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SIMPLE TERMINATION OF REWRITE SYSTEMS

Abstract

In this paper we investigate the concept of simple termination. A term rewriting system (TRS for short) is called simply terminating if its termination can be proved by means of a simplification order. We propose a new definition of simplification order and we investigate the properties of the resulting class of simply terminating systems.

1. Preliminaries

We assume the reader's familiarity with term rewriting [3, 5]. A binary relation R on terms is *closed under contexts* if $C[s]\ R\ C[t]$ whenever $s\ R\ t$, for all contexts C. We say that R is *closed under substitutions* if $s\sigma\ R\ t\sigma$ whenever $s\ R\ t$, for all substitutions σ. A *rewrite relation* is a binary relation on terms that is closed under contexts and substitutions. A rewrite relation that is also a (strict) partial order is called a *rewrite order*. A well-founded rewrite order is called a *reduction order*. We say that a TRS $(\mathcal{F}, \mathcal{R})$ and a partial order $\succ$ on $\mathcal{T}(\mathcal{F}, \mathcal{V})$ are *compatible* if $\mathcal{R}$ is contained in $\succ$, i.e., $l \succ r$ for every rewrite rule $l \rightarrow r$ of $\mathcal{R}$. It is easy to show that a TRS is terminating if and only if it is compatible with a reduction order.

The subterm relation is denoted by $\trianglelefteq$. We say that a binary relation R on terms has the *subterm property* if $C[t]\ R\ t$ for all contexts $C \neq \Box$ and terms t, i.e., if $\rhd \subseteq R$. The subterm property is closely related to *embedding*. Let $\mathcal{F}$ be a signature. The TRS $\mathcal{E}mb\,(\mathcal{F})$ consists of all rewrite rules $f(x_1, \ldots, x_n) \rightarrow x_i$ with $f \in \mathcal{F}$ a function symbol of arity $n \geqslant 1$ and $i \in \{1, \ldots, n\}$. Here $x_1, \ldots, x_n$ are pairwise different variables. We abbreviate $\rightarrow^+_{\mathcal{E}mb\,(\mathcal{F})}$ to $\rhd_{emb}$. It is easy to prove that a rewrite order has the subterm property if and only if it is compatible with $\mathcal{E}mb\,(\mathcal{F})$. As a consequence, $\rhd_{emb}$ is the smallest rewrite order with the subterm property. Embedding is a special case of *homeomorphic* embedding. Let $\succ$ be a partial order on a signature $\mathcal{F}$. The TRS $\mathcal{E}mb(\mathcal{F},\succ)$ consists of all rewrite rules of $\mathcal{E}mb(\mathcal{F})$ together with all rewrite rules $f(x_1,\ldots,x_n) \to g(x_{i_1},\ldots,x_{i_m})$ with f an n-ary function symbol in $\mathcal{F}$, g an m-ary function symbol in $\mathcal{F}$, $n \geqslant m \geqslant 0$, $f \succ g$, and $1 \leqslant i_1 < \cdots < i_m \leqslant n$ whenever $m \geqslant 1$. We abbreviate $\to^+_{\mathcal{E}mb(\mathcal{F},\succ)}$ to $\succ_{emb}$.

We conclude this preliminary section by recalling the *Tree Theorem* of Kruskal [6]. A partial order $\succ$ on a set A is called a *partial well-order* (PWO for short) if every partial order on A that extends $\succ$ (including $\succ$ itself) is well-founded. Kruskal's Tree Theorem can be phrased as follows: $\succ_{emb}$ is a PWO on $\mathcal{T}(\mathcal{F})$ whenever $\succ$ is a PWO on the signature $\mathcal{F}$. A special case states that for finite $\mathcal{F}$, $\rhd_{emb}$ is a PWO on $\mathcal{T}(\mathcal{F})$.

2. Finite Signatures

Throughout this section we are dealing with *finite* signatures only. Under this assumption a *simplification order* is defined as a rewrite order with the subterm property and we call a TRS $(\mathcal{F},\mathcal{R})$ *simply terminating* if it is compatible with a simplification order on $\mathcal{T}(\mathcal{F},\mathcal{V})$. It follows from the above-mentioned special case of Kruskal's Tree Theorem that simplification orders are well-founded ([1]). Hence every simply terminating TRS is terminating. Another well-known result states that a TRS $(\mathcal{F},\mathcal{R})$ is simply terminating if and only if $(\mathcal{F},\mathcal{R}\cup\mathcal{E}mb(\mathcal{F}))$ is terminating.

In the term rewriting literature the notion of simplification order is sometimes based on preorders instead of partial orders. Dershowitz [2] proved that a TRS $(\mathcal{F},\mathcal{R})$ is terminating if there exists a preorder $\succsim$ on $\mathcal{T}(\mathcal{F},\mathcal{V})$ which is closed under contexts, has the subterm property, and satisfies $l\sigma \succ r\sigma$ for every rewrite rule $l \to r \in \mathcal{R}$ and substitution σ. Observe that we require $l\sigma \succ r\sigma$ *for all substitutions* σ. It should be stressed that this requirement cannot be weakened to the compatibility of $\mathcal{R}$ and $\succ$ even if we additionally require that $\succsim$ is closed under substitutions. The following result, which can be proved constructively, explains why there is no reason to base the definition of simplification order on preorders.

THEOREM 1. *A TRS $(\mathcal{F}, \mathcal{R})$ is simply terminating if and only if there exists a preorder $\succsim$ on $\mathcal{T}(\mathcal{F}, \mathcal{V})$ that is closed under contexts, has the subterm property, and satisfies $l\sigma \succ r\sigma$ for every rewrite rule $l \to r \in \mathcal{R}$ and substitution σ.* □

3. Infinite Signatures

Kurihara and Ohuchi [7] were the first to use the terminology simple termination. They call a TRS $(\mathcal{F}, \mathcal{R})$ simply terminating if it is compatible with a rewrite order on $\mathcal{T}(\mathcal{F}, \mathcal{V})$ that has the subterm property. Since compatibility with a rewrite order that has the subterm property doesn't ensure the termination of TRSs over infinite signatures, this definition is clearly not the right one. Consider for instance the TRS $(\mathcal{F}, \mathcal{R})$ consisting of infinitely many constants a_i and rewrite rules $a_i \to a_{i+1}$ for all $i \geqslant 1$. The rewrite order $\to_{\mathcal{R}}^{+}$ vacuously satisfies the subterm property, but $(\mathcal{F}, \mathcal{R})$ is not terminating.

Ohlebusch [10] and others call a TRS $(\mathcal{F}, \mathcal{R})$ simply terminating if it is compatible with a *well-founded* simplification order on $\mathcal{T}(\mathcal{F}, \mathcal{V})$. The basic motivation for simple termination is that termination can be concluded without explicitly testing for well-foundedness. This motivation is not met anymore if the requirement of well-foundedness is included in the definition of simplification order. We propose instead to call a TRS that is compatible with a well-founded rewrite order that has the subterm property *pseudo-simply terminating*, and bring the definition of simple termination in accordance with Kruskal's Tree Theorem.

DEFINITION 2. A *simplification order* is a rewrite order on $\mathcal{T}(\mathcal{F}, \mathcal{V})$ that contains $\succ_{emb}$ for some PWO $\succ$ on $\mathcal{F}$. A TRS $(\mathcal{F}, \mathcal{R})$ is *simply terminating* if it is compatible with a simplification order on $\mathcal{T}(\mathcal{F}, \mathcal{V})$.

If the signature $\mathcal{F}$ is finite, the above definition of simplification order coincides with the one in the preceding section. Using Kruskal's Tree Theorem, it is not difficult to prove that simplification orders are well-founded. Hence we obtain the following result.

THEOREM 3. *Every simply terminating TRS is terminating.* □

The next result provides a useful characterization of simple termination.

THEOREM 4. *A TRS $(\mathcal{F}, \mathcal{R})$ is simply terminating if and only if the TRS $(\mathcal{F}, \mathcal{R} \cup \mathcal{E}mb\,(\mathcal{F}, \succ))$ is terminating for some PWO $\succ$ on $\mathcal{F}$.* $\square$

4. Comparison and Modularity

Let us call a TRS *simplifying* if it is compatible with a rewrite order that has the subterm property. Figure 1 shows the relationship between the classes of simplifying (1), pseudo-simply terminating (2), simply terminating (3), and terminating (4) TRSs. The two dashed areas consist of all TRSs over finite signatures. So for TRSs over finite signature the notions of simplifyingness, pseudo-simple termination, and simple termination coincide. All areas are inhabited.

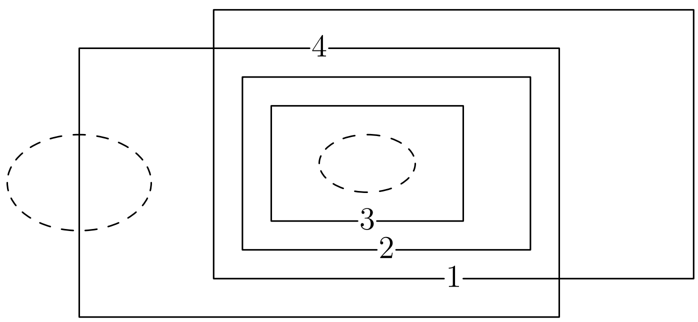

FIGURE 1.

In our paper [9] simple termination is compared with other restricted kinds of termination. There it is also argued that the class of simply terminating TRSs is not "too small". Here we explain why simple termination has a better *modular* behaviour than pseudo-simple termination. A property of TRSs is called *modular* if it is preserved under disjoint union. Toyama [12] showed that termination is not modular. One of the (finite) TRSs in his famous counterexample is not simplifying. Kurihara and Ohuchi [7] showed that simplifyingness is a modular property. Gramlich [4] showed that pseudo-simple termination is a modular property of so-called *finitely branching* TRSs. Ohlebusch [11] showed that the latter result extends to arbitrary systems. The following result is easily derived from this.

THEOREM 5. *Simple termination is a modular property of TRSs.* $\square$

Because of the disjointness requirement, modularity is a rather restricted property. If we allow the sharing of certain function symbols among TRSs, we might hope for more useful results. With every TRS $(\mathcal{F}, \mathcal{R})$ we associate the set $\mathcal{F}_{\mathcal{D}} = \{root(l) \mid l \to r \in \mathcal{R}\}$ of *defined symbols* and the set $\mathcal{F}_{\mathcal{C}} = \mathcal{F} - \mathcal{F}_{\mathcal{D}}$ of *constructors*. We say that two TRSs $(\mathcal{F}, \mathcal{R})$ and $(\mathcal{F}', \mathcal{R}')$ *share constructors* if $\mathcal{F}_{\mathcal{D}}$, $\mathcal{F}'_{\mathcal{D}}$, and $\mathcal{F}_{\mathcal{C}} \cup \mathcal{F}'_{\mathcal{C}}$ are pairwise disjoint. A property of TRSs is called *constructor sharing modular* (cs-modular for short) if the union of two TRSs that share constructors inherits the property from the two TRSs. Kurihara and Ohuchi [8] were the first to study cs-modularity. They showed that simplifyingness is cs-modular. Gramlich [4] showed that pseudo-simple termination is cs-modular for finitely branching TRSs. Surprisingly, the latter result does not extend to arbitrary TRSs, as shown by the following example of Ohlebusch [11]:

$$\begin{aligned}\mathcal{R}_1 &= \{f_i(c_i, x) \to f_{i+1}(x, x) \mid i \in \mathbb{N}\},\\ \mathcal{R}_2 &= \{a \to c_i \mid i \in \mathbb{N}\}.\end{aligned}$$

Both TRSs are pseudo-simply terminating. They share constructors $\{c_i \mid i \in \mathbb{N}\}$, but their union is not (pseudo-simply) terminating. With help of Theorem 4 it is not difficult to show that TRS $\mathcal{R}_1$ is not simply terminating. This brings us to the final result in this paper.

THEOREM 6. *Simple termination is a cs-modular property of TRSs.* □

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This paper is partly based on [9]. A full paper is in preparation.

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