Simple Termination of Rewrite Systems

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Abstract

In this paper we investigate the concept of simple termination. A term rewriting system is called simply terminating if its termination can be proved by means of a simplification order. The basic ingredient of a simplification order is the subterm property, but in the literature two different definitions are given: one based on (strict) partial orders and another one based on preorders (or quasi-orders). We argue that there is no reason to choose the second one, while the first one has certain advantages.

Simplification orders are known to be well-founded orders on terms over a finite signature. This important result no longer holds if we consider infinite signatures. Nevertheless, well-known simplification orders like the recursive path order are also well-founded on terms over infinite signatures, provided the underlying precedence is well-founded. We propose a new definition of simplification order, which coincides with the old one (based on partial orders) in case of finite signatures, but which is also well-founded over infinite signatures and covers orders like the recursive path order. We investigate the properties of the ensuing class of simply terminating systems.

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1 Introduction

One of the main problems in the theory of term rewriting is the detection of termination: for a fixed system of rewrite rules, determine whether there exist infinite reduction sequences or not. Huet and Lankford [18] showed that this problem is undecidable in general. Dauchet [2] showed that termination is undecidable even for one-rule systems. However, there are several methods for proving termination that are successful for many special cases. A well-known method for proving termination is the recursive path order (Dershowitz [4]). The basic idea of such a path order is that, starting from a given order (the so-called *precedence*) on the operation symbols, in a recursive way a well-founded order on terms is defined. If every reduction step in a term rewriting system corresponds to a decrease according to this order, one can conclude that the system is terminating. If the order is closed under contexts and substitutions then the decrease only has to be checked for the rewrite rules instead of all reduction steps. The bottleneck of this kind of method is how to prove that a relation defined recursively on terms is indeed a well-founded order. Proving irreflexivity and transitivity often turns out to be feasible, using some induction and case analysis. However, when stating an arbitrary recursive definition of such an order, well-foundedness is very hard to prove directly. Fortunately, the powerful *Tree Theorem* of Kruskal implies that if the order satisfies some simplification property, well-foundedness is obtained for free. An order satisfying this property is called a *simplification order*. This notion of simplification comprises two ingredients:

- a term decreases by removing parts of it, and
- a term decreases by replacing an operation symbol with a smaller (according to the precedence) one.

If the signature is infinite, both of these ingredients are essential for the applicability of Kruskal's Tree Theorem. It is amazing, however, that in the term rewriting literature the notion of simplification order is motivated by the applicability of Kruskal's Tree Theorem but only covers the first ingredient. For infinite signatures one easily defines non-well-founded orders that are simplification orders according to that definition. Therefore, the usual definition of simplification order is only helpful for proving termination of systems over finite signatures. Nevertheless, it is well-known that simplification orders like the recursive path order are also well-founded on terms over infinite signatures (provided the precedence on the signature is well-founded).

In this paper we propose a definition of a simplification order that matches exactly the requirements of Kruskal's Tree Theorem, since that is the basic motivation for the notion of simplification order. According to this new definition all simplification orders are well-founded, both over finite and infinite signatures. For finite signatures the new and the old notion of simplification order coincide. A term rewriting system is called *simply terminating* if there is a simplification order that orients the rewrite rules from left to right. It is straightforward from the definition that every recursive path order over a well-founded precedence can be extended to a simplification order, and hence is well-founded. Even if one is only interested in finite term rewriting systems this is of interest: *semantic labelling* (Zantema [45]) often succeeds in proving termination of a finite but "difficult" (non-simply terminating) system by transforming it into an infinite system over an infinite signature to which the recursive path order readily applies.

In the literature simplification orders are sometimes based on preorders (or quasi-orders) instead of (strict) partial orders. A main result of this paper is that there are no compelling reasons for doing so. We prove (constructively) that every term rewriting system that can be shown to be terminating by means of a simplification order based on preorders, can be shown to be terminating by means of a simplification order (based on partial orders). Since basing the notion of simplification order on preorders is more susceptible to mistakes and results in stronger proof obligations, simplification orders should be based on partial orders. (As explained in Section 4 these remarks already apply to finite signatures.) As a consequence, we prefer the partial order variant of *well-quasi-orders*, the so-called *partial well-orders*, in case of infinite signatures. By choosing partial well-orders instead of well-quasi-orders a great part of the theory is not affected, but another part becomes cleaner. For instance, in Section 5 we prove a useful result stating that a term rewriting system is simply terminating if and only if the union of the system and a particular system that captures simplification is terminating. Based on well-quasi-orders a similar result does not hold.

The remainder of the paper is organized as follows. In a preliminary section we review the basic notions of term rewriting. In Section 3 we study the subterm property—the basic ingredient of simplification orders—and the related embedding notion. Section 4 is concerned with term rewriting systems over finite signatures. In Section 5 we consider arbitrary signatures: we present our definition of simplification order and state some basic properties of the ensuing class of simply terminating term rewriting systems. In Section 6 we compare our definition of simple termination with previous proposals and other restricted notions of termination, among which the useful notion of *total* termination (see [11, 43]). For finite signature one easily shows that total termination implies simple termination. We show that for infinite signatures this does not hold any more: we construct an infinite term rewriting system whose terminating can be proved by a polynomial interpretation, but which is not simply terminating. The recursive path order and the Knuth-Bendix order, two well-known techniques for proving termination, are addressed in Section 7. We pay special attention to their behaviour over infinite signatures. In Section 8 we investigate the behaviour of simple termination under combinations of term rewriting systems. We show that our notion of simple termination is preserved under constructor sharing combinations. This is not true for the earlier notion of simple termination (Ohlebusch [36]). In two appendices we present some useful facts about partial well-orders and, for completeness sake, a proof of Kruskal's Tree Theorem.

2 Preliminaries

In order to fix our notations and terminology, we start with a very brief introduction to term rewriting. Term rewriting is surveyed in Dershowitz and Jouannaud [7] and Klop [21].

A *signature* is a set $\mathcal{F}$ of *function symbols*. Associated with every $f \in \mathcal{F}$ is a natural number denoting its arity. Function symbols of arity 0 are called *constants*. Let $\mathcal{T}(\mathcal{F}, \mathcal{V})$ be the set of all terms built from $\mathcal{F}$ and a countably infinite set $\mathcal{V}$ of *variables*, disjoint from $\mathcal{F}$. The set of variables occurring in a term t is denoted by $\mathcal{V}\mathrm{ar}(\mathrm{t})$. A term t is called *ground* if $\mathcal{V}\mathrm{ar}(\mathrm{t}) = \varnothing$. The set of all ground terms is denoted by $\mathcal{T}(\mathcal{F})$. The *root symbol* of a term t is defined as follows: $\mathrm{root}(\mathrm{t}) = \mathrm{t}$ if t is a variable and $\mathrm{root}(\mathrm{t}) = \mathrm{f}$ if $t = f(t_1, \ldots, t_n)$.

We introduce a fresh constant symbol $\Box$, named *hole*. A *context* C is a term in $\mathcal{T}(\mathcal{F} \cup \{\Box\}, \mathcal{V})$ containing precisely one hole. The designation *term* is restricted to members of $\mathcal{T}(\mathcal{F}, \mathcal{V})$. If C is a context and t a term then $C[t]$ denotes the result of replacing the hole in C by t. A term s is a *subterm* of a term t, denoted by $s \trianglelefteq t$, if there exists a context C such that $t = C[s]$. A subterm s of t is *proper* if $s \neq t$. The proper subterm relation is denoted by $\triangleleft$. We assume familiarity with the *position* formalism for describing subterm occurrences. A *substitution* is a map σ from $\mathcal{V}$ to $\mathcal{T}(\mathcal{F}, \mathcal{V})$ with the property that the set $\{x \in \mathcal{V} \mid \sigma(x) \neq x\}$ is finite. If σ is a substitution and t a term then $t\sigma$ denotes the result of applying σ to t. We call $t\sigma$ an *instance* of t. A binary relation R on terms is *closed under contexts* if $C[s]\ R\ C[t]$ whenever $s\ R\ t$, for all contexts C. A binary relation R on terms is *closed under substitutions* if $s\sigma\ R\ t\sigma$ whenever $s\ R\ t$, for all substitutions σ. A *rewrite relation* is a binary relation on terms that is closed under contexts and substitutions.

A *rewrite rule* is a pair (l, r) of terms such that the left-hand side l is not a variable and variables which occur in the right-hand side r occur also in l, i.e., $\mathcal{V}\mathrm{ar}(\mathrm{r}) \subseteq \mathcal{V}\mathrm{ar}(\mathrm{l})$. Since we are interested in (simple) termination in this paper, these two restrictions rule out only trivial cases. Rewrite rules (l, r) will henceforth be written as $l \to r$.

A *term rewriting system* (TRS for short) is a pair $(\mathcal{F}, \mathcal{R})$ consisting of a signature $\mathcal{F}$ and a set $\mathcal{R}$ of rewrite rules between terms in $\mathcal{T}(\mathcal{F}, \mathcal{V})$. We often present a TRS as a set of rewrite rules, without making explicit its signature, assuming that the signature consists of the function symbols occurring in the rewrite rules. The smallest rewrite relation on $\mathcal{T}(\mathcal{F}, \mathcal{V})$ that contains $\mathcal{R}$ is denoted by $\to_{\mathcal{R}}$. So $s \to_{\mathcal{R}} t$ if there exists a rewrite rule $l \to r$ in $\mathcal{R}$, a substitution σ, and a context C such that $s = C[l\sigma]$ and $t = C[r\sigma]$. The subterm $l\sigma$ of s is called a *redex* and we say that s rewrites to t by *contracting* redex $l\sigma$. We call $s \to_{\mathcal{R}} t$ a *rewrite* or *reduction step*. The transitive closure of $\to_{\mathcal{R}}$ is denoted by $\to_{\mathcal{R}}^+$ and $\to_{\mathcal{R}}^*$ denotes the transitive and reflexive closure of $\to_{\mathcal{R}}$. If $s \to_{\mathcal{R}}^* t$ we say that s *reduces* to t. The converse of $\to_{\mathcal{R}}^*$ is denoted by $\leftarrow_{\mathcal{R}}^*$. A TRS $(\mathcal{F}, \mathcal{R})$ is called *terminating* if there are no infinite reduction sequences $t_1 \to_{\mathcal{R}} t_2 \to_{\mathcal{R}} t_3 \to_{\mathcal{R}} \cdots$ of terms in $\mathcal{T}(\mathcal{F}, \mathcal{V})$. A TRS $(\mathcal{F}, \mathcal{R})$ is called *cyclic* if $t \to_{\mathcal{R}}^+ t$ for some term $t \in \mathcal{T}(\mathcal{F}, \mathcal{V})$. Clearly every terminating TRS is acyclic.

A (strict) *partial order* $\succ$ is a transitive and irreflexive relation. The reflexive closure of $\succ$ is denoted by $\succcurlyeq$. The converse of $\succcurlyeq$ is denoted by $\preccurlyeq$. A partial order $\succ$ on a set A is *well-founded* if there are no infinite descending sequences $a_1 \succ a_2 \succ \cdots$ of elements of A. A partial order $\succ$ on A is *total* if for all different elements $a, b \in A$ either $a \succ b$ or $b \succ a$. A *preorder* (or *quasi-order*) $\succsim$ is a transitive and reflexive relation. The converse of $\succsim$ is denoted by $\precsim$. The *strict part* of a preorder $\succsim$ is the partial order $\succ$ defined as $\succsim \setminus \precsim$. Every preorder $\succsim$ induces an equivalence relation $\sim$ defined as $\succsim \cap \precsim$. It is easy to see that $\succ = \succsim \setminus \sim$. A preorder is said to be well-founded if its strict part is a well-founded partial order.

A rewrite relation that is also a partial order is called a *rewrite order*. A well-founded rewrite order is called a *reduction order*. We say that a TRS $(\mathcal{F}, \mathcal{R})$ and a partial order $\succ$ on $\mathcal{T}(\mathcal{F}, \mathcal{V})$ are *compatible* if $\mathcal{R}$ is contained in $\succ$, i.e., $l \succ r$ for every rewrite rule $l \to r$ of $\mathcal{R}$. It is easy to show that a TRS is terminating if and only if it is compatible with a reduction order.

3 Subterm Property and Embedding

Definition 3.1 We say that a binary relation R on terms has the *subterm property* if $C[t]\ R\ t$ for all contexts $C \neq \Box$ and terms t.

The subterm property of a relation R can be expressed more concisely by the inclusion $\rhd \subseteq R$. The task of showing that a given *transitive* relation R has the subterm property amounts to verifying $f(t_1, \ldots, t_n)\ R\ t_i$ for all function symbols f of arity $n \geqslant 1$, terms $t_1, \ldots, t_n$, and $i \in \{1, \ldots, n\}$. This observation will be used freely in the sequel.

Definition 3.2 Let $\mathcal{F}$ be a signature. The TRS $\mathcal{E}\mathrm{mb}(\mathcal{F})$ consists of all rewrite rules

$$f(x_1, \ldots, x_n) \quad \to \quad x_i$$

with $f \in \mathcal{F}$ a function symbol of arity $n \geqslant 1$ and $i \in \{1, \ldots, n\}$. Here $x_1, \ldots, x_n$ are pairwise different variables. We abbreviate $\to^+_{\mathcal{E}\mathrm{mb}(\mathcal{F})}$ to $\rhd_{\mathrm{emb}}$ and $\leftarrow^*_{\mathcal{E}\mathrm{mb}(\mathcal{F})}$ to $\unlhd_{\mathrm{emb}}$. The latter relation is called *embedding*.

The following easy result relates the subterm property to embedding.

Lemma 3.3 *A rewrite order $\succ$ on $\mathcal{T}(\mathcal{F}, \mathcal{V})$ has the subterm property if and only if $\rhd_{\mathrm{emb}} \subseteq \succ$.*

Proof The "if" direction is trivial because $\succ$ inherits the subterm property from $\rhd_{\mathrm{emb}}$. For the "only if" direction we reason as follows. Since x_i is a proper subterm of $f(x_1, \ldots, x_n)$ the TRS $\mathcal{E}\mathrm{mb}(\mathcal{F})$ is compatible with $\succ$. Because $\succ$ is a transitive rewrite relation we obtain $\rhd_{\mathrm{emb}} = \to^+_{\mathcal{E}\mathrm{mb}(\mathcal{F})} \subseteq \succ$. $\Box$

It follows that $\rhd_{\mathrm{emb}}$ is the *smallest* rewrite order with the subterm property. Note that $\rhd$ is not a rewrite order as it lacks closure under contexts.

Embedding is a special case of *homeomorphic* embedding.

Definition 3.4 Let $\succ$ be a partial order on a signature $\mathcal{F}$. The TRS $\mathcal{E}\mathrm{mb}(\mathcal{F},\succ)$ consists of all rewrite rules of $\mathcal{E}\mathrm{mb}(\mathcal{F})$ together with all rewrite rules

$$f(x_1,\ldots,x_n) \quad\rightarrow\quad g(x_{i_1},\ldots,x_{i_m})$$

with f an n-ary function symbol in $\mathcal{F}$, g an m-ary function symbol in $\mathcal{F}$, $n \geqslant m \geqslant 0$, $f \succ g$, and $1 \leqslant i_1 < \cdots < i_m \leqslant n$ whenever $m \geqslant 1$. Here $x_1,\ldots,x_n$ are pairwise different variables. We abbreviate $\rightarrow^+_{\mathcal{E}\mathrm{mb}(\mathcal{F},\succ)}$ to $\succ_{\mathrm{emb}}$ and $\leftarrow^*_{\mathcal{E}\mathrm{mb}(\mathcal{F},\succ)}$ to $\preccurlyeq_{\mathrm{emb}}$. The latter relation is called *homeomorphic embedding*. We denote $\mathcal{E}\mathrm{mb}(\mathcal{F},\succ) \setminus \mathcal{E}\mathrm{mb}(\mathcal{F})$ by $\mathcal{E}\mathrm{mb}^*(\mathcal{F},\succ)$.

Since $\mathcal{E}\mathrm{mb}(\mathcal{F},\varnothing) = \mathcal{E}\mathrm{mb}(\mathcal{F})$, homeomorphic embedding generalizes embedding. Consider for instance the signature $\mathcal{F}$ consisting of constants a and b, a unary function symbol g, and binary functions symbols f and h. Define the partial order $\succ$ on $\mathcal{F}$ by $a \succ b \succ f \succ g \succ h$. In the TRS

$$\mathcal{E}\mathrm{mb}(\mathcal{F},\succ) = \mathcal{E}\mathrm{mb}(\mathcal{F}) \cup \left\{ \begin{array}{rcl} a & \rightarrow & b \\ f(x,y) & \rightarrow & g(x) \\ f(x,y) & \rightarrow & g(y) \\ f(x,y) & \rightarrow & h(x,y) \end{array} \right\}$$

we have the reduction sequence $f(h(a,b),g(a)) \rightarrow f(a,g(a)) \rightarrow f(a,a) \rightarrow f(a,b)$, hence the term $f(a,b)$ is homeomorphically embedded in $f(h(a,b),g(a))$. Since there is no reduction sequence in the TRS $\mathcal{E}\mathrm{mb}(\mathcal{F})$ from $f(h(a,b),g(a))$ to $f(a,b)$, the term $f(a,b)$ is not embedded in $f(h(a,b),g(a))$.

4 Simple Termination — Finite Signatures

Throughout this section we are dealing with *finite* signatures only.

Definition 4.1 A *simplification order* is a rewrite order with the subterm property. A TRS $(\mathcal{F},\mathcal{R})$ is *simply terminating* if it is compatible with a simplification order on $\mathcal{T}(\mathcal{F},\mathcal{V})$.

Since we are only interested in signatures consisting of function symbols with fixed arity, we have no need for the *deletion property* (cf. [4]). It should also be noted that many authors (e.g. [3, 4, 5, 14, 19, 39]) do not require that simplification orders are closed under substitutions. Since we don't really want to check whether a simplification order orients *all instances* of rewrite rules from left to right in order to conclude termination, and concrete simplification orders like the recursive path order are closed under substitutions, closure under substitutions should be part of the definition. Moreover, it is easy to show that the class of simply terminating TRSs is not affected by imposing closure under substitutions.

Dershowitz [3, 4] showed that every simply terminating TRS is terminating. The proof is based on the beautiful Tree Theorem of Kruskal [26].

Definition 4.2 An infinite sequence t_1, t_2, t_3, $\ldots$ of terms in $\mathcal{T}(\mathcal{F},\mathcal{V})$ is *self-embedding* if there exist $1 \leqslant i < j$ such that $t_i \trianglelefteq_{\mathrm{emb}} t_j$.

Theorem 4.3 (Kruskal's Tree Theorem—Finite Version) *Every infinite sequence of ground terms is self-embedding.* □

We refrain from proving Theorem 4.3 since it is a special case of the general version of Kruskal's Tree Theorem, which is presented in the next section and proved in Appendix B.

Theorem 4.4 *Simplification orders are well-founded.* □

Observe that simplification orders are well-founded on arbitrary—not necessarily ground—terms over a finite signature. In the next section we generalize this result to terms over arbitrary signatures.

Corollary 4.5 *Every simply terminating TRS is terminating.* □

The following well-known result is especially useful for showing that a given TRS is *not* simply terminating. For instance, the terminating one-rule TRS $\mathcal{R} = \{f(f(x)) \rightarrow f(g(f(x)))\}$ is not simply terminating because $\mathcal{R} \cup \{f(x) \rightarrow x, g(x) \rightarrow x\}$ admits a cycle: $f(f(x)) \rightarrow f(g(f(x))) \rightarrow f(f(x))$.

Lemma 4.6 *The following statements are equivalent.*
(1) *The TRS* $(\mathcal{F}, \mathcal{R})$ *is simply terminating.*
(2) *The TRS* $(\mathcal{F}, \mathcal{R} \cup \mathcal{E}\text{mb}(\mathcal{F}))$ *is terminating.*
(3) *The TRS* $(\mathcal{F}, \mathcal{R} \cup \mathcal{E}\text{mb}(\mathcal{F}))$ *is acyclic.*

Proof

(1) ⇒ (2) Let $(\mathcal{F}, \mathcal{R})$ be compatible with the simplification order $\succ$ on $\mathcal{T}(\mathcal{F}, \mathcal{V})$. From Lemma 3.3 we learn that $\rhd_{\text{emb}} \subseteq \succ$ and hence $\succ$ is compatible with the TRS $\mathcal{E}\text{mb}(\mathcal{F})$. Therefore $(\mathcal{F}, \mathcal{R} \cup \mathcal{E}\text{mb}(\mathcal{F}))$ is a simply terminating TRS. Corollary 4.5 yields its termination.

(2) ⇒ (3) Obvious.

(3) ⇒ (1) Let $\succ$ be the transitive closure of the rewrite relation of the TRS $(\mathcal{F}, \mathcal{R} \cup \mathcal{E}\text{mb}(\mathcal{F}))$. Because $(\mathcal{F}, \mathcal{R} \cup \mathcal{E}\text{mb}(\mathcal{F}))$ is acyclic, $\succ$ is irreflexive and hence a rewrite order. Since $\rhd_{\text{emb}} \subseteq \succ$, $\succ$ is a simplification order. Since the TRS $(\mathcal{F}, \mathcal{R})$ is compatible with $\succ$, it is simply terminating.

□

In the term rewriting literature the notion of simplification order is sometimes based on preorders instead of partial orders. Dershowitz [4] obtained the following result.

Theorem 4.7 *Let* $(\mathcal{F}, \mathcal{R})$ *be a TRS. Let* $\succsim$ *be a preorder on* $\mathcal{T}(\mathcal{F}, \mathcal{V})$ *which is closed under contexts and has the subterm property. If* $l\sigma \succ r\sigma$ *for every rewrite rule* $l \rightarrow r \in \mathcal{R}$ *and substitution* σ *then* $(\mathcal{F}, \mathcal{R})$ *is terminating.* □

A preorder that is closed under contexts and has the subterm property is sometimes called a *quasi-simplification order*. Observe that we require $l\sigma \succ r\sigma$ *for all substitutions* σ in Theorem 4.7. It should be stressed that this requirement cannot be weakened to the compatibility of $(\mathcal{F}, \mathcal{R})$ and $\succ$ (i.e., $l \succ r$ for all rules $l \to r \in \mathcal{R}$) if we additionally require that $\succsim$ is closed under substitutions, as is incorrectly done in Dershowitz and Jouannaud [7]. For instance, the relation $\to_{\mathcal{R}}^*$ associated with the TRS

$$\mathcal{R} = \begin{cases} f(g(x)) & \to & f(f(x)) \\ f(g(x)) & \to & g(g(x)) \\ f(x) & \to & x \\ g(x) & \to & x \end{cases}$$

is a rewrite relation with the subterm property (because $\mathcal{E}\mathrm{mb}(\{\mathrm{f}, \mathrm{g}\}) \subseteq \mathcal{R}$). Moreover, $l \to_{\mathcal{R}}^* r$ but not $r \to_{\mathcal{R}}^* l$, for every rewrite rule $l \to r \in \mathcal{R}$. So $\mathcal{R}$ is included in the strict part of $\to_{\mathcal{R}}^*$. Nevertheless, $\mathcal{R}$ is not terminating:

$$f(g(g(x))) \to_{\mathcal{R}} f(f(g(x)) \to_{\mathcal{R}} f(g(g(x))) \to_{\mathcal{R}} \cdots.$$

The point is that the strict part of $\to_{\mathcal{R}}^*$ is not closed under substitutions. Hence to conclude termination from compatibility with $\succsim$ it is essential that both $\succ$ and $\succsim$ are closed under substitutions. A simpler TRS illustrating the same point, due to Enno Ohlebusch (personal communication), is $\{f(x) \to f(a),\ f(x) \to x\}$.

Dershowitz [4] writes that Theorem 4.7 generalizes Theorem 4.5. We have the following result.

Theorem 4.8 *A TRS* $(\mathcal{F}, \mathcal{R})$ *is simply terminating* if and only if *there exists a preorder* $\succsim$ *on* $\mathcal{T}(\mathcal{F}, \mathcal{V})$ *that is closed under contexts, has the subterm property, and satisfies* $l\sigma \succ r\sigma$ *for every rewrite rule* $l \to r \in \mathcal{R}$ *and substitution* σ. □

The proof is given in Section 5, where the above theorem is generalized to TRSs over arbitrary, not necessarily finite, signatures.

So every TRS whose termination can be shown by means of Theorem 4.7 is simply terminating, i.e., its termination can be shown by a simplification order. Since it is easier to check $l \succ r$ for finitely many rewrite rules $l \to r$ than $l\sigma \succsim r\sigma$ but not $r\sigma \succsim l\sigma$ for finitely many rewrite rules $l \to r$ and infinitely many substitutions σ, there is no reason to base the definition of simplification order on preorders.

5 Simple Termination — Infinite Signatures

Kurihara and Ohuchi [27] were the first to use the terminology simple termination. They call a TRS $(\mathcal{F}, \mathcal{R})$ simply terminating if it is compatible with a rewrite order on $\mathcal{T}(\mathcal{F}, \mathcal{V})$ that has the subterm property. Since compatibility with a rewrite order that has the subterm property doesn't ensure the termination of TRSs over infinite signatures, this definition of simple termination is clearly not the right one. Consider for instance the TRS $(\mathcal{F}, \mathcal{R})$ consisting of infinitely many constants a_i and rewrite rules $a_i \to a_{i+1}$ for all $i \in \mathbb{N}$. The rewrite order $\to_{\mathcal{R}}^+$ vacuously satisfies the subterm property, but $(\mathcal{F}, \mathcal{R})$ is not terminating.

Ohlebusch [35] and others call a TRS $(\mathcal{F}, \mathcal{R})$ simply terminating if it is compatible with a *well-founded* rewrite order on $\mathcal{T}(\mathcal{F}, \mathcal{V})$ that has the subterm property. The basic motivation for simple termination is that termination can be concluded without explicitly testing for well-foundedness. This motivation is not met anymore if the requirement of well-foundedness is included in the definition.

We propose instead to bring the definition of simple termination in accordance with (the general version of) Kruskal's Tree Theorem.

Theorem 5.1 (Kruskal's Tree Theorem—General Version) *If $\succ$ is a PWO on a signature $\mathcal{F}$ then $\succ_{\text{emb}}$ is a PWO on $\mathcal{T}(\mathcal{F})$.* $\square$

A partial order $\succ$ on a set A is called a *partial well-order* (PWO) if for every infinite sequence a_1, a_2, a_3, ... of elements of A there exist indices $1 \leqslant i < j$ such that $a_i \preccurlyeq a_j$. This is equivalent to stating that every partial order on A that extends $\succ$ (including $\succ$ itself) is well-founded. In Appendix A several other equivalent formulations of PWO are given. Using the terminology of PWOs, Theorem 4.3 states that $\rhd_{\text{emb}}$ is a PWO on $\mathcal{T}(\mathcal{F})$ whenever $\mathcal{F}$ is finite. Appendix B contains a proof of Theorem 5.1.

Definition 5.2 A *simplification order* is a rewrite order on $\mathcal{T}(\mathcal{F},\mathcal{V})$ that contains $\succ_{\text{emb}}$ for some PWO $\succ$ on $\mathcal{F}$. A TRS $(\mathcal{F},\mathcal{R})$ is *simply terminating* if it is compatible with a simplification order on $\mathcal{T}(\mathcal{F},\mathcal{V})$.

Because the empty relation $\varnothing$ is a PWO on any finite $\mathcal{F}$ and $\varnothing_{\text{emb}} = \rhd_{\text{emb}}$, this definition coincides with the one in Section 4 in case of finite signatures.

Theorem 5.3 *Simplification orders are well-founded.*

Proof Let $\sqsupset$ be a simplification order on $\mathcal{T}(\mathcal{F},\mathcal{V})$. So there exists a PWO $\succ$ on $\mathcal{F}$ such that $\succ_{\text{emb}} \subseteq \sqsupset$. First we show that $\mathcal{V}\text{ar}(\text{t}) \subseteq \mathcal{V}\text{ar}(\text{s})$ whenever $s \sqsupset t$. Suppose to the contrary that there exists a variable $x \in \mathcal{V}\text{ar}(\text{t}) \setminus \mathcal{V}\text{ar}(\text{s})$. Define $\sigma = \{x \mapsto s\}$. Closure under substitutions of $\sqsupset$ yields $s = s\sigma \sqsupset t\sigma$. Since $s \unlhd t\sigma$ and thus $s \preccurlyeq_{\text{emb}} t$ we also have $s \sqsubseteq t\sigma$, contradicting the fact that $\sqsupset$ is a partial order. Now consider an infinite sequence $t_1 \sqsupset t_2 \sqsupset t_3 \sqsupset \cdots$ of terms in $\mathcal{T}(\mathcal{F},\mathcal{V})$. Let $\mathcal{V}\text{ar}(\text{t}_1) = \{\text{x}_1,\ldots,\text{x}_\text{n}\}$. According to the above observation we have $\mathcal{V}\text{ar}(\text{t}_\text{i}) \subseteq \{\text{x}_1,\ldots,\text{x}_\text{n}\}$ for all $i \geqslant 1$. Choose fresh constants $c_1,\ldots,c_n$ and define the substitution $\tau = \{x_i \mapsto c_i \mid 1 \leqslant i \leqslant n\}$. The infinite sequence $t_1\tau,\ t_2\tau,\ t_3\tau,\ \ldots$ contains only terms in $\mathcal{T}(\mathcal{F} \cup \{c_1,\ldots,c_n\})$. From Kruskal's Tree Theorem we learn the existence of indices i, j with $1 \leqslant i < j$ such that $t_i\tau \preccurlyeq_{\text{emb}} t_j\tau$. It is not difficult to see that $t_i\tau \preccurlyeq_{\text{emb}} t_j\tau$ is equivalent to $t_i \preccurlyeq_{\text{emb}} t_j$. Therefore $t_i \sqsubseteq t_j$. Since $i < j$ we also have $t_i \sqsupset t_j$. This is impossible. We conclude that $\sqsupset$ is well-founded. $\square$

Corollary 5.4 *Every simply terminating TRS is terminating.* $\square$

The following result extends the very useful Lemma 4.6 to arbitrary TRSs.

Lemma 5.5 *The following statements are equivalent.*

(1) *The TRS $(\mathcal{F},\mathcal{R})$ is simply terminating.*

(2) *The TRS $(\mathcal{F},\mathcal{R} \cup \mathcal{E}\text{mb}(\mathcal{F},\succ))$ is terminating for some PWO $\succ$ on $\mathcal{F}$.*

(3) *The TRS* $(\mathcal{F}, \mathcal{R} \cup \mathcal{E}\text{mb}(\mathcal{F}, \succ))$ *is acyclic for some PWO* $\succ$ *on* $\mathcal{F}$*.*

Proof Essentially the same as the proof of Lemma 4.6. □

In the remainder of this section we generalize Theorem 4.8 (and hence Theorem 4.7) to arbitrary TRSs. Our proof is based on the elegant proof sketch of Theorem 4.7 given by Plaisted [39]. The proof employs *multiset extensions* of preorders. A *multiset* is a collection in which elements are allowed to occur more than once. If A is a set then the set of all finite multisets over A is denoted by $\mathcal{M}(A)$. The *multiset extension* of a partial order $\succ$ on A is the partial order $\succ_{\text{mul}}$ on $\mathcal{M}(A)$ defined as follows: $M_1 \succ_{\text{mul}} M_2$ if $M_2 = (M_1 - X) \uplus Y$ for some multisets $X, Y \in \mathcal{M}(A)$ that satisfy $\varnothing \neq X \subseteq M_1$ and for all $y \in Y$ there exists an $x \in X$ such that $x \succ y$. Using Higman's Lemma, it is quite easy to show that multiset extension preserves PWO. From this we infer that the multiset extension of a well-founded partial order is well-founded, using the well-known facts that (1) every well-founded partial order can be extended to a total well-founded order (in particular a PWO) and (2) multiset extension is monotonic (i.e., if $\succ \subseteq \sqsupset$ then $\succ_{\text{mul}} \subseteq \sqsupset_{\text{mul}}$). Using König's Lemma, Dershowitz and Manna [8] gave a direct proof that multiset extension preserves well-founded partial orders.

Definition 5.6 Let $\succsim$ be a preorder on a set A. For every $a \in A$, let $[a]$ denote the equivalence class with respect to the equivalence relation $\sim$ containing a. Let $A/{\sim} = \{[a] \mid a \in A\}$ be the set of all equivalence classes of A. The preorder $\succsim$ on A induces a partial order $\succ$ on $A/{\sim}$ as follows: $[a] \succ [b]$ if and only if $a \succ b$. (The latter $\succ$ denotes the strict part of the preorder $\succsim$.) For every multiset $M \in \mathcal{M}(A)$, let $[M] \in \mathcal{M}(A/{\sim})$ denote the multiset obtained from M by replacing every element a by $[a]$. We now define the *multiset extension* $\succsim_{\text{mul}}$ of the preorder $\succsim$ as follows: $M_1 \succsim_{\text{mul}} M_2$ if and only if $[M_1] \succ_{\text{mul}}^{=} [M_2]$ where $\succ_{\text{mul}}^{=}$ denotes the reflexive closure of the multiset extension of the partial order $\succ$ on $A/{\sim}$.

It is easy to show that $\succsim_{\text{mul}}$ is a preorder on $\mathcal{M}(A)$. The associated equivalence relation $\sim_{\text{mul}} = \succsim_{\text{mul}} \cap \precsim_{\text{mul}}$ can be characterized in the following simple way: $M_1 \sim_{\text{mul}} M_2$ if and only if $[M_1] = [M_2]$. Likewise, its strict part $\gnsim_{\text{mul}} = \succsim_{\text{mul}} \setminus \precsim_{\text{mul}} = \succsim_{\text{mul}} \setminus \sim_{\text{mul}}$ has the following simple characterization: $M_1 \gnsim_{\text{mul}} M_2$ if and only if $[M_1] \succ_{\text{mul}} [M_2]$. Observe that we denote the strict part of $\succsim_{\text{mul}}$ by $\gnsim_{\text{mul}}$ in order to avoid confusion with the multiset extension $\succ_{\text{mul}}$ of the strict part $\succ$ of $\succsim$, which is a smaller relation.

The above definition of multiset extension of a preorder can be shown to be equivalent to the more operational ones in Dershowitz [5] and Gallier [14], but since we define the multiset extension of a preorder in terms of the well-known multiset extension of a partial order, we get all desired properties basically for free. In particular, using the fact that multiset extension preserves well-founded partial orders, it is very easy to show that the multiset extension of a well-founded preorder is well-founded.

Definition 5.7 If $t \in \mathcal{T}(\mathcal{F}, \mathcal{V})$ then $S(t) \in \mathcal{M}(\mathcal{T}(\mathcal{F}, \mathcal{V}))$ denotes the finite multiset of all subterm occurrences in t and $F(t) \in \mathcal{M}(\mathcal{F})$ denotes the finite multiset of all function symbol occurrences in t. Formally,

$$S(t) = \begin{cases} \{t\} & \text{if } t \text{ is a variable,} \\ \{t\} \uplus \biguplus_{i=1}^{n} S(t_i) & \text{if } t = f(t_1, \ldots, t_n), \end{cases}$$

$$F(t) = \begin{cases} \varnothing & \text{if } t \text{ is a variable,} \\ \{f\} \uplus \biguplus_{i=1}^{n} F(t_i) & \text{if } t = f(t_1, \ldots, t_n). \end{cases}$$

Lemma 5.8 *Let $\succsim$ be a preorder on $\mathcal{T}(\mathcal{F}, \mathcal{V})$ with the subterm property. If $s \succ t$ then $S(s) \succnsim_{\text{mul}} S(t)$.*

Proof We show that $s \succ t'$ for all $t' \in S(t)$. This implies $\{s\} \succnsim_{\text{mul}} S(t)$ and hence also $S(s) \succnsim_{\text{mul}} S(t)$. If $t' = t$ then $s \succ t'$ by assumption. Otherwise t' is a proper subterm of t and hence $t \succsim t'$ by the subterm property. Combining this with $s \succ t$ yields $s \succ t'$. □

Lemma 5.9 *Let $\succsim$ be a preorder on $\mathcal{T}(\mathcal{F}, \mathcal{V})$ which is closed under contexts. Suppose $s \succsim t$ and let C be an arbitrary context.*
(1) *If $S(s) \succnsim_{\text{mul}} S(t)$ then $S(C[s]) \succnsim_{\text{mul}} S(C[t])$.*
(2) *If $S(s) \succsim_{\text{mul}} S(t)$ then $S(C[s]) \succsim_{\text{mul}} S(C[t])$.*

Proof Let $S_1 = S(C[s]) - S(s)$ and $S_2 = S(C[t]) - S(t)$. For both statements it suffices to prove that $S_1 \succsim_{\text{mul}} S_2$. Let $p \in \mathcal{P}\text{os}(\text{C[s]})$ be the position of the displayed s in $C[s]$. There is a one-to-one correspondence between terms in S_1 (S_2) and positions in $\mathcal{P}\text{os}(\text{C}) \setminus \{\text{p}\}$. Hence it suffices to show that $s' \succsim t'$ where $s' = C[s]_{|q}$ and $t' = C[t]_{|q}$ are the terms in S_1 and S_2 corresponding to position q, for all $q \in \mathcal{P}\text{os}(\text{C}) \setminus \{\text{p}\}$. If p and q are disjoint positions then $s' = t'$. Otherwise $q < p$ and there exists a context C' such that $s' = C'[s]$ and $t' = C'[t]$. By assumption $s \succsim t$. Closure under contexts yields $s' \succsim t'$. We conclude that $S_1 \succsim_{\text{mul}} S_2$. □

After these two preliminary results we are ready for the generalization of Theorem 4.8 to arbitrary TRSs.

Theorem 5.10 *A TRS $(\mathcal{F}, \mathcal{R})$ is simply terminating if and only if there exists a preorder $\succsim$ on $\mathcal{T}(\mathcal{F}, \mathcal{V})$ that is closed under contexts, contains the relation $\sqsupset_{\text{emb}}$ for some PWO $\sqsupset$ on $\mathcal{F}$, and satisfies $l\sigma \succ r\sigma$ for every rewrite rule $l \to r \in \mathcal{R}$ and substitution σ.*

Proof The "only if" direction is obvious since the reflexive closure $\succcurlyeq$ of the simplification order $\succ$ used to prove simple termination is a preorder with the desired properties. For the "if" direction it suffices to show that $(\mathcal{F}, \mathcal{R} \cup \mathcal{E}\text{mb}(\mathcal{F}, \sqsupset))$ is a terminating TRS, according to Lemma 5.5. First we show that either $S(s) \succnsim_{\text{mul}} S(t)$ or $S(s) \sim_{\text{mul}} S(t)$ and $F(s) \sqsupset_{\text{mul}} F(t)$ whenever $s \to t$ is a reduction step in the TRS $(\mathcal{F}, \mathcal{R} \cup \mathcal{E}\text{mb}(\mathcal{F}, \sqsupset))$. So let $s = C[l\sigma]$ and $t = C[r\sigma]$ with $l \to r \in \mathcal{R} \cup \mathcal{E}\text{mb}(\mathcal{F}, \sqsupset)$. We distinguish three cases.

(1) If $l \to r \in \mathcal{R}$ then $l\sigma \succ r\sigma$ by assumption and $S(l\sigma) \gtrsim_{\text{mul}} S(r\sigma)$ according to Lemma 5.8. The first part of Lemma 5.9 yields $S(s) \gtrsim_{\text{mul}} S(t)$.
(2) If $l \to r \in \mathcal{E}\text{mb}(\mathcal{F})$ then $l\sigma = f(t_1, \ldots, t_n)$ and $r\sigma = t_i$ for some $i \in \{1, \ldots, n\}$. Therefore $S(l\sigma) \gtrsim_{\text{mul}} S(r\sigma)$ since $S(t_i)$ is properly contained in $S(f(t_1, \ldots, t_n))$. Clearly $l\sigma \sqsupset_{\text{emb}} r\sigma$ and thus also $l\sigma \gtrsim r\sigma$. An application of the first part of Lemma 5.9 yields $S(s) \gtrsim_{\text{mul}} S(t)$.
(3) If $l \to r \in \mathcal{E}\text{mb}^*(\mathcal{F}, \sqsupset)$ then $l\sigma = f(t_1, \ldots, t_n)$ and $r\sigma = g(t_{i_1}, \ldots, t_{i_m})$ with $f \sqsupset g$, $n \geqslant m \geqslant 0$, and $1 \leqslant i_1 < \cdots < i_m \leqslant n$ whenever $m \geqslant 1$. We have of course $l\sigma \sqsupset_{\text{emb}} r\sigma$ and thus also $l\sigma \gtrsim r\sigma$. Since the multiset $\{t_{i_1}, \ldots, t_{i_m}\}$ is contained in the multiset $\{t_1, \ldots, t_n\}$, we obtain $S(l\sigma) \gtrsim_{\text{mul}} S(r\sigma)$ and $F(l\sigma) \sqsupset_{\text{mul}} F(r\sigma)$. The second part of Lemma 5.9 yields $S(s) \gtrsim_{\text{mul}} S(t)$. We obtain $F(s) \sqsupset_{\text{mul}} F(t)$ from $F(l\sigma) \sqsupset_{\text{mul}} F(r\sigma)$.

Kruskal's Tree Theorem shows that $\sqsupset_{\text{emb}}$ is a PWO on $\mathcal{T}(\mathcal{F})$. Hence $\gtrsim$ is a well-founded preorder on $\mathcal{T}(\mathcal{F})$. Since multiset extension preserves well-founded preorders, $\gtrsim_{\text{mul}}$ is a well-founded preorder on $\mathcal{M}(\mathcal{T}(\mathcal{F}))$. Because $\sqsupset$ is a PWO on the signature $\mathcal{F}$ it is a well-founded partial order. Hence its multiset extension $\sqsupset_{\text{mul}}$ is a well-founded partial order on $\mathcal{M}(\mathcal{F})$. We conclude that $(\mathcal{F}, \mathcal{R} \cup \mathcal{E}\text{mb}(\mathcal{F}, \sqsupset))$ is a terminating TRS. $\square$

6 Comparison

In this section we investigate the relationships between our definition of simple termination, the previous definitions of simple termination [27, 35], and other restricted kinds of termination as introduced in [43]. Let us first rename the previous notions of simple termination.

Definition 6.1 A TRS $(\mathcal{F}, \mathcal{R})$ is *simplifying* if it is compatible with a rewrite order on $\mathcal{T}(\mathcal{F}, \mathcal{V})$ that has the subterm property. We call $(\mathcal{F}, \mathcal{R})$ *pseudo-simply terminating* if it is compatible with a well-founded rewrite order on $\mathcal{T}(\mathcal{F}, \mathcal{V})$ that has the subterm property.

The following well-known lemma (e.g. [27]) states that simplifyingness is equivalent to property (3) in Lemma 4.6.

Lemma 6.2 *A TRS $(\mathcal{F}, \mathcal{R})$ is simplifying if and only if the TRS $(\mathcal{F}, \mathcal{R} \cup \mathcal{E}\text{mb}(\mathcal{F}))$ is acyclic.* $\square$

Pseudo-simple termination is equivalent to property (2) in Lemma 4.6.

Lemma 6.3 *A TRS $(\mathcal{F}, \mathcal{R})$ is pseudo-simply terminating if and only if the TRS $(\mathcal{F}, \mathcal{R} \cup \mathcal{E}\text{mb}(\mathcal{F}))$ is terminating.* $\square$

Figure 1 shows the relationship between the classes of simplifying (S), pseudo-simply terminating (PST), simply terminating (ST), and terminating (T) TRSs. The two dashed areas consist of all TRSs over finite signatures. So for TRSs over finite signatures the notions of simplifyingness, pseudo-simple termination, and simple termination coincide. All areas are inhabited. The

TRS $\mathcal{R}_1 = \{a_i \to a_{i+1} \mid i \in \mathbb{N}\}$ we encountered before. For $\mathcal{R}_2$ we can take $\{f_i(a) \to f_{i+1}(g(a)) \mid i \in \mathbb{N}\}$. This TRS, due to Ohlebusch [35, 36], is simplifying but not pseudo-simply terminating because the extension with the embedding rules $\{f_i(x) \to x \mid i \in \mathbb{N}\} \cup \{g(x) \to x\}$ results in an acyclic TRS that is not terminating. Clearly $\mathcal{R}_2$ is terminating. Note that not every pseudo-simply terminating TRS is simply terminating. Later in this section and in the final section we present examples of such TRSs, among which $\mathcal{R}_3$.

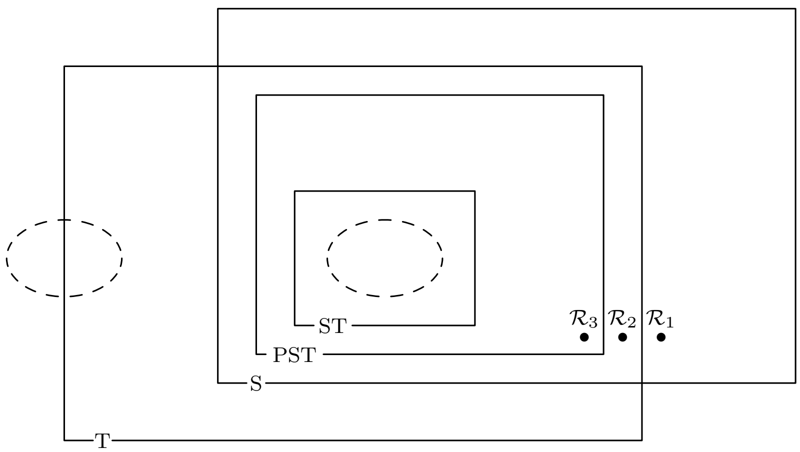

Figure 1: Comparison between different notions of simple termination.

Before we can compare simple termination to other restricted notions of termination we give a semantic characterization of termination. Let $\mathcal{F}$ be a signature. A *monotone* $\mathcal{F}$-algebra $(\mathcal{A}, \succ)$ consists of a non-empty $\mathcal{F}$-algebra $\mathcal{A} = (A, \{f_\mathcal{A}\}_{f \in \mathcal{F}})$ and a partial order $\succ$ on the carrier A of $\mathcal{A}$ such that every algebra operation is strictly monotone in all its coordinates, i.e., if $f \in \mathcal{F}$ has arity n then

$$f_\mathcal{A}(a_1, \ldots, a_i, \ldots, a_n) \;\succ\; f_\mathcal{A}(a_1, \ldots, b, \ldots, a_n)$$

for all $a_1, \ldots, a_n, b \in A$ with $a_i \succ b$ ($i \in \{1, \ldots, n\}$). A monotone $\mathcal{F}$-algebra $(\mathcal{A}, \succ)$ is said to be *well-founded* if $\succ$ is well-founded. Every monotone $\mathcal{F}$-algebra $(\mathcal{A}, \succ)$ induces a rewrite order $\succ_\mathcal{A}$ on $\mathcal{T}(\mathcal{F}, \mathcal{V})$ as follows: $s \succ_\mathcal{A} t$ if $[\alpha](s) \succ [\alpha](t)$ for all assignments $\alpha\colon \mathcal{V} \to A$. Here $[\alpha]$ denotes the homomorphic extension of α, i.e.,

$$[\alpha](t) = \begin{cases} \alpha(t) & \text{if } t \text{ is a variable,} \\ f([\alpha](t_1), \ldots, [\alpha](t_n)) & \text{if } t = f(t_1, \ldots, t_n). \end{cases}$$

If $(\mathcal{A}, \succ)$ is in addition well-founded then $\succ_\mathcal{A}$ is a reduction order. We say that a TRS $(\mathcal{F}, \mathcal{R})$ and a monotone $\mathcal{F}$-algebra $(\mathcal{A}, \succ)$ are compatible if and only if $(\mathcal{F}, \mathcal{R})$ and $\succ_\mathcal{A}$ are compatible. It is straightforward to show that a TRS $(\mathcal{F}, \mathcal{R})$ is terminating if and only if it is compatible with a well-founded monotone $\mathcal{F}$-algebra. Simple termination can be characterized semantically as follows.

Definition 6.4 A monotone $\mathcal{F}$-algebra is called *simple* if it is compatible with the TRS $\mathcal{E}\text{mb}(\mathcal{F}, \succ)$ for some PWO $\succ$ on $\mathcal{F}$.

It is straightforward to show that a TRS $(\mathcal{F}, \mathcal{R})$ is simply terminating if and only if it is compatible with a simple monotone $\mathcal{F}$-algebra.

Definition 6.5 A TRS $(\mathcal{F}, \mathcal{R})$ is called *totally terminating* if it is compatible with a well-founded monotone $\mathcal{F}$-algebra $(\mathcal{A}, \succ)$ such that $\succ$ is a total order on the carrier set of $\mathcal{A}$. If the carrier set of $\mathcal{A}$ is the set of natural numbers and $\succ$ is the standard order then the TRS is called *ω-terminating*. If in addition the operation $f_{\mathcal{A}}$ is a polynomial for every $f \in \mathcal{F}$, the TRS is called *polynomially terminating*.

Total termination has been extensively studied in [11, 12]. In [12] the following non-semantical characterization is proved: a TRS $(\mathcal{F}, \mathcal{R})$ is totally terminating if and only if it admits a compatible total reduction order on ground terms $\mathcal{T}(\mathcal{F})$. Here $\mathcal{F}$ has to be extended by a constant if it does not contain one.

Clearly every polynomially terminating TRS is ω-terminating and every ω-terminating TRS is totally terminating. For both assertions the converse does not hold, as can be shown by the counterexamples $\mathcal{R}_4 = \{f(g(h(x))) \to g(f(h(g(x))))\}$ and $\mathcal{R}_5 = \{f(g(x)) \to g(f(f(x)))\}$ respectively. An easy observation ([43]) shows that every totally terminating TRS is pseudo-simply terminating. Hence every totally terminating TRS over a finite signature is simply terminating. Again the converse does not hold as is shown by the well-known example $\mathcal{R}_6 = \{f(a) \to f(b), g(b) \to g(a)\}$. Somewhat surprisingly, for infinite signatures total termination does no longer imply simple termination: we prove that the non-simply terminating TRS $(\mathcal{F}, \mathcal{R}_7)$ is even polynomially terminating. Here $\mathcal{F}$ is the signature $\{f_i, g_i \mid i \in \mathbb{N}\}$ and $\mathcal{R}_7$ consists of all rewrite rules $f_i(g_j(x)) \to f_j(g_j(x))$ where $i, j \in \mathbb{N}$ with $i < j$. First we prove that $(\mathcal{F}, \mathcal{R}_7)$ is not simply terminating. Let $\succ$ be any PWO on $\mathcal{F}$. Consider the infinite sequence $(f_i)_{i \geqslant 1}$. Since $\succ$ is a PWO we have $f_j \succ f_i$ for some $i < j$. Hence $\mathcal{E}\text{mb}(\mathcal{F}, \succ)$ contains the rewrite rule $f_j(x) \to f_i(x)$, yielding the cycle $f_i(g_j(x)) \to f_j(g_j(x)) \to f_i(g_j(x))$ in the TRS $(\mathcal{F}, \mathcal{R}_7 \cup \mathcal{E}\text{mb}(\mathcal{F}, \succ))$. Lemma 5.5 shows that $(\mathcal{F}, \mathcal{R}_7)$ is not simply terminating. For proving polynomial termination of $(\mathcal{F}, \mathcal{R}_7)$, interpret the function symbols as the following polynomials over $\mathbb{N}$:

$$\begin{aligned} f_{i\mathcal{A}}(x) &= x^3 - ix^2 + i^2 x, \\ g_{i\mathcal{A}}(x) &= x + 2i \end{aligned}$$

for all $i, x \in \mathbb{N}$. Let $i \in \mathbb{N}$. The interpretation $g_{i\mathcal{A}}$ of g_i is clearly strictly monotone in its single argument. The same holds for the interpretation of f_i since

$$\begin{aligned} f_{i\mathcal{A}}(x+1) - f_{i\mathcal{A}}(x) &= (x+1-i)^2 + 2x^2 + x + i \\ &> 0 \end{aligned}$$

for all $x \in \mathbb{N}$. It remains to show that $f_{i\mathcal{A}}(g_{j\mathcal{A}}(x)) > f_{j\mathcal{A}}(g_{j\mathcal{A}}(x))$ for all $i, j, x \in \mathbb{N}$ with $i < j$. Fix i, j, x and let $y = g_{j\mathcal{A}}(x) = x + 2j$. Then

$$\begin{aligned} f_{i\mathcal{A}}(g_{j\mathcal{A}}(x)) - f_{j\mathcal{A}}(g_{j\mathcal{A}}(x)) &= f_{i\mathcal{A}}(y) - f_{j\mathcal{A}}(y) \\ &= y(j-i)(y-j-i) \\ &> 0 \end{aligned}$$

since $j > i$ and $y \geqslant 2j > j + i > 0$. We conclude that $(\mathcal{F}, \mathcal{R}_7)$ is polynomially terminating.

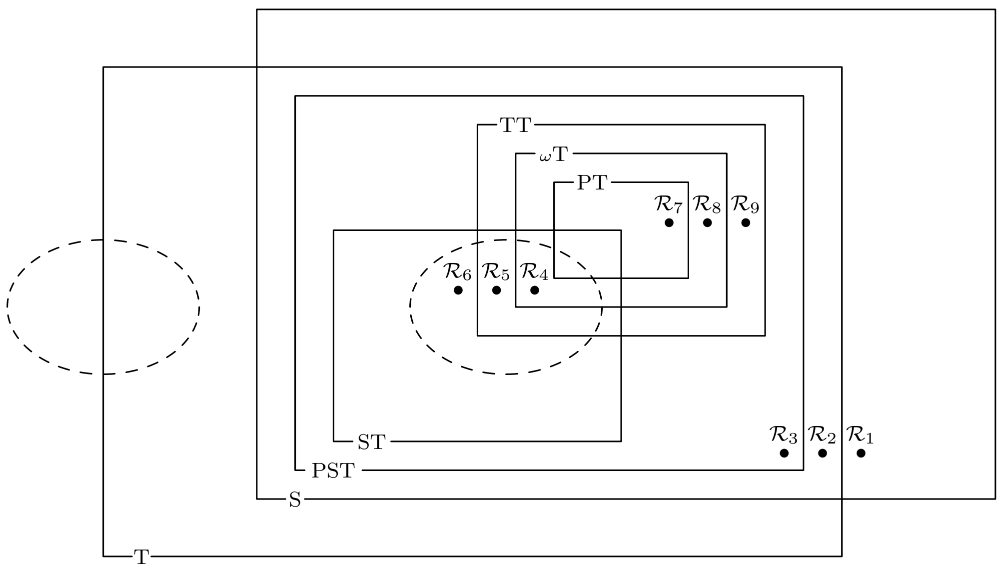

Figure 2: Comparison between different notions of termination.

Incorporating total termination (TT), ω-termination (ωT), and polynomial termination (PT) into Figure 1 gives us Figure 2; for $\mathcal{R}_3$, $\mathcal{R}_8$, and $\mathcal{R}_9$ we simply take the union of $\mathcal{R}_7$ with $\mathcal{R}_6$, $\mathcal{R}_4$, and $\mathcal{R}_5$ respectively. Uwe Waldmann (personal communication) was the first to prove total termination of a non-simply terminating system similar to $\mathcal{R}_4$, using a much more complicated total well-founded order.

We conclude this section with a few remarks on (un)decidability. In the introduction we already mentioned that termination is an undecidable property of one-rule TRSs (Dauchet [2]). Caron [1] showed the undecidability of termination for the class of length-preserving string rewriting systems. Since for length-preserving string rewriting systems termination and simple termination coincide, simple termination is an undecidable property. Middeldorp and Gramlich [30] showed that simple termination is undecidable for one-rule TRSs. Recently, Zantema [44] showed the undecidability of total termination for finite TRSs.

7 Examples of Simplification Orders

In this section we discuss some well-known simplification orders suitable for mechanizing termination proofs: the recursive path order and the Knuth-Bendix order. Several extensions of these orders, in particular of the recursive path order, have been proposed; see Steinbach [41] for an extensive overview. The power of these orders is that they are computable: given a finite TRS it is decidable and practically feasible to check whether an instance of the order exists for which all left-hand sides of the TRS are greater than the corresponding right-hand sides. If such an instance has been found, termination of the TRS is established. For the recursive path order this decision procedure is straightforward from the definition, for the basic version of the Knuth-Bendix order a procedure is described in [9]. Rather than presenting all variations of the orders as in [41] we concentrate on the general behaviour of these two typical orders. In particular we are interested in infinite signatures and in the comparison with the restricted kinds of termination discussed in the previous section.

Both recursive path order and Knuth-Bendix order depend on an order $\succ$ on the signature, the so-called *precedence*. We restrict to the case where this precedence is a (strict) partial order; it can easily be generalized to quasi-orders. Further, a *status* function τ is assumed, mapping every $f \in \mathcal{F}$ to either mul or lex_π for some permutation π on n elements, where n is the arity of f. For a partial order $\succ$ on terms the partial order $\succ^{\tau(f)}$ is defined on sequences of length n: $\tau(f) = \text{mul}$ describes multiset extension and $\tau(f) = \text{lex}_\pi$ describes lexicographic comparison according to the permutation π. Note that any status satisfies the following monotonicity properties:

- if $s \succ t$ then $(\ldots, s, \ldots) \succ^{\tau(f)} (\ldots, t, \ldots)$,
- if $\phi\colon \mathcal{T}(\mathcal{F}, \mathcal{V}) \to \mathcal{T}(\mathcal{F}, \mathcal{V})$ is strictly increasing and $(s_1, \ldots, s_n) \succ^{\tau(f)} (t_1, \ldots, t_n)$ then $(\phi(s_1), \ldots, \phi(s_n)) \succ^{\tau(f)} (\phi(t_1), \ldots, \phi(t_n))$.

7.1 Recursive Path Order

The recursive path order with only multiset status goes back to Dershowitz [3]; its generalization to arbitrary status was first described in Kamin and Lévy [20].

Definition 7.1 For a precedence $\succ$ on $\mathcal{F}$ and a status τ the *recursive path order* $\succ_{\text{rpo}}$ on $\mathcal{T}(\mathcal{F}, \mathcal{V})$ is recursively defined as follows: $s \succ_{\text{rpo}} t$ if and only if $s = f(s_1, \ldots, s_n)$ and

- $s_i = t$ or $s_i \succ_{\text{rpo}} t$ for some $1 \leqslant i \leqslant n$, or
- $t = g(t_1, \ldots, t_m)$, $s \succ_{\text{rpo}} t_i$ for all $1 \leqslant i \leqslant m$, and either
  - $f \succ g$, or
  - $f = g$ and $(s_1, \ldots, s_n) \succ_{\text{rpo}}^{\tau(f)} (t_1, \ldots, t_n)$.

This relation is well-defined, irreflexive, transitive, and closed under substitutions and contexts. In particular well-definedness is not trivial: what is meant by a multiset lifting or a lexicographic lifting of a relation that is still to be defined? A proof of all these properties using some CPO-theory is given in Ferreira [10, section 4.2]; there the notion of status is generalized to an arbitrary lifting of relations satisfying some preservation properties and a continuity requirement. Anyhow, we conclude that $\succ_{\text{rpo}}$ is a rewrite order. By definition it satisfies the subterm property. Hence for finite signatures it is a simplification order, and thus well-founded.

For infinite signatures at least well-foundedness of the precedence $\succ$ is necessary for concluding that $\succ_{\text{rpo}}$ is well-founded: if $c_1 \succ c_2 \succ c_3 \succ \cdots$ then also $c_1 \succ_{\text{rpo}} c_2 \succ_{\text{rpo}} c_3 \succ_{\text{rpo}} \cdots$. The next theorem states that well-foundedness of $\succ$ is also sufficient. First a lemma.

Lemma 7.2 *For every well-founded precedence $\succ$ on $\mathcal{F}$ there exists a PWO $\sqsupset$ on $\mathcal{F}$ satisfying $\succ \subseteq \sqsupset$ and $\succ_{\text{rpo}} \subseteq \sqsupset_{\text{rpo}}$.*

Proof (sketch) By structural induction it can be proved that if $\succ \subseteq \sqsupset$ then $\succ_{\text{rpo}} \subseteq \sqsupset_{\text{rpo}}$. (This well-known property is known as the *incrementality* of the recursive path order.) Next one can prove that every well-founded precedence is contained in a total well-founded precedence; this statement is equivalent to the Axiom of Choice. Now the lemma follows since every total well-founded order is a PWO. □

Theorem 7.3 *If $\succ$ is a well-founded precedence then $\succ_{\text{rpo}}$ is a reduction order.*

Proof As remarked above $\succ_{\text{rpo}}$ is a rewrite order. It remains to prove well-foundedness. This follows directly from Lemma 7.2, Theorem 5.3, and the following theorem. □

A direct proof of Theorem 7.3, independent of Lemma 7.2 and Kruskal's Tree Theorem, is given in Ferreira and Zantema [13].

Theorem 7.4 *If $\succ$ is a PWO on $\mathcal{F}$ then $\succ_{\text{rpo}}$ is a simplification order.*

Proof It suffices to show that $\succ_{\text{emb}} \subseteq \succ_{\text{rpo}}$. We already observed that $\succ_{\text{rpo}}$ has the subterm property. Hence it remains to show that $f(x_1, \ldots, x_n) \succ_{\text{rpo}} g(x_{i_1}, \ldots, x_{i_m})$ if $f \succ g$ and $1 \leqslant i_1 < \cdots < i_m \leqslant n$, where n and m are the arities of f and g. This is immediate from the definition. □

Let us call a TRS *RPO-terminating* if it is compatible with $\succ_{\text{rpo}}$ for some well-founded precedence $\succ$ and a status τ. From Lemma 7.2 and Theorem 7.4 we conclude that RPO-termination implies simple termination. It was shown in Ferreira and Zantema [12] that RPO-termination implies total termination. If the TRS is finite then RPO-termination implies ω-termination, provided all function symbols have multiset status (Hofbauer [17]).

The latter result does not extend to infinite TRSs. Consider for example the TRS $\mathcal{R}$ consisting of the rules

$$g(f(x)) \;\rightarrow\; \underbrace{f(\cdots(f}_{n}(g(x)))\cdots)$$

for all $n \in \mathbb{N}$. RPO-termination of $\mathcal{R}$ follows by choosing the precedence $g \succ f$. If $\mathcal{R}$ is ω-terminating then there exist strictly increasing functions $f, g\colon \mathbb{N} \rightarrow \mathbb{N}$ satisfying $g(f(x)) > f^n(g(x))$ for all $n, x \in \mathbb{N}$. From $g(f(x)) > g(x)$ one concludes $f(x) > x$, for all $x \in \mathbb{N}$. Hence

$$g(f(0)) > f^n(g(0)) > f^{n-1}(g(0)) > \cdots > f(g(0))$$

for all n, which is impossible in $\mathbb{N}$. Hence $\mathcal{R}$ is not ω-terminating.

RPO-termination does not imply polynomial termination, not even for one-rule string rewriting systems. As an example we mention

$$f(g(h(x))) \quad \rightarrow \quad g(f(h(g(x)))).$$

RPO-termination can be shown by the precedence $f \succ g \succ h$. In [43] it was shown that this TRS is not polynomially terminating.

Conversely, neither ω-termination nor polynomial termination implies RPO-termination: the TRS $\{f(f(x)) \rightarrow g(x), g(x) \rightarrow f(x)\}$ is not RPO-terminating, while $f_{\mathcal{A}}(x) = x + 2$, $g_{\mathcal{A}}(x) = x + 3$ is a very simple polynomial interpretation for this system.

7.2 Knuth-Bendix Order

The order we describe here is a generalization of the original Knuth-Bendix order (Knuth and Bendix [22]). An essentially similar version as the one described here has been mentioned in Dershowitz [5].

A *weakly monotone* $\mathcal{F}$-algebra $(\mathcal{A}, \sqsupset)$ consists of a non-empty $\mathcal{F}$-algebra $\mathcal{A} = (A, \{f_{\mathcal{A}}\}_{f \in \mathcal{F}})$ and a partial order $\sqsupset$ on the carrier A of $\mathcal{A}$ such that

$$f_{\mathcal{A}}(a_1, \ldots, a_i, \ldots, a_n) \quad \sqsupseteq \quad f_{\mathcal{A}}(a_1, \ldots, b, \ldots, a_n)$$

for all n-ary $f \in \mathcal{F}$, $i \in \{1, \ldots, n\}$, and $a_1, \ldots, a_n, b \in A$ with $a_i \succ b$. Here $\sqsupseteq$ stands for the reflexive closure of $\sqsupset$. The rewrite order $\sqsupset_{\mathcal{A}}$ is defined as in Section 6. We write $s \sqsupseteq_{\mathcal{A}} t$ if $[\alpha](s) \sqsupseteq [\alpha](t)$ for all assignments $\alpha\colon \mathcal{V} \rightarrow A$. We say that $(\mathcal{A}, \sqsupset)$ has the *subterm property* if $f_{\mathcal{A}}(a_1, \ldots, a_n) \sqsupset a_i$ for every n-ary $f \in \mathcal{F}$, $a_1, \ldots, a_n \in A$, and $i \in \{1, \ldots, n\}$.

Definition 7.5 For a precedence $\succ$ on $\mathcal{F}$, a weakly monotone $\mathcal{F}$-algebra $(\mathcal{A}, \sqsupset)$, and a status τ, the *generalized Knuth-Bendix order* $\succ_{\text{kbo}}$ on $\mathcal{T}(\mathcal{F}, \mathcal{V})$ is defined as follows: $s \succ_{\text{kbo}} t$ if and only if $s = f(s_1, \ldots, s_n)$ and

- $s \sqsupset_{\mathcal{A}} t$, or
- $s \sqsupseteq_{\mathcal{A}} t$, $t = g(t_1, \ldots, t_m)$, and either
  - $f \succ g$, or
  - $f = g$ and $(s_1, \ldots, s_n) \succ_{\text{kbo}}^{\tau(f)} (t_1, \ldots, t_n)$.

Theorem 7.6 *The relation $\succ_{\text{kbo}}$ is a rewrite order.*

Proof (sketch) Irreflexivity and transitivity follow by induction on the structure of terms, using irreflexivity and transitivity of $\sqsupset$, $\succ$, and $\succ_{\text{kbo}}^{\tau(f)}$. Closure under contexts of $\succ_{\text{kbo}}$ follows from weak monotonicity of $(\mathcal{A}, \sqsupset)$ and the first monotonicity property of the status τ. For closure under substitutions we need the property that $[\alpha](t\sigma) = [[\alpha] \circ \sigma](t)$, which is easily proved by induction on the term t. Closure under substitutions of $\succ_{\text{kbo}}$ then follows from the second monotonicity property of τ by induction on the structure of terms. □

Theorem 7.7 *If $\mathcal{F}$ is finite and $(\mathcal{A}, \sqsupset)$ satisfies the subterm property then $\succ_{\text{kbo}}$ is a simplification order.*

Proof It is easy to see that $\succ_{\text{kbo}}$ inherits the subterm property from $(\mathcal{A}, \sqsupset)$. Hence $\succ_{\text{kbo}}$ is a simplification order. □

In the original Knuth-Bendix order [22] for every $f \in \mathcal{F}$ a weight $w(f) \in \mathbb{N}$ is defined, while $w(x) = N$ for some positive constant N for every $x \in \mathcal{V}$. The resulting order is a special case of our order by choosing $\mathcal{A}$ to be the set of natural numbers greater than or equal to N equipped with the usual order $>$ and the interpretations

$$f_{\mathcal{A}}(m_1, \ldots, m_n) = w(f) + \sum_{i=1}^{n} m_i$$

for all n-ary $f \in \mathcal{F}$ and $m_1, \ldots, m_n \geqslant N$, and τ to be lexicographic (from left to right) status. For this case one easily verifies

$$s >_{\mathcal{A}} t \quad \text{if and only if} \quad V(t) \subseteq V(s) \text{ and } W(s) > W(t),$$

where $W(u)$ is defined to be the total weight of a term u and $V(u)$ denotes the multiset of variables occurrences in u. For the well-definedness of $\mathcal{A}$ we need the requirement that $w(c) \geqslant N$ for all constants c; for the subterm property of $(\mathcal{A}, >)$ we need the requirement that $w(f) > 0$ for unary function symbols f. These are exactly the requirements as they appear in the original Knuth-Bendix order. Actually, the order defined in [22] is somewhat stronger: for at most one unary symbol f_0 it is allowed that $w(f_0) = 0$, provided that $f_0 \succ g$ for all $g \in \mathcal{F} \setminus \{f_0\}$. In this case the clause

- $s \geqslant_{\mathcal{A}} t$, $t \in \mathcal{V}$, and $s = f_0^k(t)$ for some $k > 0$

is added to the definition of $\succ_{\mathrm{kbo}}$ in order to achieve the subterm property. However, restricted to ground terms the order is not affected by adding this clause and is still a special case of our definition.

Since $\succ_{\mathrm{kbo}}$ is a simplification order (for terms over finite signatures), it is well-founded and thus suitable for giving termination proofs. For using it for mechanizing termination proofs, one needs a procedure to find suitable $\mathcal{A}$, $\sqsupset$, $\succ$, and τ such that $l \succ_{\mathrm{kbo}} r$ for every rewrite rule $l \to r$. For the restricted version described above such a procedure has been given in Dick *et al.* [9], based on the simplex method from linear programming.

In Theorem 7.7 we don't require that $\sqsupset$ is a well-founded order on the carrier of $\mathcal{A}$. Rather, the subterm property of $(\mathcal{A}, \sqsupset)$ turns out to be essential. For instance, let $\mathcal{A}$ consist of the natural numbers with the usual order $>$ and the interpretations $a_{\mathcal{A}} = 1$, $b_{\mathcal{A}} = 0$, and $f_{\mathcal{A}}(m, n) = m + n$ for $m, n \in \mathbb{N}$. This (weakly) monotone algebra is well-founded but it doesn't have the subterm property. Now, if $\tau(f)$ compares lexicographically from left to right then for any precedence $\succ$ we have the following infinite descending sequence:

$$f(a, a) \succ_{\mathrm{kbo}} f(b, f(a, a)) \succ_{\mathrm{kbo}} f(b, f(b, f(a, a))) \succ_{\mathrm{kbo}} \cdots .$$

An interesting question is how the generalized Knuth-Bendix order behaves for infinite signatures. We have the following results.

Theorem 7.8 *Let $\succ$ be a well-founded order on $\mathcal{F}$ and $\sqsupset$ a well-founded order on the carrier of $\mathcal{A}$. If $(\mathcal{A}, \sqsupset)$ has the subterm property then $\succ_{\mathrm{kbo}}$ is a reduction order.*

Proof We have to prove well-foundedness. Suppose to the contrary that there are infinite descending sequences with respect to the order $\succ_{\text{kbo}}$. Let us call a term *well-founded* if it is not the first element of an infinite descending sequence. So there exist non-well-founded terms. We construct a particular infinite descending sequence $t_1 \succ_{\text{kbo}} t_2 \succ_{\text{kbo}} \cdots$ inductively as follows:

> For t_1 we take any non-well-founded term of minimal size. Suppose we already chose the first n terms $t_1, \ldots, t_n$ ($n \geqslant 1$). Define t_{n+1} to be a smallest non-well-founded term u such that $t_n \succ_{\text{kbo}} u$.

Choose $\alpha: \mathcal{V} \to A$ arbitrarily. Since $[\alpha](t_i) \sqsupseteq [\alpha](t_{i+1})$ for all $i \geqslant 1$ and $\sqsupset$ is well-founded, there exists an index $N \geqslant 1$ such that $[\alpha](t_i) = [\alpha](t_{i+1})$ for all $i \geqslant N$. For every $i \geqslant N$, let f_i be the root symbol of t_i. Let $i \geqslant N$. Since $[\alpha](t_i) = [\alpha](t_{i+1})$ we obtain from the definition of $\succ_{\text{kbo}}$ that $f_i \succ f_{i+1}$ or $f_i = f_{i+1}$. Since $\succ$ is a well-founded order on $\mathcal{F}$, there exist $M \geqslant N$ and $f \in \mathcal{F}$ such that $f_i = f$ for all $i \geqslant M$. Let n be the arity of f and write $t_i = f(u_{i,1}, \ldots, u_{i,n})$ for $i \geqslant M$. From the definition of $\succ_{\text{kbo}}$ we conclude that

$$(u_{i,1}, \ldots, u_{i,n}) \quad \succ_{\text{kbo}}^{\tau(f)} \quad (u_{i+1,1}, \ldots, u_{i+1,n}) \tag{1}$$

for all $i \geqslant M$. We claim that every $u_{i,j}$ is well-founded: if $u_{i,j}$ is non-well-founded for some $i \geqslant M$ and $1 \leqslant j \leqslant n$ then we obtain a contradiction with the minimality of t_i as $t_i \succ_{\text{kbo}} u_{i,j}$ by the subterm property of $\succ_{\text{kbo}}$. Let $U = \{u_{i,j} \mid i \geqslant M \text{ and } 1 \leqslant j \leqslant n\}$, so the restriction of $\succ_{\text{kbo}}$ to U is well-founded. Since $\tau(f)$ is either the multiset extension or a lexicographic extension, it preserves well-foundedness. Hence the restriction of $\succ_{\text{kbo}}^{\tau(f)}$ to U^n is well-founded. This contradicts (1). $\square$

This theorem can also be proved using the more general theorems in [13]. The minimality construction is inspired by the proof of Higman's Lemma as given in Appendix B.

The question arises whether $\succ_{\text{kbo}}$ is a simplification order. Without further restrictions this is not the case, even if both $(\mathcal{F}, \succ)$ and $(\mathcal{A}, \sqsupset)$ are total orders. Consider for example the signature $\mathcal{F} = \{f_i, g_i \mid i \in \mathbb{N}\}$ and let $\mathcal{A}$ consist of the natural numbers with the usual order $>$ and the interpretations

$$\begin{aligned} f_{i\mathcal{A}}(x) &= x^3 - ix^2 + i^2x, \\ g_{i\mathcal{A}}(x) &= x + 2i \end{aligned}$$

for all $i, x \in \mathbb{N}$. In Section 6 we proved that $f_i(g_j(x)) >_{\mathcal{A}} f_j(g_j(x))$ and hence $f_i(g_j(x)) \succ_{\text{kbo}} f_j(g_j(x))$ for all $i < j$. Therefore, independent of τ and $\succ$, $\succ_{\text{kbo}}$ is neither a simplification order nor contained in one.

However, if we require the additive behaviour of the weights as in the original Knuth-Bendix order, we can conclude that the order is a simplification order. Before we can state this we need a precise definition of this additive behaviour.

Definition 7.9 A weakly monotone algebra $(\mathcal{A}, \sqsupset)$ is called *additive* if there exists a $c \in A$ such that for every n-ary $f \in \mathcal{F}$ and m-ary $g \in \mathcal{F}$ with $f_{\mathcal{A}}(c, \ldots, c) \sqsupseteq g_{\mathcal{A}}(c, \ldots, c)$ we have

$$f_{\mathcal{A}}(a_1, \ldots, a_n) \quad \sqsupseteq \quad g_{\mathcal{A}}(a_{i_1}, \ldots, a_{i_m})$$

for all $a_1, \ldots, a_n \in A$ and $1 \leqslant i_1 < \cdots < i_m \leqslant n$.

Clearly the weakly monotone algebra $(\mathcal{A}, >)$ induced by the original Knuth-Bendix order is additive as $f_{\mathcal{A}}(m_1, \ldots, m_n) = w(f) + m_1 + \cdots + m_n$.

Theorem 7.10 *Let $\succ$ be a PWO on $\mathcal{F}$ and $\sqsupset$ a PWO on the carrier of $\mathcal{A}$. If $(\mathcal{A}, \sqsupset)$ is additive and has the subterm property then $\succ_{\mathrm{kbo}}$ is a simplification order.*

Proof We define a new precedence $\succ'$ on $\mathcal{F}$ as follows: $f \succ' g$ if and only if $f \succ g$ and $f_{\mathcal{A}}(c, \ldots, c) \sqsupseteq g_{\mathcal{A}}(c, \ldots, c)$. Since $(\mathcal{F}, \succ)$ and $(A, \sqsupset)$ are PWOs we can apply Lemma A.4 in Appendix A, choosing $\varphi(f) = f_{\mathcal{A}}(c, \ldots, c)$, to conclude that $\succ'$ is again a PWO. We shall prove that $\succ_{\mathrm{kbo}}$ is a simplification order with respect to the PWO $\succ'$. We already observed that $\succ_{\mathrm{kbo}}$ satisfies the subterm property. Let $f \succ' g$ and $1 \leqslant i_1 < \cdots < i_m \leqslant n$, where n and m are the arities of f and g, respectively. It remains to show that $f(x_1, \ldots, x_n) \succ_{\mathrm{kbo}} g(x_{i_1}, \ldots, x_{i_m})$. Let $\alpha\colon \mathcal{V} \to A$ be an arbitrary assignment. Because $f \succ' g$ we have $f_{\mathcal{A}}(c, \ldots, c) \sqsupseteq g_{\mathcal{A}}(c, \ldots, c)$. Hence, using the fact that $\mathcal{A}$ is additive, we obtain

$$
\begin{aligned}
[\alpha](f(x_1, \ldots, x_n)) &= f_{\mathcal{A}}([\alpha](x_1), \ldots, [\alpha](x_n)) \\
&\sqsupseteq g_{\mathcal{A}}([\alpha](x_{i_1}), \ldots, [\alpha](x_{i_m})) \\
&= [\alpha](g(x_{i_1}, \ldots, x_{i_m})).
\end{aligned}
$$

Since we also have $f \succ g$ we conclude that

$$f(x_1, \ldots, x_n) \quad \succ_{\mathrm{kbo}} \quad g(x_{i_1}, \ldots, x_{i_m}).$$

□

Observe that in the above proof we show that $\succ_{\mathrm{kbo}}$ is a simplification order with respect to a *restriction* of the given precedence $\succ$. This is essential because under the conditions of this theorem the inclusion $\succ_{\mathrm{emb}} \subseteq \succ_{\mathrm{kbo}}$ does not hold in general. For instance, if $f \succ g$ for unary function symbols f and g with $w(g) > w(f)$, then the required inequality $f(x) \succ_{\mathrm{kbo}} g(x)$ does not hold.

This subsection is concluded by comparing the Knuth-Bendix order with other kinds of termination. Termination of any simply terminating TRS can be proved by means of the generalized Knuth-Bendix order by choosing $\mathcal{A}$ to be any compatible simple monotone algebra, choosing $\succ$ to be an arbitrary well-founded precedence, and applying Theorem 7.8. A TRS is totally terminating if and only if it is compatible with a generalized Knuth-Bendix order induced by a total well-founded precedence and a total monotone algebra. The "if" part was essentially proved in [12]. The "only if" part follows by taking $\mathcal{A}$ to be a compatible total monotone algebra. In case the subterm property is not satisfied it can easily be forced by taking the lexicographic product with the algebra in which a term is interpreted by its size.

Of more interest is a decidable version of the Knuth-Bendix order. We take the original version extended to arbitrary status: a TRS $(\mathcal{F}, \mathcal{R})$ is called *KBO-terminating* if it is compatible with $\succ_{\mathrm{kbo}}$ for some well-founded precedence $\succ$, some status τ, and a monotone $\mathcal{F}$-algebra $(\mathcal{A}, >)$ consisting of the natural numbers greater than or equal to some $N \in \mathbb{N}$, equipped with the usual order $>$ and the interpretations

$$f_{\mathcal{A}}(m_1, \ldots, m_n) = w(f) + \sum_{i=1}^{n} m_i$$

for all n-ary $f \in \mathcal{F}$ and $m_1, \ldots, m_n \geqslant N$. Here $w(f)$ has to be non-negative, for constants it has to be at least N, and for only one unary symbol f_0 it is allowed that $w(f_0) = 0$, provided that $f_0 \succ g$ for all $g \in \mathcal{F} \setminus \{f_0\}$. In the case that such an f_0 occurs, an extra clause is added to definition, as described before. Now KBO-termination implies both simple and total termination. However, KBO-termination is incomparable with any of the notions RPO, ω, and polynomial termination, as shown by the following two examples. The TRS $\{f(g(x)) \to g(g(f(x)))\}$ is not KBO-terminating, but it is RPO-terminating by choosing $f \succ g$, and polynomially (and hence ω-)terminating by choosing $f_{\mathcal{A}}(x) = 3x$ and $g_{\mathcal{A}}(x) = x + 1$. The TRS $\{f(g(x)) \to g(f(f(x)))\}$ is KBO-terminating by choosing $w(f) = 0$, $w(g) = 1$, and $f \succ g$, but it is not ω-terminating (and hence not polynomially terminating) as was shown in [43].

8 Modularity

In this section we explain why simple termination has a better modular behaviour than pseudo-simple termination. We refer to Ohlebusch [37] for a recent overview of the area of modularity.

Definition 8.1 A property of TRSs is called *modular* if the union of two TRSs that do not share function symbols inherits the property from the two TRSs.

Toyama [42] showed that termination is not modular by means of the following celebrated example:

$$\begin{aligned} \mathcal{R}_1 &= \{f(a, b, x) \to f(x, x, x)\}, \\ \mathcal{R}_2 &= \{g(x, y) \to x, g(x, y) \to y\}. \end{aligned}$$

Kurihara and Ohuchi [27] observed that $\mathcal{R}_1$ is not simplifying. They proved the following result.

Theorem 8.2 *Simplifyingness is modular.* □

Hence (pseudo-)simple termination is modular for TRSs over finite signatures. Gramlich [15] showed that pseudo-simple termination is modular for *finitely branching* TRSs. A TRS $(\mathcal{F}, \mathcal{R})$ is called finitely branching if the set $\{t \mid s \to_{\mathcal{R}} t\}$ of one-step reducts of s is finite, for any term $s \in \mathcal{T}(\mathcal{F}, \mathcal{V})$. Ohlebusch [36] extended this result to arbitrary TRSs.

Theorem 8.3 *Pseudo-simple termination is modular.* □

We have the following result. We refrain from giving the proof because later we prove a more general result.

Theorem 8.4 *Simple termination is modular.* □

Because of the disjointness requirement, modularity is a rather restricted property. If we allow the sharing of certain function symbols among TRSs, we might hope for more useful results.

Definition 8.5 With every TRS $(\mathcal{F}, \mathcal{R})$ we associate the set $\mathcal{D} = \{\mathrm{root}(\mathrm{l}) \mid \mathrm{l} \to \mathrm{r} \in \mathcal{R}\}$ of *defined symbols* and the set $\mathcal{C} = \mathcal{F} \setminus \mathcal{D}$ of *constructors*. We say that two TRSs $(\mathcal{F}_1, \mathcal{R}_1)$ and $(\mathcal{F}_2, \mathcal{R}_2)$ *share constructors* if $\mathcal{D}_1$, $\mathcal{D}_2$, and $\mathcal{C}_1 \cup \mathcal{C}_2$ are pairwise disjoint. A property of TRSs is called *constructor sharing modular* if the union of two TRSs that share constructors inherits the property from the two TRSs.

A constructor sharing modular property is clearly modular. Kurihara and Ohuchi [28] were the first to study constructor sharing modularity. They proved the following result.

Theorem 8.6 *Simplifyingness is constructor sharing modular.* □

So (pseudo-)simple termination is constructor sharing modular for TRSs over finite signatures. Gramlich [15] showed that pseudo-simple termination is constructor sharing modular for finitely branching TRSs. Surprisingly, the latter result does not extend to arbitrary TRSs, as shown by the following example of Ohlebusch [36]:

$$\begin{aligned}\mathcal{R}_1 &= \{f_i(c_i, x) \to f_{i+1}(x, x) \mid i \in \mathbb{N}\},\\ \mathcal{R}_2 &= \{a \to c_i \mid i \in \mathbb{N}\}.\end{aligned}$$

Both TRSs are pseudo-simply terminating. Actually they are polynomially terminating. For $\mathcal{R}_2$ this is obvious, for $\mathcal{R}_1$ this can be shown by the following polynomials over $\mathbb{N}$:

$$\begin{aligned}f_{i\mathcal{A}}(x, y) &= x + y^3 - iy^2 + i^2 y,\\ c_{i\mathcal{A}} &= i^2 + 2i + 2\end{aligned}$$

for all $i, x, y \in \mathbb{N}$. The two TRSs share constructors c_i for $i \in \mathbb{N}$, but their union is not (pseudo-simply) terminating:

$$f_1(c_1, a) \to_{\mathcal{R}_1} f_2(a, a) \to_{\mathcal{R}_2} f_2(c_2, a) \to_{\mathcal{R}_1} f_3(a, a) \to_{\mathcal{R}_2} \cdots$$

Observe that $\mathcal{R}_2$ is not finitely branching. We claim that $\mathcal{R}_1$ is not simply terminating. Let $\succ$ be an arbitrary PWO on the signature $\mathcal{F}$ of $\mathcal{R}_1$. We must have $f_j \succ f_i$ for some $i < j$. Hence $\mathcal{E}\mathrm{mb}(\mathcal{F}, \succ)$ contains the rewrite rule $f_j(x, y) \to f_i(x, y)$. Now consider the term $t = f_1(c_i, f_1(c_{i+1}, f_1(\ldots, c_{j-1})))$. Since $t \to^+_{\mathcal{E}\mathrm{mb}(\mathcal{F})} c_k$ for all $i \leqslant k \leqslant j-1$, the term $f_i(t, t)$ is cyclic in the TRS $\mathcal{R}_1 \cup \mathcal{E}\mathrm{mb}(\mathcal{F}, \succ)$:

$$f_i(t, t) \to^+ f_i(c_i, t) \to f_{i+1}(t, t) \to^+ \cdots \to f_j(t, t) \to f_i(t, t).$$

According to Lemma 5.5 $\mathcal{R}_1$ is not simply terminating. We show below that simple termination is constructor sharing modular for arbitrary TRSs. Actually, we show a stronger result.

Definition 8.7 Let $(\mathcal{F}, \mathcal{R})$ be a TRS and $\mathcal{F}'$ be a set of function symbols. We denote the set $\{l \to r \in \mathcal{R} \mid \mathrm{root}(l) \in \mathcal{F}'\}$ by $\mathcal{R} \mid \mathcal{F}'$. So $\mathcal{R} \mid \mathcal{F}'$ consists of those rules of $\mathcal{R}$ that define the symbols in $\mathcal{F}'$. We say that two TRSs $(\mathcal{F}_1, \mathcal{R}_1)$ and $(\mathcal{F}_2, \mathcal{R}_2)$ are *composable* if $\mathcal{R}_1 \mid \mathcal{F}_2 = \mathcal{R}_2 \mid \mathcal{F}_1$. A property of TRSs is called *decomposable* if the union of two composable TRSs inherits the property from the two TRSs.

This definition originates from Middeldorp and Toyama [31]. There it was defined for *constructor systems*. A constructor system is a TRS with the property that the arguments $t_1, \ldots, t_n$ of the left-hand side $f(t_1, \ldots, t_n)$ of every rewrite rule do not contain defined symbols. It is not difficult to see that composable TRSs may share not only constructors but also defined symbols, provided the common defined symbols have the same defining rewrite rules in both TRSs. Hence every decomposable property is constructor sharing modular. Ohlebusch [37, 38] extended Theorem 8.6 to composable TRSs.

Theorem 8.8 *Simplifyingness is decomposable.* □

Very recently Kurihara and Ohuchi [29] showed that pseudo-simple termination is a decomposable property of finitely branching TRSs, thereby extending Gramlich's result.

Theorem 8.9 *Simple termination is decomposable.*

Proof Let $(\mathcal{F}_1, \mathcal{R}_1)$ and $(\mathcal{F}_2, \mathcal{R}_2)$ be composable and simply terminating TRSs. According to Lemma 5.5 there exist PWOs $\succ_1$ on $\mathcal{F}_1$ and $\succ_2$ on $\mathcal{F}_2$ such that the TRSs $\mathcal{R}_1 \cup \mathcal{E}\mathrm{mb}(\mathcal{F}_1, \succ_1)$ and $\mathcal{R}_2 \cup \mathcal{E}\mathrm{mb}(\mathcal{F}_2, \succ_1)$ are acyclic. For $i \in \{1, 2\}$ let $\sqsupset_i^c$ be the restriction of $\succ_i$ to $\mathcal{F}_1 \cap \mathcal{F}_2$ and $\sqsupset_i$ the restriction of $\succ_i$ to $\mathcal{F}_i \setminus (\mathcal{F}_1 \cap \mathcal{F}_2)$. These four relations are clearly PWOs. Because PWOs are closed under intersection (Corollary A.5), the relation $\sqsupset_c = \sqsupset_1^c \cap \sqsupset_2^c$ is a PWO on $\mathcal{F}_1 \cap \mathcal{F}_2$. Let $\sqsupset$ be the (disjoint) union of $\sqsupset_1$, $\sqsupset_2$, and $\sqsupset_c$. It is easy to see that $\sqsupset$ is a PWO on the combined signature $\mathcal{F} = \mathcal{F}_1 \cup \mathcal{F}_2$. We claim that the TRS $\mathcal{R}_1 \cup \mathcal{R}_2 \cup \mathcal{E}\mathrm{mb}(\mathcal{F}, \sqsupset)$ is acyclic, thereby establishing the simple termination of $\mathcal{R}_1 \cup \mathcal{R}_2$ using Lemma 5.5. Clearly $\mathcal{E}\mathrm{mb}(\mathcal{F}, \sqsupset)$ is the union of $\mathcal{E}\mathrm{mb}^*(\mathcal{F}_1, \sqsupset_1) \cup \mathcal{E}\mathrm{mb}^*(\mathcal{F}_2, \sqsupset_2) \cup \mathcal{E}\mathrm{mb}^*(\mathcal{F}_1 \cap \mathcal{F}_2, \sqsupset_\mathrm{c})$ and $\mathcal{E}\mathrm{mb}(\mathcal{F})$. Define the TRSs $(\mathcal{F}_1, \mathcal{S}_1)$ and $(\mathcal{F}_2, \mathcal{S}_2)$ as follows:

$$\begin{aligned} \mathcal{S}_1 &= \mathcal{R}_1 \cup \mathcal{E}\mathrm{mb}^*(\mathcal{F}_1, \sqsupset_1) \cup \mathcal{E}\mathrm{mb}^*(\mathcal{F}_1 \cap \mathcal{F}_2, \sqsupset_\mathrm{c}), \\ \mathcal{S}_2 &= \mathcal{R}_2 \cup \mathcal{E}\mathrm{mb}^*(\mathcal{F}_2, \sqsupset_2) \cup \mathcal{E}\mathrm{mb}^*(\mathcal{F}_1 \cap \mathcal{F}_2, \sqsupset_\mathrm{c}). \end{aligned}$$

We have $\mathcal{S}_i \cup \mathcal{E}\mathrm{mb}(\mathcal{F}_\mathrm{i}) \subseteq \mathcal{R}_\mathrm{i} \cup \mathcal{E}\mathrm{mb}(\mathcal{F}_\mathrm{i}, \succ_\mathrm{i})$ for $i \in \{1, 2\}$. Hence $\mathcal{S}_1$ and $\mathcal{S}_2$ are simplifying. They are also composable:

$$\begin{aligned} \mathcal{S}_1 \mid \mathcal{F}_2 &= \mathcal{R}_1 \mid \mathcal{F}_2 \cup \mathcal{E}\mathrm{mb}^*(\mathcal{F}_1, \sqsupset_1) \mid \mathcal{F}_2 \cup \mathcal{E}\mathrm{mb}^*(\mathcal{F}_1 \cap \mathcal{F}_2, \sqsupset_\mathrm{c}) \mid \mathcal{F}_2 \\ &= \mathcal{R}_1 \mid \mathcal{F}_2 \cup \varnothing \cup \mathcal{E}\mathrm{mb}^*(\mathcal{F}_1 \cap \mathcal{F}_2, \sqsupset_\mathrm{c}) \\ &= \mathcal{R}_2 \mid \mathcal{F}_1 \cup \varnothing \cup \mathcal{E}\mathrm{mb}^*(\mathcal{F}_1 \cap \mathcal{F}_2, \sqsupset_\mathrm{c}) \\ &= \mathcal{R}_2 \mid \mathcal{F}_1 \cup \mathcal{E}\mathrm{mb}^*(\mathcal{F}_2, \sqsupset_2) \mid \mathcal{F}_1 \cup \mathcal{E}\mathrm{mb}^*(\mathcal{F}_1 \cap \mathcal{F}_2, \sqsupset_\mathrm{c}) \mid \mathcal{F}_1 \\ &= \mathcal{S}_2 \mid \mathcal{F}_1. \end{aligned}$$

According to Theorem 8.8 $\mathcal{S}_1 \cup \mathcal{S}_2$ is simplifying. Lemma 6.2 shows that $\mathcal{S}_1 \cup \mathcal{S}_2 \cup \mathcal{E}\text{mb}(\mathcal{F})$ is acyclic. Since $\mathcal{S}_1 \cup \mathcal{S}_2 \cup \mathcal{E}\text{mb}(\mathcal{F}) = \mathcal{R}_1 \cup \mathcal{R}_2 \cup \mathcal{E}\text{mb}(\mathcal{F}, \sqsupset)$ we are done. □

Concerning the other restricted termination notions introduced in Section 6, it is very easy to see that polynomial and ω-termination are modular. At present it is an open problem whether total termination is a modular property. Ferreira and Zantema [11] showed that the disjoint union of totally terminating TRSs is totally terminating whenever one the systems lacks *duplicating* rules. A rewrite rule $l \to r$ is called duplicating if its right-hand side r contains more occurrences of some variable than its left-hand side l. Using completely different techniques, Rubio [40] obtained the same result. None of the properties polynomial, ω, and total termination is constructor sharing modular, as shown by partitioning the non-totally terminating TRS $\mathcal{R} = \{f(a) \to f(b), g(b) \to g(a)\}$ into the polynomially terminating and constructor sharing TRSs $\mathcal{R}_1 = \{f(a) \to f(b)\}$ and $\mathcal{R}_2 = \{g(b) \to g(a)\}$.

Recently *hierarchical* combinations of TRSs entered the spotlight of modularity research [6, 25, 23, 24]. Krishna Rao [23] showed that simplifyingness is modular for a certain class of hierarchical combinations. This result can be used to prove the modularity of simple termination for the same class of hierarchical combinations, similar to the proof of Theorem 8.8 (Krishna Rao, personal communication).

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A Partial Well-Orders

Throughout this appendix and the next we deal with infinite sequences of some kind. We find it convenient to abbreviate an infinite sequence $(a_i)_{i \geqslant 1} = a_1, a_2, a_3, \ldots$ to $\mathbf{a}$. Moreover, we denote $(f(a_i))_{i \geqslant 1}$ by $f(\mathbf{a})$, $(a_{\psi(i)})_{i \geqslant 1}$ by $\mathbf{a}_\psi$, and $(a_i)_{i \geqslant n}$ by $\mathbf{a}_{\geqslant n}$.

Definition A.1 Let $\succ$ be a partial order on a set A and suppose that $\mathbf{a}$ is an infinite sequence of elements of A. The sequence $\mathbf{a}$ is called *good* if there exist indices $1 \leqslant i < j$ with $a_i \preccurlyeq a_j$, otherwise it is called *bad*. We say that $\mathbf{a}$ is a *chain* if $a_i \preccurlyeq a_{i+1}$ for all $i \geqslant 1$. We say that $\mathbf{a}$ contains a chain if it has a subsequence that is a chain. The sequence $\mathbf{a}$ is called an *antichain* if neither $a_i \preccurlyeq a_j$ nor $a_j \preccurlyeq a_i$, for all $1 \leqslant i < j$.

Lemma A.2 *Let $\succ$ be a partial order on a set A. The following statements are equivalent.*

(1) *Every partial order that extends $\succ$ (including $\succ$ itself) is well-founded.*
(2) *Every infinite sequence over A is good.*
(3) *Every infinite sequence over A contains a chain.*
(4) *The partial order $\succ$ is well-founded and does not admit antichains.*

Proof

(1) $\Rightarrow$ (2) Suppose $\mathbf{a}$ is a bad sequence. Define $\sqsupset = (\succ \cup \{(a_i, a_{i+1}) \mid i \geqslant 1\})^+$. Assume $a \sqsupset a$ for some $a \in A$. Since $\succ$ is irreflexive there is a non-empty sequence of numbers $i_1, \ldots, i_n$ such that

$$a \succcurlyeq a_{i_1},\ a_{i_1+1} \succcurlyeq a_{i_2},\ a_{i_2+1} \succcurlyeq a_{i_3},\ \ldots,\ a_{i_{n-1}+1} \succcurlyeq a_{i_n},\ a_{i_n+1} \succcurlyeq a.$$

Since $\mathbf{a}$ is bad, $a_i \succcurlyeq a_j$ is only possible for $i \leqslant j$. Hence we obtain the impossible

$$i_1 < i_1 + 1 \leqslant i_2 < i_2 + 1 \leqslant i_3 < \cdots < i_{n-1} + 1 \leqslant i_n < i_n + 1 \leqslant i_1.$$

We conclude that $\sqsupset$ is irreflexive. By definition it is transitive, hence it is a partial order extending $\succ$. However, since $a_1 \sqsupset a_2 \sqsupset a_3 \sqsupset \cdots$, it is not well-founded.

(2) $\Rightarrow$ (3) Let $\mathbf{a}$ be any infinite sequence over A. Consider the subsequence consisting of all elements a_i with the property that $a_i \preccurlyeq a_j$ holds for no $j > i$. If this subsequence is infinite then it is a bad sequence, contradicting (2). Hence it is finite, and thus there exists an index $N \geqslant 1$ such that for every $i \geqslant N$ there exists a $j > i$ with $a_i \preccurlyeq a_j$. Define inductively

$$\phi(i) = \begin{cases} N & \text{if } i = 1, \\ \min\{j \mid j > \phi(i-1) \text{ and } a_{\phi(i-1)} \preccurlyeq a_j\} & \text{if } i > 1. \end{cases}$$

Now $\mathbf{a}_\phi$ is a chain.

(3) $\Rightarrow$ (4) If $\succ$ is not well-founded then there exists an infinite sequence $a_1 \succ a_2 \succ \cdots$. Clearly $a_i \preccurlyeq a_j$ doesn't hold for any $1 \leqslant i < j$. Hence this sequence doesn't contain a chain. If $\succ$ admits an antichain then this antichain is an infinite sequence not containing a chain.

(4) $\Rightarrow$ (1) For a proof by contradiction, let $\succ$ be a well-founded order that doesn't satisfy (1). So there is an extension $\sqsupset$ of $\succ$ that is not well-founded. Hence there exists an infinite sequence $a_1 \sqsupset a_2 \sqsupset \cdots$. Since $\succ$ is well-founded, the sequence $\mathbf{a}$ contains an element a_i with the property that for no $j > i$ $a_i \succ a_j$ holds. Actually, $\mathbf{a}$ contains infinitely many such elements. We claim that the infinite subsequence $\mathbf{a}_\phi$ consisting of those elements is an antichain (with respect to $\succ$). Let $1 \leqslant i < j$. By construction $a_{\phi(i)} \succ a_{\phi(j)}$ is impossible. If $a_{\phi(i)} \preccurlyeq a_{\phi(j)}$ then also $a_{\phi(i)} \sqsubseteq a_{\phi(j)}$, contradicting $a_{\phi(i)} \sqsupset a_{\phi(j)}$. Hence $\succ$ admits a anti-chain.

□

Definition A.3 A partial order $\succ$ on a set A is called a *partial well-order* (PWO for short) if it satisfies one of the four equivalent assertions of Lemma A.2.

By definition every PWO is a well-founded order, but the reverse does not hold. For instance, the empty relation on an infinite set is a well-founded order but not a PWO. Clearly every total well-founded order (or well-order) is a PWO. Any partial order extending a PWO is a PWO. The following lemma states how new PWOs can be obtained by restricting existing PWOs.

Lemma A.4 *Let $\succ$ be a PWO on a set A and let $\sqsupset$ be a PWO on a set B. Let $\varphi\colon A \to B$ be any function. The partial order $\succ'$ on A defined by $a \succ' b$ if and only if $a \succ b$ and $\varphi(a) \sqsupseteq \varphi(b)$ is a PWO.*

Proof Let $\mathbf{a}$ be any infinite sequence over A. Since $\succ$ is a PWO this sequence admits a chain $\mathbf{a}_\phi$. Since $\sqsupset$ is a PWO on B there exist $1 \leqslant i < j$ with $\varphi(a_{\phi(i)}) \sqsubseteq \varphi(a_{\phi(j)})$. Transitivity of $\preccurlyeq$ yields $a_{\phi(i)} \preccurlyeq a_{\phi(j)}$. Hence $a_{\phi(i)} \preccurlyeq' a_{\phi(j)}$, while $\phi(i) < \phi(j)$. We conclude that $\mathbf{a}$ is a good sequence with respect to $\succ'$, so $\succ'$ is a PWO. □

Corollary A.5 *The intersection of two PWOs on a set A is a PWO on A.*

Proof Choose the function φ in Lemma A.4 to be the identity on A. □

B Kruskal's Tree Theorem

For the sake of completeness, below we present a proof of this beautiful theorem, even though it is very similar to the proof of the Kruskal's Tree Theorem formulated in terms of *well-quasi-orders* (see e.g. Gallier [14]). First we show a related result for strings, known as *Higman's Lemma* (Higman [16]).

Definition B.1 Let $\succ$ be a partial order on a set A. We define a relation $\succ^*$ on A^* as follows: if $w_1 = a_1 a_2 \cdots a_n$ and $w_2 = b_1 b_2 \cdots b_m$ are elements of A^* then $w_1 \succ^* w_2$ if and only if $w_1 \neq w_2$ and either $m = 0$, or $n \geqslant m > 0$ and there exist indices $i_1, \ldots, i_m$ such that $1 \leqslant i_1 < \cdots < i_m \leqslant n$ and $a_{i_j} \succcurlyeq b_j$ for all $1 \leqslant j \leqslant m$.

The next result can be viewed as an alternative definition of $\succ^*$.

Lemma B.2 *Let $\succ$ be a partial order on a set A. The relation $\succ^*$ is the least partial order $\sqsupset$ on A^* satisfying the following two properties:*
(1) $w_1 a w_2 \sqsupset w_1 w_2$ for all $w_1, w_2 \in A^*$ and $a \in A$,
(2) $w_1 a w_2 \sqsupset w_1 b w_2$ for all $w_1, w_2 \in A^*$ and $a, b \in A$ with $a \succ b$.

Proof First we show that $\succ^*$ is a partial order. Irreflexivity is obvious. Let $w_1 = a_1 \cdots a_n$, $w_2 = b_1 \cdots b_m$, and $w_3 = c_1 \cdots c_l$ be elements of A^* such that $w_1 \succ^* w_2 \succ^* w_3$. If $l = 0$ then $m > 0$ (because $w_2 \neq w_3$) and $n \geqslant m > 0$. Hence $w_1 \succ^* w_3$. Suppose $l > 0$. We have $n \geqslant m \geqslant l$. There exist indices $i_1, \ldots, i_l$ and $j_1, \ldots, j_m$ such that $1 \leqslant i_1 < \cdots < i_l \leqslant m$, $b_{i_k} \succcurlyeq c_k$ for all $1 \leqslant k \leqslant l$, $1 \leqslant j_1 < \cdots < j_m \leqslant n$, and $a_{j_k} \succcurlyeq b_k$ for all $1 \leqslant k \leqslant m$. Since $1 \leqslant j_{i_1} < \cdots < j_{i_l} \leqslant n$ and $a_{j_{i_k}} \succcurlyeq b_{i_k} \succcurlyeq c_k$ for all $1 \leqslant k \leqslant l$, we have $w_1 \succ^* w_3$. This concludes the proof of the transitivity of $\succ^*$. It is very easy to see that $\succ^*$ satisfies properties (1) and (2). Conversely, let $\sqsupset$ be any partial order on A^* that satisfies properties (1) and (2). We will show that $\succ^* \subseteq \sqsupset$. Suppose $w_1 = a_1 \cdots a_n \succ^* b_1 \cdots b_m = w_2$. If $m = 0$ then $n > 0$ and hence the sequence $w_1 = a_1 \cdots a_n \sqsupset a_2 \cdots a_n \sqsupset \cdots \sqsupset a_n \sqsupset \varepsilon = w_2$ is non-empty, showing that $w_1 \sqsupset w_2$. If $n \geqslant m > 0$ then there exist indices $i_1, \ldots, i_m$ such that $1 \leqslant i_1 < \cdots < i_m \leqslant n$ and $a_{i_j} \succcurlyeq b_j$ for all $1 \leqslant j \leqslant m$. Let $w_3 = a_{i_1} \cdots a_{i_m}$. We have $w_1 \sqsupseteq w_3$ by successively removing elements a_i from w_1 whose index i does not belong to the set $\{i_1, \ldots, i_m\}$. (Clearly $w_1 = w_3$ if and only if $n = m$.) We have $w_3 \sqsupseteq w_2$ by replacing a_{i_j} with b_j whenever $a_{i_j} \succ b_j$. Therefore $w_1 \sqsupseteq w_2$ and since $w_1 \neq w_2$ we obtain $w_1 \sqsupset w_2$. □

Lemma B.3 (Higman's Lemma) *If $\succ$ is a PWO on a set A then $\succ^*$ is a PWO on A^*.*

Proof The following proof is essentially due to Nash-Williams [34]. We have to show that there are no bad sequences over A^*. Suppose to the contrary that there exist bad sequences over A^*. We construct a *minimal bad sequence* $\mathbf{w}$ as follows:

> Suppose we already chose the first $n-1$ strings $w_1, \ldots, w_{n-1}$. Define w_n to be a shortest string such that there are bad sequences that start with $w_1, \ldots, w_n$.

Because $\varepsilon \preccurlyeq^* w$ for all $w \in A^*$, we have $w_i \neq \varepsilon$ for all $i \geqslant 1$. Hence we may write $w_i = a_i v_i$ ($i \geqslant 1$). Since $\succ$ is a PWO on A, the infinite sequence $\mathbf{a}$ contains a chain, say $\mathbf{a}_\phi$. Because $v_{\phi(1)}$ is shorter than $w_{\phi(1)}$, the sequence $w_1, \ldots, w_{\phi(1)-1}, \mathbf{v}_\phi$ must be good. Clearly $w_i \preccurlyeq^* w_j$ ($1 \leqslant i < j \leqslant \phi(1) - 1$) is impossible as $(w_i)_{i \geqslant 1}$ is bad. Likewise, $w_i \preccurlyeq^* v_{\phi(j)}$ ($1 \leqslant i \leqslant \phi(1) - 1$ and $1 \leqslant j$) contradicts the badness of $\mathbf{w}$ since $v_{\phi(j)} \preccurlyeq^* w_{\phi(j)}$ and therefore $w_i \preccurlyeq^* w_{\phi(j)}$. Hence we must have $v_{\phi(i)} \preccurlyeq^* v_{\phi(j)}$ for some $1 \leqslant i < j$. Combining this with $a_{\phi(i)} \preccurlyeq a_{\phi(j)}$ easily yields $w_{\phi(i)} = a_{\phi(i)} v_{\phi(i)} \preccurlyeq^* a_{\phi(j)} v_{\phi(j)} = w_{\phi(j)}$, contradicting the badness of $\mathbf{w}$. We conclude that there are no bad sequences over A^*. □

Proof of Kruskal's Tree Theorem—General Version The proof, essentially due to Nash-Williams [34], has the same structure as the proof of Higman's Lemma. We have to show that there are no bad sequences of terms in $\mathcal{T}(\mathcal{F})$. Suppose to the contrary that there exist bad sequences of ground terms. We construct a minimal bad sequence $\mathbf{t}$ as follows:

> Suppose we already chose the first $n-1$ terms $t_1, \ldots, t_{n-1}$. Define t_n to be a smallest (with respect to size) term such that there are bad sequences that start with $t_1, \ldots, t_n$.

For every $i \geqslant 1$, let f_i be the root symbol of t_i and let A_i be the set of arguments of t_i (if t_i is a constant then $A_i = \varnothing$). Moreover, let w_i be the string of arguments (from left to right) of t_i. Finally, let $A = \bigcup_{i \geqslant 1} A_i$.

We claim that $\succ_{\text{emb}}$ is a PWO on the subset A of $\mathcal{T}(\mathcal{F})$. For a proof by contradiction, suppose $\mathbf{a}$ is a bad sequence over A. Let $a_1 \in A_k$. Since $A' = \bigcup_{i=1}^{k-1} A_i$ is a finite set and all elements of $\mathbf{a}$ are different, only finitely many elements of $\mathbf{a}$ belong to A'. Thus there exists an index $l > 1$ such that $a_i \in A \setminus A'$ for all $i \geqslant l$. Because a_1 is a proper subterm of t_k, the sequence $t_1, \ldots, t_{k-1}, a_1, \mathbf{a}_{\geqslant l}$ must be good. Clearly $t_i \preccurlyeq_{\text{emb}} t_j$ ($1 \leqslant i < j \leqslant k-1$) is impossible as $\mathbf{t}$ is bad. Likewise, $t_i \preccurlyeq_{\text{emb}} a_j$ ($1 \leqslant i \leqslant k-1$ and $j = 1$ or $l \leqslant j$) contradicts the badness of $\mathbf{t}$ since $a_j \preccurlyeq_{\text{emb}} t_m$ for some $m \geqslant k$—recall that a_1 is a proper subterm of t_k and if $j \geqslant l$ then $a_j \in A \setminus A'$—and thus $t_i \preccurlyeq_{\text{emb}} t_j$. Hence we must have $a_i \preccurlyeq_{\text{emb}} a_j$ for some $1 \leqslant i < j$ (and $i, j \notin \{2, \ldots, l-1\}$), contradicting the badness of $\mathbf{a}$. Hence $\succ_{\text{emb}}$ is a PWO on A. From Higman's Lemma we infer that $\succ_{\text{emb}}^*$ is a PWO on A^*.

Since $\succ$ is a PWO on $\mathcal{F}$, the infinite sequence $\mathbf{f}$ contains a chain, say $\mathbf{f}_\phi$. Consider the infinite sequence $\mathbf{w}_\phi$ over A^*. Since $\succ^*_{\text{emb}}$ is a PWO on A^*, we have $w_{\phi(i)} \preccurlyeq^*_{\text{emb}} w_{\phi(j)}$ for some $1 \leqslant i < j$. A straightforward case analysis reveals that $f_{\phi(i)} \preccurlyeq f_{\phi(j)}$ and $w_{\phi(i)} \preccurlyeq^*_{\text{emb}} w_{\phi(j)}$ imply $t_{\phi(i)} \preccurlyeq_{\text{emb}} t_{\phi(j)}$. Hence we obtained a contradiction with the badness of $\mathbf{t}$. We conclude that there are no bad sequences over $\mathcal{T}(\mathcal{F})$. $\square$

Kruskal's Tree Theorem is usually presented in terms of WQOs. A *well-quasi-order* (WQO) is a preorder that contains a PWO. This definition is equivalent to all other definitions of WQO found in the literature. The WQO version of Kruskal's Tree Theorem is not more powerful than the PWO version: notwithstanding the fact that the strict part of a WQO is not necessarily a PWO, it is very easy to show that the WQO version of Kruskal's Tree Theorem is a corollary of Theorem 5.1, and vice-versa.

Let $\succ$ be a PWO on a signature $\mathcal{F}$. A natural question is whether we can restrict $\succ_{\text{emb}}$ while retaining the property of being a PWO on $\mathcal{T}(\mathcal{F})$. In particular, do we really need all rewrite rules in $\mathcal{E}\text{mb}(\mathcal{F},\succ)$? In case there is a uniform bound on the arities of the function symbols in $\mathcal{F}$, we can greatly reduce the set $\mathcal{E}\text{mb}(\mathcal{F},\succ)$. That is, suppose there exists an $N \geqslant 0$ such that all function symbols in $\mathcal{F}$ have arity less than or equal to N. Now we can apply Lemma A.4: choose φ to be the function that assigns to every function symbol its arity and take $\sqsupset$ to be the empty relation on $\{1,\ldots,N\}$. Hence the partial order $\succ'$ on $\mathcal{F}$ defined by $f \succ' g$ if and only if f and g have the same arity and $f \succ g$ is a PWO. The corresponding set $\mathcal{E}\text{mb}(\mathcal{F},\succ')$ consists, besides all rewrite rules of the form $f(x_1,\ldots,x_n) \to x_i$, of all rewrite rules $f(x_1,\ldots,x_n) \to g(x_1,\ldots,x_n)$ with f and g n-ary function symbols such that $f \succ g$. This construction does not work if the arities of function symbols in $\mathcal{F}$ are not uniformly bounded. Consider for instance a signature $\mathcal{F}$ consisting of a constant a and n-ary function symbols f_n for every $n \geqslant 1$ (and let $\succ$ be any PWO on $\mathcal{F}$). The sequence

$$f_1(a),\ f_2(a,a),\ f_3(a,a,a),\ \ldots$$

is bad with respect to $\succ'_{\text{emb}}$. Finally, one may wonder whether the restriction to all rewrite rules $f(x_1,\ldots,x_n) \to g(x_{i+1},\ldots,x_{i+m})$ with f an n-ary function symbol, g an m-ary function symbol, $n \geqslant m \geqslant 0$, $n-m \geqslant i \geqslant 0$, and $f \succ g$ is sufficient. This is also not the case, as can be seen by extending the previous signature with a constant b and considering the sequence

$$f_2(b,b),\ f_3(b,a,b),\ f_4(b,a,a,b),\ \ldots\ .$$

Of course, if the signature $\mathcal{F}$ is finite then the rules of $\mathcal{E}\text{mb}(\mathcal{F})$ are sufficient since the empty relation is a PWO on any finite set.