $\mathcal{R}$ consisting of the rules

$$\mathsf{f}(x) \to \mathsf{h}(\mathsf{e}(x), x) \qquad \mathsf{h}(\mathsf{c}, x) \to \mathsf{g}(\mathsf{f}(x)) \qquad \mathsf{e}(x) \to x \qquad \mathsf{i}(y, y) \to \mathsf{a} \qquad \mathsf{i}(y, \mathsf{g}(y)) \to \mathsf{b}.$$

This TRS is compatible* with $\mathcal{S}$. On order-sorted terms, it is locally confluent and terminating and thus confluent (note that x may not be instantiated by c due to the sort constraints). It is not confluent on arbitrary terms because

$$\mathsf{a} \leftarrow \mathsf{i}(\mathsf{f}(\mathsf{c}), \mathsf{f}(\mathsf{c})) \to^* \mathsf{i}(\mathsf{f}(\mathsf{c}), \mathsf{g}(\mathsf{f}(\mathsf{c}))) \to \mathsf{b}.$$

Note that any compatible* TRS is strongly compatible (cf. Definition 6.2), unless it is neither left-linear nor bounded duplicating, and contains a collapsing rule. Indeed, the TRS $\mathcal{R}$ of Example 7.2 has all these features. Ultimately, the culprit is the collapsing rule $\mathsf{e}(x) \to x$, causing fusion from above (cf. Figure 3). This case is not considered in the proof of Aoto and Toyama [1996, Proposition 3.9]. Definition 6.2 takes care of the problem with collapsing rules in Definition 7.1. Furthermore, it puts fewer constraints on the right-hand sides in case of left-linear or bounded duplicating systems, which is beneficial (cf. Example 6.5).

7.2. Modularity

We compare the proof setups of Klop et al. [1994] and van Oostrom [2008].

The first difference concerns the decomposition of terms. Whereas Klop et al. split a term into its max-top and aliens, van Oostrom splits it into a base context and a sequence of tall aliens. This is the key for making the proof constructive: while fusion of an alien may cause many new aliens to appear, none of them will be tall, so they do not have to be tracked explicitly. In contrast, Klop et al. start by constructing witnesses, and thus prevent aliens from fusing while establishing confluence.

The other ingredients of the proofs are quite similar: the proof setup is an induction on the rank of the starting term. One distinguishes inner ($\to_i^*$, acting on aliens) and outer ($\to_o^*$, acting on the max-top) steps (Klop et al.) or tall ($\rhd\!\rhd_\iota$, acting on the tall aliens) and short ($\blacktriangleright\!\blacktriangleright$, acting on the base context) steps (van Oostrom). One then argues as follows:

(1) Outer (short) steps are confluent because one can replace the principal (tall) subterms by suitable variables in the top (base) context and then invoke the induction hypothesis.
(2) Inner (tall) steps are confluent because they only act on principal subterms (tall aliens). In joining these subterms, one can ensure that any equalities between them are preserved (we call such sequences of inner steps *balanced*). In van Oostrom's

proof, the resulting joining sequences may involve fusion and therefore short steps, but by ranking short steps below tall steps, a locally decreasing diagram is obtained.

(3) Balanced inner steps (tall steps) and outer steps (short steps) commute (can be joined decreasingly). The idea is to replace the principal subterms (tall aliens) of source and target of the inner steps by the same variables, so that the outer steps can be simulated on the result. In van Oostrom's proof, the target term has to be balanced (with respect to the source) first.

When specialized to modularity, the same differences and similarities can be encountered when comparing [Felgenhauer et al. 2011] to the present work. Short steps differ in two ways from [van Oostrom 2008]. The imbalance is defined differently and the underlying rewrite sequences are less restricted here. Nevertheless, they define the same relation on native terms. This covers Theorem 4.6. For Theorems 4.1 and 4.3, our proof deals with a new effect, namely, fusion from above. This makes confluence of short short steps (▶▶) a nontrivial matter.

We remark that layer systems according to Definition 3.3 differ from those in [Felgenhauer et al. 2011]. The latter are closer to variable-restricted layer systems (Definition 6.8). Since the weakened condition (L_2') is only needed for the order-sorted setting, we decided to base the theory on the easier condition (L_2) instead and then derive the main results for variable-restricted layer systems separately (cf. Section 6.3). Furthermore, we remark that the notions of weakly layered and layered (which are related to weakly consistent and consistent in [Felgenhauer et al. 2011]) have changed in an incomparable way, even for variable-restricted layer systems. This is due to the new condition (C_2), which is required for our constructive proof, as shown in Example 4.35.

7.3. Constructivity

We say that a TRS is *constructively confluent* if there is a procedure that, given a peak $t \;{}^*\!\!\leftarrow s \rightarrow^* u$, constructs a valley $t \rightarrow^* v \;{}^*\!\!\leftarrow u$. In [van Oostrom 2008], constructive confluence is proved to be a modular property for disjoint TRSs.

Most previous proofs of modularity and related results rely on the reduction of terms until they allow no further fusion, which requires checking whether the top layer of a term may collapse, a property that is undecidable. This includes the proofs by [Toyama 1987], [Klop et al. 1994], [Ohlebusch 1994b], [Kahrs 1995], [Aoto and Toyama 1996, 1997], [Jouannaud and Toyama 2008], and [Jouannaud and Liu 2012]. Interestingly, Lüth's proof [Lüth 1996] is constructive, but not applicable in general as observed at the beginning of this section.

The key observation for obtaining a constructive result is that our main tool for establishing confluence, the decreasing diagrams technique, is constructive: if any given *local* peak can be joined decreasingly in a constructive way, then any conversion becomes joinable by exhaustively replacing local peaks by smaller conversions until none are left.

For our proofs to be constructive, the TRS needs to be constructively confluent on terms of rank one. Furthermore, the proofs rely on the decomposition of arbitrary terms into their max-top and aliens. Consequently, we must be able to decide whether a given context $C \sqsubseteq t$ is a max-top of t. In the applications from Section 5, this is indeed the case.

If these two assumptions are satisfied, then our proofs are constructive and we obtain the following corollary.

COROLLARY 7.3. *Let $\mathcal{R}$ be a TRS. Assume that $\mathcal{R}$ is left-linear and weakly layered, or bounded duplicating and weakly layered, or layered. If $\mathcal{R}$ is constructively confluent on terms of rank one and for any context C and term t it is decidable if C is a max-top of t, then $\mathcal{R}$ is constructively confluent.*

We remark that the previous corollary extends to variable-restricted layer systems, and thus to the order-sorted application in Section 6.

8. CONCLUSION

In this article, we have presented an abstract layer framework that covers several known results about the modularity and persistence of confluence. The framework enabled us to correct the result claimed in [Aoto and Toyama 1996] on order-sorted persistence and, by placing weaker conditions on left-linear or bounded duplicating systems, to increase its applicability. We have incorporated a decomposition technique based on order-sorted persistence (Theorem 6.3) into CSI [Zankl et al. 2011a], our confluence prover. In the implementation, we approximate bounded duplication by nonduplication. We also showed how Kahrs's confluence result for curried systems is obtained as an instance of our layer framework.

As future work, we plan to investigate how to apply layer systems to other properties of TRSs, like termination or having unique normal forms. Finally, we worked out the technical details of our main results to prepare for future certification efforts in a theorem prover like Isabelle or Coq. For the latter, it is essential that here (compared to our previous work [Felgenhauer et al. 2011]) we based our setting on the constructive modularity proof in van Oostrom [2008]. The underlying proof technique, decreasing diagrams, has already been formalized in Isabelle [Zankl 2013a, 2013b].

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