Termination Tools in Automated Reasoning*

PhD defense

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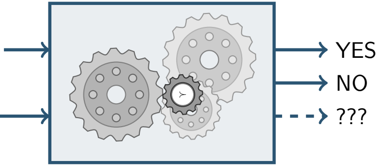
Automated Reasoning

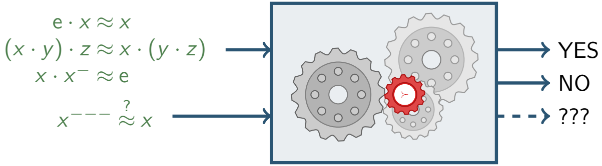
$$e \cdot x \approx x$$
$$(x \cdot y) \cdot z \approx x \cdot (y \cdot z)$$
$$x \cdot x^{-} \approx e$$

$$x^{---} \stackrel{?}{\approx} x$$



- deduction calculi formalize reasoning

Automated Reasoning



- deduction calculi formalize reasoning, relying on reduction order $\succ$

Challenge

- crucial parameter, but hard to fix appropriately in advance
- classical reduction orders have limited computational power

Automated Reasoning with Termination Tools

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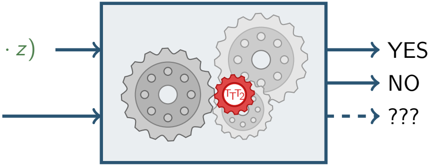

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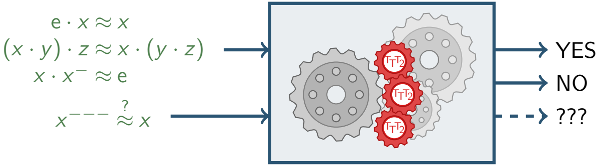
PhD Project

- Idea 1: apply automatic termination tools instead of $\succ$

Automated Reasoning with Termination Tools



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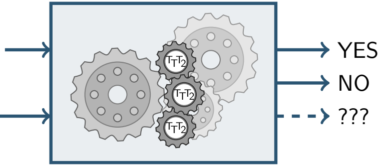

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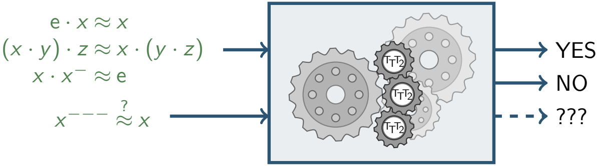
- Idea 2: deduction as optimization problem over constraints

Automated Reasoning with Termination Tools



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Outline

Term Rewriting

Knuth-Bendix Completion

Ordered Completion

Normalized Completion

Results

Term Rewriting

Example

$$\begin{aligned} \mathsf{sort}([]) &\to [] \\ \mathsf{sort}(x : y) &\to \mathsf{ins}(x, \mathsf{sort}(y)) \\ \mathsf{ins}(x, []) &\to x : [] \\ \mathsf{ins}(x, y : z) &\to \mathsf{c}(x, y : z, x, y) \end{aligned}$$

$$\begin{aligned} \mathsf{c}(x, y : z, w, 0) &\to x : (y : z) \\ \mathsf{c}(x, y : z, 0, w) &\to y : \mathsf{ins}(x, z) \\ \mathsf{c}(x, y : z, \mathsf{s}(v), \mathsf{s}(w)) &\to \mathsf{c}(x, y : z, v, w) \end{aligned}$$

Term Rewriting

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$\mathsf{sort}(\mathsf{s}(0) : (0 : []))$

term

Term Rewriting

Example

$$\mathsf{sort}([]) \to [] \qquad \mathsf{c}(x, y : z, w, 0) \to x : (y : z)$$
$$\mathsf{sort}(x : y) \to \mathsf{ins}(x, \mathsf{sort}(y)) \qquad \mathsf{c}(x, y : z, 0, w) \to y : \mathsf{ins}(x, z)$$
$$\mathsf{ins}(x, []) \to x : [] \qquad \mathsf{c}(x, y : z, \mathsf{s}(v), \mathsf{s}(w)) \to \mathsf{c}(x, y : z, v, w)$$
$$\mathsf{ins}(x, y : z) \to \mathsf{c}(x, y : z, x, y)$$

$\mathsf{sort}(\mathsf{s}(0) : (0 : []))$

ground term (no variables)

Term Rewriting

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$$\mathsf{sort}(\mathsf{s}(0) : (0 : [])) \to_{\mathcal{R}} \mathsf{ins}(\mathsf{s}(0), \mathsf{sort}(0 : []))$$

rewrite step

Term Rewriting

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$$\mathsf{sort}(\mathsf{s}(0) : (0 : [])) \to_{\mathcal{R}} \mathsf{ins}(\mathsf{s}(0), \mathsf{sort}(0 : [])) \to^{*}_{\mathcal{R}} 0 : (\mathsf{s}(0) : [])$$

many rewrite steps

Term Rewriting

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normal form

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$\mathsf{sort}(\mathsf{s}(0) : (0 : [])) \leftrightarrow^*_{\mathcal{R}} \mathsf{sort}(0 : (\mathsf{s}(0) : []))$

many rewrite steps in both directions (equational theory)

Term Rewriting

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Definition

TRS $\mathcal{R}$ is

- terminating if $\exists$ no infinite sequence $t_1 \to_{\mathcal{R}} t_2 \to_{\mathcal{R}} t_3 \to_{\mathcal{R}} \dots$

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- terminating if $\exists$ no infinite sequence $t_1 \to_{\mathcal{R}} t_2 \to_{\mathcal{R}} t_3 \to_{\mathcal{R}} \dots$
- confluent if $\forall\, s \;{}_{\mathcal{R}}^{*}\!\!\leftarrow u \to_{\mathcal{R}}^{*} t \quad \exists v\; s \to_{\mathcal{R}}^{*} v \;{}_{\mathcal{R}}^{*}\!\!\leftarrow t$

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- confluent if $\forall\, s\ {}_{\mathcal{R}}^{*}\!\leftarrow u \to_{\mathcal{R}}^{*} t \quad \exists v\ s \to_{\mathcal{R}}^{*} v\ {}_{\mathcal{R}}^{*}\!\leftarrow t$
- convergent if confluent and terminating

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Fact

for convergent $\mathcal{R}$ have $s \leftrightarrow_{\mathcal{R}}^{*} t$ iff s and t have same $\mathcal{R}$-normal form

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Definition

TRS $\mathcal{R}$ is

- terminating if $\exists$ no infinite sequence $t_1 \to_{\mathcal{R}} t_2 \to_{\mathcal{R}} t_3 \to_{\mathcal{R}} \ldots$
- ground confluent if $\quad \forall$ ground $s \; {}_{\mathcal{R}}^{*}\!\!\leftarrow u \to_{\mathcal{R}}^{*} t \quad \exists v \; s \to_{\mathcal{R}}^{*} v \; {}_{\mathcal{R}}^{*}\!\!\leftarrow t$
- ground convergent if ground confluent and terminating

Fact

for convergent $\mathcal{R}$ have $s \leftrightarrow_{\mathcal{R}}^{*} t$ iff s and t have same $\mathcal{R}$-normal form

Knuth-Bendix Completion

$$\underset{\text{equations}}{\mathcal{E}} + \underset{\text{reduction order}}{\succ} \longrightarrow_{KB} \underset{\text{rewrite system}}{\mathcal{R}}$$

- succeeds if $\mathcal{R}$ is convergent and $\leftrightarrow^*_{\mathcal{E}} = \leftrightarrow^*_{\mathcal{R}}$
- may also fail or loop

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Group Theory

$$\begin{array}{c} \mathsf{e} \cdot x \approx x \\ x^- \cdot x \approx \mathsf{e} \\ (x \cdot y) \cdot z \approx x \cdot (y \cdot z) \end{array} \quad \rightarrow_{KB} \quad \begin{array}{rr} \mathsf{e} \cdot x \rightarrow x & x \cdot \mathsf{e} \rightarrow x \\ x^- \cdot x \rightarrow \mathsf{e} & x \cdot x^- \rightarrow \mathsf{e} \\ x^{--} \rightarrow x & (x \cdot y) \cdot z \rightarrow x \cdot (y \cdot z) \\ \mathsf{e}^- \rightarrow \mathsf{e} & (x \cdot y)^- \rightarrow y^- \cdot x^- \\ x \cdot (x^- \cdot y) \rightarrow y & x^- \cdot (x \cdot y) \rightarrow y \end{array}$$

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Does $x^{---} \leftrightarrow^*_{\mathcal{R}} x^-$ hold?

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$$x^{---} \rightarrow^!_{\mathcal{R}} x^-$$

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Does $x^{---} \leftrightarrow^*_{\mathcal{R}} x^-$ hold? Yes!

$$x^{---} \to^!_{\mathcal{R}} x^-$$

Knuth-Bendix Completion

Definition (KB)

$\mathcal{E}$: set of equations $\mathcal{R}$: set of rewrite rules $\succ$: reduction order

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inference system KB consists of six rules:

orient $$\frac{\mathcal{E} \cup \{s \approx t\}, \mathcal{R}}{\mathcal{E}, \mathcal{R} \cup \{s \to t\}}$$ if $s \succ t$

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deduce
$$\frac{\mathcal{E}, \mathcal{R}}{\mathcal{E} \cup \{s \approx t\}, \mathcal{R}}$$
if $s \;{}_{\mathcal{R}}{\leftarrow}\; u \to_{\mathcal{R}} t$

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deduce $\dfrac{\mathcal{E}, \mathcal{R}}{\mathcal{E} \cup \{s \approx t\}, \mathcal{R}}$ if $s \;{}_{\mathcal{R}}\!\leftarrow u \to_{\mathcal{R}} t$

delete $\dfrac{\mathcal{E} \cup \{s \approx s\}, \mathcal{R}}{\mathcal{E}, \mathcal{R}}$

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if $s \succ t$

deduce
$$\frac{\mathcal{E}, \mathcal{R}}{\mathcal{E} \cup \{s \approx t\}, \mathcal{R}}$$
if $s \;{}_{\mathcal{R}}\!\leftarrow u \to_{\mathcal{R}} t$

delete
$$\frac{\mathcal{E} \cup \{s \approx s\}, \mathcal{R}}{\mathcal{E}, \mathcal{R}}$$

simplify
$$\frac{\mathcal{E} \cup \{s \approx t\}, \mathcal{R}}{\mathcal{E} \cup \{s \approx u\}, \mathcal{R}}$$
if $t \to_{\mathcal{R}} u$

compose
$$\frac{\mathcal{E}, \mathcal{R} \cup \{s \to t\}}{\mathcal{E}, \mathcal{R} \cup \{s \to u\}}$$
if $t \to_{\mathcal{R}} u$

collapse
$$\frac{\mathcal{E}, \mathcal{R} \cup \{t \to s\}}{\mathcal{E} \cup \{u \approx s\}, \mathcal{R}}$$
if $t \to_{\mathcal{R}} u$

Knuth-Bendix Completion

Definition (KB)

$\mathcal{E}$: set of equations $\mathcal{R}$: set of rewrite rules $\succ$: reduction order

inference system KB consists of six rules:

orient $$\frac{\mathcal{E} \cup \{s \approx t\}, \mathcal{R}}{\mathcal{E}, \mathcal{R} \cup \{s \to t\}}$$ if $s \succ t$

deduce $$\frac{\mathcal{E}, \mathcal{R}}{\mathcal{E} \cup \{s \approx t\}, \mathcal{R}}$$ if $s \;{}_{\mathcal{R}}\!\leftarrow u \to_{\mathcal{R}} t$

delete $$\frac{\mathcal{E} \cup \{s \approx s\}, \mathcal{R}}{\mathcal{E}, \mathcal{R}}$$

simplify $$\frac{\mathcal{E} \cup \{s \approx t\}, \mathcal{R}}{\mathcal{E} \cup \{s \approx u\}, \mathcal{R}}$$ if $t \to_{\mathcal{R}} u$

compose $$\frac{\mathcal{E}, \mathcal{R} \cup \{s \to t\}}{\mathcal{E}, \mathcal{R} \cup \{s \to u\}}$$ if $t \to_{\mathcal{R}} u$

collapse $$\frac{\mathcal{E}, \mathcal{R} \cup \{t \to s\}}{\mathcal{E} \cup \{u \approx s\}, \mathcal{R}}$$ if $t \to_{\mathcal{R}} u$

Theorem

Knuth & Bendix 1970

Let $(\mathcal{E}_0, \varnothing) \vdash \ldots \vdash (\varnothing, \mathcal{R}_n)$ satisfy $\mathrm{CP}(\mathcal{R}_n) \subseteq \bigcup_{i \geqslant 0} \mathcal{E}_i$.
Then $\mathcal{R}_n$ is convergent.

Knuth-Bendix Completion

Definition (KBtt)

$\mathcal{E}$: set of equations $\qquad$ $\mathcal{R}, \mathcal{C}$: sets of rewrite rules

inference system KBtt consists of six rules:

orient $$\frac{\mathcal{E} \cup \{s \approx t\}, \mathcal{R}, \mathcal{C}}{\mathcal{E}, \mathcal{R} \cup \{s \to t\}, \mathcal{C} \cup \{s \to t\}}$$
if $\mathcal{C} \cup \{s \to t\}$ terminates

deduce $$\frac{\mathcal{E}, \mathcal{R}, \mathcal{C}}{\mathcal{E} \cup \{s \approx t\}, \mathcal{R}, \mathcal{C}}$$
if $s \;{}_{\mathcal{R}}\!\leftarrow u \to_{\mathcal{R}} t$

delete $$\frac{\mathcal{E} \cup \{s \approx s\}, \mathcal{R}, \mathcal{C}}{\mathcal{E}, \mathcal{R}, \mathcal{C}}$$

simplify $$\frac{\mathcal{E} \cup \{s \approx t\}, \mathcal{R}, \mathcal{C}}{\mathcal{E} \cup \{s \approx u\}, \mathcal{R}, \mathcal{C}}$$
if $t \to_{\mathcal{R}} u$

compose $$\frac{\mathcal{E}, \mathcal{R} \cup \{s \to t\}, \mathcal{C}}{\mathcal{E}, \mathcal{R} \cup \{s \to u\}, \mathcal{C}}$$
if $t \to_{\mathcal{R}} u$

collapse $$\frac{\mathcal{E}, \mathcal{R} \cup \{t \to s\}, \mathcal{C}}{\mathcal{E} \cup \{u \approx s\}, \mathcal{R}, \mathcal{C}}$$
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Definition (KBtt)

$\mathcal{E}$: set of equations $\qquad \mathcal{R}, \mathcal{C}$: sets of rewrite rules

inference system KBtt consists of six rules:

orient
$$\frac{\mathcal{E} \cup \{s \approx t\}, \mathcal{R}, \mathcal{C}}{\mathcal{E}, \mathcal{R} \cup \{s \to t\}, \mathcal{C} \cup \{s \to t\}}$$
if $\mathcal{C} \cup \{s \to t\}$ terminates

deduce
$$\frac{\mathcal{E}, \mathcal{R}, \mathcal{C}}{\mathcal{E} \cup \{s \approx t\}, \mathcal{R}, \mathcal{C}}$$
if $s \mathrel{{}_{\mathcal{R}}{\leftarrow}} u \to_{\mathcal{R}} t$

delete
$$\frac{\mathcal{E} \cup \{s \approx s\}, \mathcal{R}, \mathcal{C}}{\mathcal{E}, \mathcal{R}, \mathcal{C}}$$

simplify
$$\frac{\mathcal{E} \cup \{s \approx t\}, \mathcal{R}, \mathcal{C}}{\mathcal{E} \cup \{s \approx u\}, \mathcal{R}, \mathcal{C}}$$
if $t \to_{\mathcal{R}} u$

compose
$$\frac{\mathcal{E}, \mathcal{R} \cup \{s \to t\}, \mathcal{C}}{\mathcal{E}, \mathcal{R} \cup \{s \to u\}, \mathcal{C}}$$
if $t \to_{\mathcal{R}} u$

collapse
$$\frac{\mathcal{E}, \mathcal{R} \cup \{t \to s\}, \mathcal{C}}{\mathcal{E} \cup \{u \approx s\}, \mathcal{R}, \mathcal{C}}$$
if $t \to_{\mathcal{R}} u$

Theorem

Wehrman *et al* 2006

Let $(\mathcal{E}_0, \varnothing, \varnothing) \vdash \ldots \vdash (\varnothing, \mathcal{R}_n, \mathcal{C}_n)$ satisfy $\mathrm{CP}(\mathcal{R}_n) \subseteq \bigcup_{i \geqslant 0} \mathcal{E}_i$.
Then $\mathcal{R}_n$ is convergent.

Knuth-Bendix Completion with Termination Tools

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delete
$$\frac{\mathcal{E} \cup \{s \approx s\}, \mathcal{R}, \mathcal{C}}{\mathcal{E}, \mathcal{R}, \mathcal{C}}$$

ask termination tools
like $\mathsf{T_TT_2}$

plify
$$\frac{\mathcal{E} \cup \{s \approx t\}, \mathcal{R}, \mathcal{C}}{\mathcal{E} \cup \{s \approx u\}, \mathcal{R}, \mathcal{C}}$$
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compose
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Example

Example (CGE_2)

KBtt-based tool Slothrop was first to complete theory of two commuting group endomorphisms:

$$\mathsf{e} \cdot x \approx x \qquad x^{-} \cdot x \approx \mathsf{e} \qquad (x \cdot y) \cdot z \approx x \cdot (y \cdot z)$$

$$\mathsf{f}(x \cdot y) \approx \mathsf{f}(x) \cdot \mathsf{f}(y) \qquad \mathsf{g}(x \cdot y) \approx \mathsf{g}(x) \cdot \mathsf{g}(y) \qquad \mathsf{f}(x) \cdot \mathsf{g}(y) \approx \mathsf{g}(y) \cdot \mathsf{f}(x)$$

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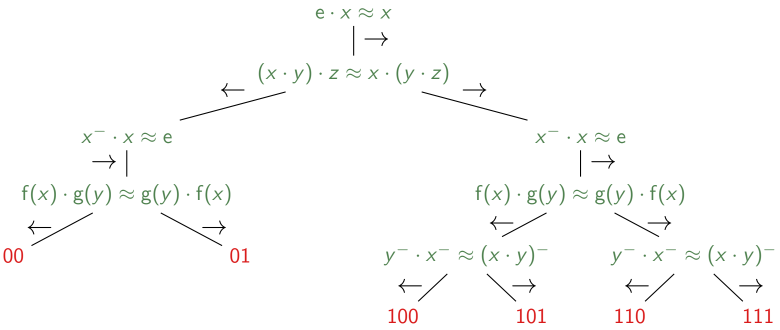
\end{array}
$$

Which Way to Go?

Example



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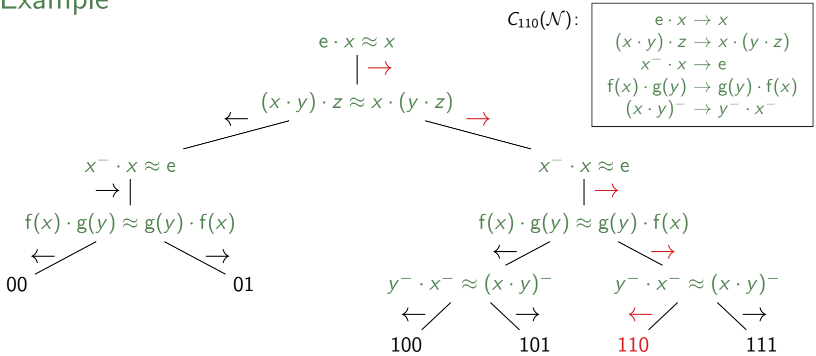

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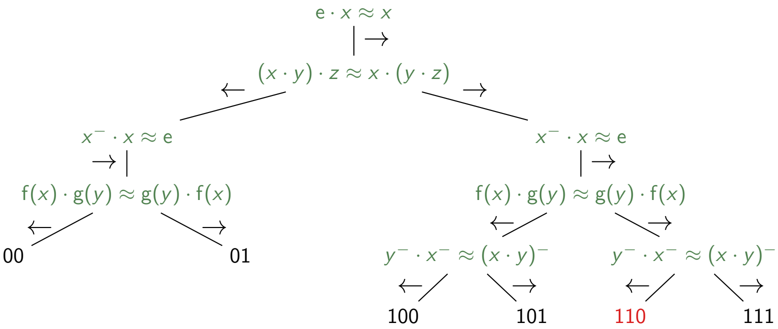

Idea: Multi-Completion

Kondo & Kurihara 99

- branches correspond to processes

Which Way to Go?

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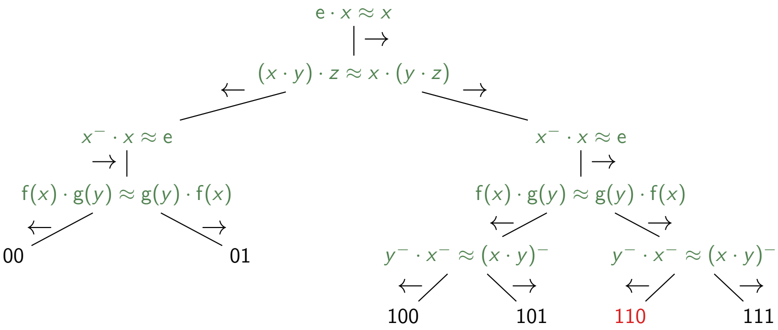

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Kondo & Kurihara 99

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- simulate multiple processes in parallel

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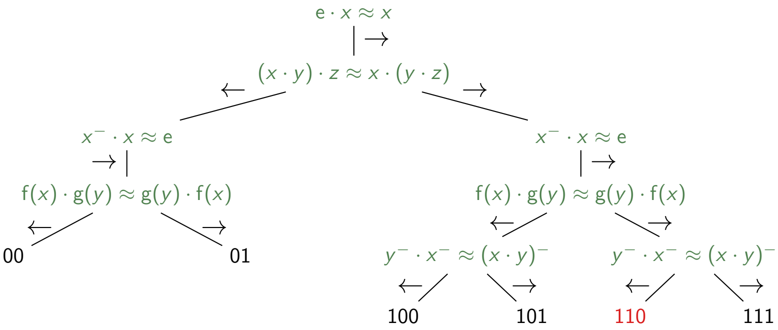

Idea: Multi-Completion

Kondo & Kurihara 99

- branches correspond to processes
- simulate multiple processes in parallel
- exploit sharing to gain efficiency

Multi-Completion with Termination Tools

Definition (MKBtt node)

node is tuple $\langle s : t, R_0, R_1, E, C_0, C_1 \rangle$ such that

- s, t are terms
- labels R_0, R_1, E, C_0, C_1 are sets of processes

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- labels if $p \in R_1$ then $t \to s \in \mathcal{R}_p$ sets of processes

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Definition (orient in MKBtt)

orient

$$\frac{\mathcal{N} \cup \{\langle s : t, R_0, R_1, E, C_0, C_1 \rangle\}}{\{\langle s : t, R_0 \qquad , R_1 \qquad , \qquad , C_0 \qquad , C_1 \qquad \rangle\}}$$

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- $E_{lr}, E_{rl} \subseteq E$

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- $E_{lr}, E_{rl} \subseteq E$
- $\mathcal{C}_p(\mathcal{N}) \cup \{s \to t\}$ terminates for all $p \in E_{lr}$,
  $\mathcal{C}_p(\mathcal{N}) \cup \{t \to s\}$ terminates for all $p \in E_{rl}$

Multi-Completion with Termination Tools

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orient $$\frac{\mathcal{N} \cup \{\langle s : t, R_0, R_1, E, C_0, C_1 \rangle\}}{\{\langle s : t, R_0 \qquad , R_1 \qquad , E', C_0 \qquad , C_1 \qquad \rangle\}}$$

- $E_{lr}, E_{rl} \subseteq E$ and $E' = E \setminus (E_{lr} \cup E_{rl})$
- $\mathcal{C}_p(\mathcal{N}) \cup \{s \to t\}$ terminates for all $p \in E_{lr}$,
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Multi-Completion with Termination Tools

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- $R_{lr} = (E_{lr} \setminus E_{rl}) \cup \{p0 \mid p \in (E_{rl} \cap E_{lr})\}$,
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orient $$\frac{\mathcal{N} \cup \{\langle s : t, R_0, R_1, E, C_0, C_1 \rangle\}}{\mathit{split}(\mathcal{N}) \cup \{\langle s : t, R_0 \cup R_{lr}, R_1 \cup R_{rl}, E', C_0 \cup R_{lr}, C_1 \cup R_{rl} \rangle\}}$$

- $E_{lr}, E_{rl} \subseteq E$ and $E' = E \setminus (E_{lr} \cup E_{rl})$
- $\mathcal{C}_p(\mathcal{N}) \cup \{s \to t\}$ terminates for all $p \in E_{lr}$,
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- $R_{lr} = (E_{lr} \setminus E_{rl}) \cup \{p0 \mid p \in (E_{rl} \cap E_{lr})\}$,
  $R_{rl} = (E_{rl} \setminus E_{lr}) \cup \{p1 \mid p \in (E_{rl} \cap E_{lr})\}$
- $\mathit{split}(\mathcal{N})$ replaces every $p \in E_{rl} \cap E_{lr}$ by $p0$, $p1$

Examples

Example (CGE_2)

KBtt-based tool Slothrop was first to complete theory of two commuting group endomorphisms:

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Example (CGE_n)

mkb_{TT} completes theory CGE_n for $n \leq 5$:

$$\begin{array}{ccc} e \cdot x \approx x & x^- \cdot x \approx e & (x \cdot y) \cdot z \approx x \cdot (y \cdot z) \\ f_i(x \cdot y) \approx f_i(x) \cdot f_i(y) & \text{for } 1 \leq i \leq n & \\ f_i(x) \cdot f_j(y) \approx f_j(y) \cdot f_i(x) & \text{for } 1 \leq i < j \leq n & \end{array}$$

Examples

Example (CGE_2)

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$$\begin{array}{ccc} \mathsf{e} \cdot x \approx x & x^- \cdot x \approx \mathsf{e} & (x \cdot y) \cdot z \approx x \cdot (y \cdot z) \\ \mathsf{f}(x \cdot y) \approx \mathsf{f}(x) \cdot \mathsf{f}(y) & \mathsf{g}(x \cdot y) \approx \mathsf{g}(x) \cdot \mathsf{g}(y) & \mathsf{f}(x) \cdot \mathsf{g}(y) \approx \mathsf{g}(y) \cdot \mathsf{f}(x) \end{array}$$

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- used in decision procedures for equality with uninterpreted functions

Ordered Completion

Limitation

KB fails if unorientable equation like $x \cdot y \approx y \cdot x$ persists – even if convergent TRS exists!

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such that $\mathcal{E}_\succ \cup \mathcal{R}$ is ground-convergent and $\leftrightarrow^*_{\mathcal{E}_0} = \leftrightarrow^*_{\mathcal{E} \cup \mathcal{R}}$

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orientable instances of $\mathcal{E}$

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Example (TPTP-GRP451-1)

can be handled by OKB using transfinite KBO $\succ$:

$$\begin{aligned} y &\approx \mathsf{d}(\mathsf{d}(\mathsf{d}(x,x),\mathsf{d}(x,\mathsf{d}(y,\mathsf{d}(\mathsf{d}(\mathsf{d}(x,x),x),z)))),z) \\ x \cdot y &\approx \mathsf{d}(x,\mathsf{d}(\mathsf{d}(z,z),y)) \\ x^{-1} &\approx \mathsf{d}(\mathsf{d}(y,y),x) \end{aligned}$$

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Bachmair *et al* 1989

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if $\mathcal{C} \cup \{s \to t\}$ is totally terminating

collapse$_2$ $$\frac{\mathcal{E}, \mathcal{R} \cup \{t \to s\}, \mathcal{C}}{\mathcal{E} \cup \{u \approx s\}, \mathcal{R}, \mathcal{C} \cup \{\ell \to r\}}$$
if $t \to_{\mathcal{E}_\succ} u$ using $\ell \to r$ and $\mathcal{C} \cup \{\ell \to r\}$ is totally terminating

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Theorem

Let $(\mathcal{E}_0, \varnothing, \varnothing) \vdash \ldots \vdash (\mathcal{E}_n, \mathcal{R}_n, \mathcal{C}_n)$ satisfy $\mathrm{CP}_\rhd(\mathcal{R}_n) \subseteq \bigcup_{i \geqslant 0} \mathcal{E}_i$.
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Let $(\mathcal{E}_0, \varnothing, \varnothing) \vdash \dots \vdash (\mathcal{E}_n, \mathcal{R}_n, \mathcal{C}_n)$ satisfy $\mathrm{CP}_\rhd(\mathcal{R}_n) \subseteq \bigcup_{i \geqslant 0} \mathcal{E}_i$.
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Remarks

- OKBtt can be combined with multi-completion

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Theorem

Let $(\mathcal{E}_0, \varnothing, \varnothing) \vdash \ldots \vdash (\mathcal{E}_n, \mathcal{R}_n, \mathcal{C}_n)$ satisfy $\mathrm{CP}_\rhd(\mathcal{R}_n) \subseteq \bigcup_{i \geqslant 0} \mathcal{E}_i$.
Then $(\mathcal{E}_n)_\succ \cup \mathcal{R}_n$ is ground-convergent for $\succ = \to^+_{\mathcal{C}_n}$.

Remarks

- OKBtt can be combined with multi-completion
- applicable termination techniques are severely restricted, critical pairs have to be over-approximated

Normalized Completion

Limitation

if input contains $\mathrm{AC} = \{x \cdot y \approx y \cdot x,\ x \cdot (y \cdot z) \approx (x \cdot y) \cdot z\}$ then KB fails and OKB is inefficient

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- succeeds if $\mathcal{R}$ is $\mathcal{S}$-convergent and $\leftrightarrow^*_{\mathcal{E} \cup \mathcal{S} \cup \mathsf{AC}} = \leftrightarrow^*_{\mathcal{R} \cup \mathcal{S} \cup \mathsf{AC}}$
- may also fail or loop

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- may also fail or loop

Definition (Normalized Rewriting)

- $t \to_{\mathcal{R}\backslash\mathcal{S}} u$ if $t \to^!_{\mathcal{S}/\mathsf{AC}} \cdot \to_{\ell\to r/\mathsf{AC}} u$ for some $\ell \to r$ in $\mathcal{R}$

Normalized Completion

Limitation

if input contains $AC = \{x \cdot y \approx y \cdot x,\ x \cdot (y \cdot z) \approx (x \cdot y) \cdot z\}$ then KB fails and OKB is inefficient

$$\underset{\text{AC-convergent theory}}{\mathcal{S}} + \underset{\text{equations}}{\mathcal{E}} + \underset{\text{AC-compatible reduction order}}{\succ} \longrightarrow_{NKB} \underset{\text{rewrite system}}{\mathcal{R}}$$

- succeeds if $\mathcal{R}$ is $\mathcal{S}$-convergent and $\leftrightarrow^*_{\mathcal{E}\cup\mathcal{S}\cup\mathsf{AC}} = \leftrightarrow^*_{\mathcal{R}\cup\mathcal{S}\cup\mathsf{AC}}$
- may also fail or loop

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- $\mathcal{R}$ is $\mathcal{S}$-convergent for $\mathcal{E}$ if $\to_{\mathcal{R}\backslash\mathcal{S}}$ is AC-terminating and $\leftrightarrow^*_{\mathcal{E}\,\cup\,\mathcal{S}\cup\,\mathsf{AC}} = \to^!_{\mathcal{R}\backslash\mathcal{S}} \cdot \leftrightarrow^*_{\mathcal{S}\cup\mathsf{AC}} \cdot {}_{\mathcal{R}\backslash\mathcal{S}}^{\,!}\!\leftarrow$

Definition (NKB)

$\mathcal{E}$: set of equations $\mathcal{R}$: set of rewrite rules $\succ$: AC-reduction order

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inference system NKB consists of seven rules, including:

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(Θ, Ψ) form $\mathcal{S}$-normalizing pair

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Theorem

Marché 1996

Let $(\mathcal{E}_0, \varnothing) \vdash \ldots \vdash (\varnothing, \mathcal{R}_n)$ satisfy $\mathrm{CP}_{\mathrm{AC}}(\mathcal{R}_n, \mathcal{R}_n^e) \subseteq \bigcup_{i \geqslant 0} \mathcal{E}_i$.
Then $\mathcal{R}_n$ is $\mathcal{S}$-convergent.

Definition (NKBtt)

$\mathcal{E}$: set of equations $\quad \mathcal{R}, \mathcal{C}$: sets of rewrite rules

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if $\mathcal{C} \cup \mathcal{S} \cup \Psi(s,t)$ is AC-terminating

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Then $\mathcal{R}_n$ is $\mathcal{S}$-convergent.

Remark

NKBtt can be combined with multi-completion

Example: CGA

$\mathsf{mkb_{TT}}$ using MuTerm can apply normalized completion to handle theory of commuting group action:

$$
\begin{array}{ccc}
x \cdot \mathsf{e} \approx x & (x \cdot y) \cdot z \approx x \cdot (y \cdot z) & x \cdot x^{-1} \approx \mathsf{e} \\
\phi(\mathsf{e}, x) \approx x & \phi(x, \phi(y, z)) \approx \phi(x \cdot y, z) & \phi(\mathsf{f}(x), \mathsf{g}(y)) \approx \phi(\mathsf{g}(y), \mathsf{f}(x)) \\
\mathsf{f}(\mathsf{e}) \approx \mathsf{e} & \mathsf{f}(x \cdot y) \approx \mathsf{f}(x) \cdot \mathsf{f}(y) & x \cdot y \approx y \cdot x \\
\mathsf{g}(\mathsf{e}) \approx \mathsf{e} & \mathsf{g}(x \cdot y) \approx \mathsf{g}(x) \cdot \mathsf{g}(y) &
\end{array}
$$

Implementation: mkb_{TT}

- standard, normalized, and ordered multi-completion with termination tools

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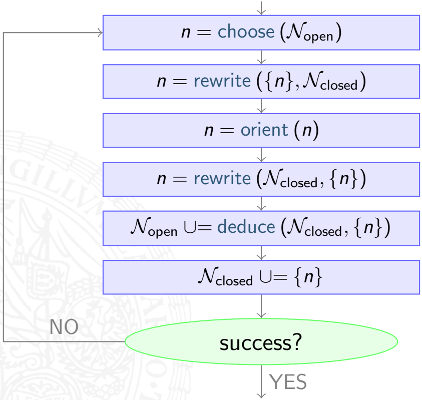

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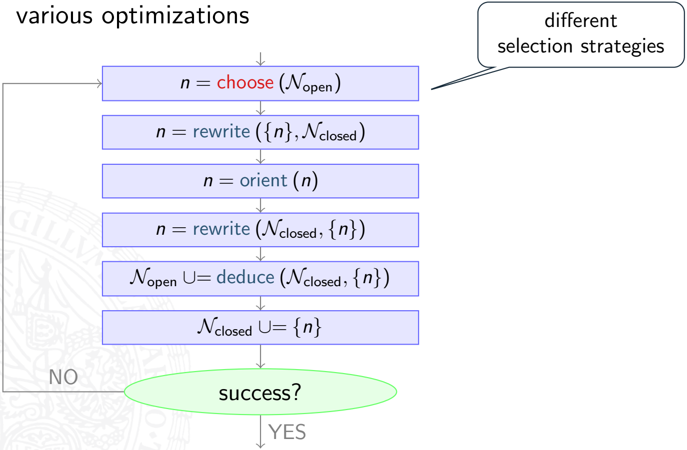

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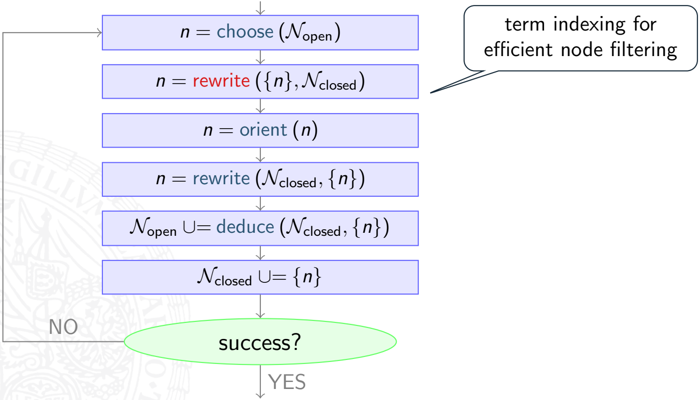

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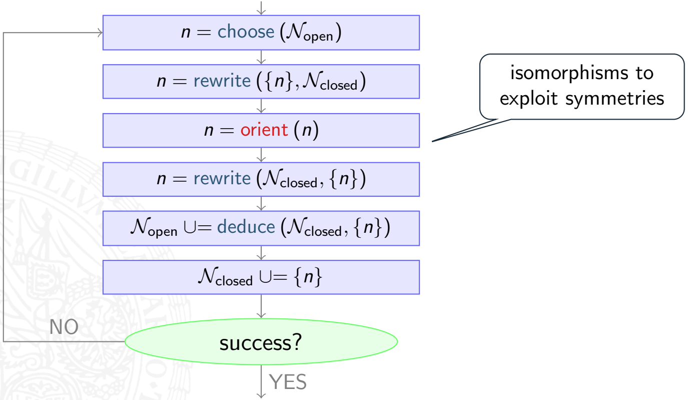

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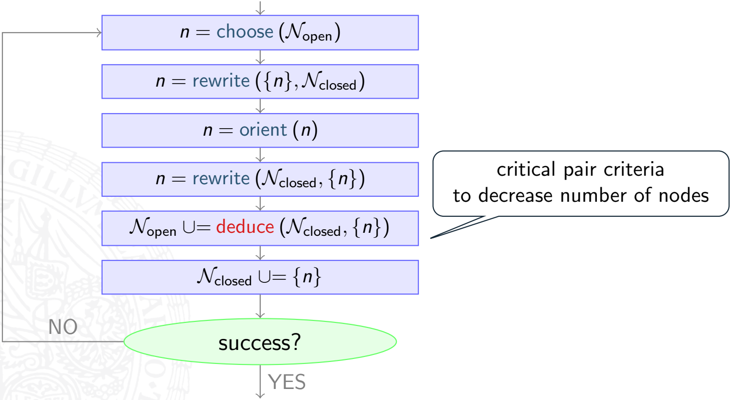

Results

Knuth-Bendix Completion

	mkb_{TT}	Slothrop	KBCV	Maxcomp
BGK94-M_{12}	∞	38.8	6.0	39.6
SK90-3.26	∞	∞	20.9	∞
SK90-3.28	223.8	436.6	∞	15.9
TPTP-GRP454-1	9.6	∞	6.2	2.0
WS06-proofreduct	237.9	208.2	∞	∞
WSW06-equiv-proofs	7.3	33.5	∞	∞
#successes (A)	87	76	87	86

- completion time in seconds, ∞ is timeout (600 seconds)
- (A) 115 problems collected from the literature

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#successes (A)	87	76	87	86
#successes (B)	1109			821

- completion time in seconds, ∞ is timeout (600 seconds)
- (A) 115 problems collected from the literature
  (B) all 3061 non-convergent TRSs in standard category of TPDB 7

Results

Ordered Completion

	lpo	kbo	tkbo	lpo+kbo	total
KH11-fib	1.8	∞	∞	2.6	15.7
KH11-rl-theory	4.3	244.9	293.2	4.5	10.5
TPTP-GRP445-1	∞	5.8	11.4	5.5	11.7
TPTP-GRP452-1	∞	∞	192.1	∞	∞
Example 13	∞	∞	0.2	∞	0.1
#successes (C)	89	88	81	96	90

- completion time in seconds, ∞ is timeout (600 seconds)
- (C) 138 problems collected from the literature

Results

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Theorem Proving

	total	kbo	lpo	dp+lpo	Waldmeister
#successes (D)	116	148	152	121	> 400
#successes (E)	149	163	164	138	> 200

- completion time in seconds, ∞ is timeout (600 seconds)
- (C) 138 problems collected from the literature
- (D) 565 *difficult* and (E) 215 *easy* UEQ problems in TPTP 3.6.0

Results

Normalized Completion

theory $\mathcal{S}$	mkb$_{TT}$ AC	mkb$_{TT}$ AG	mkb$_{TT}$ auto	C*i*ME
G94-abelian groups (AG)	1.6	0.1	0.1	0.05
AG + homomorphism	181.7	4.8	4.8	0.05
LS96-G0	1.9	0.1	0.1	?
LS96-G1	∞	12.4	12.5	?
G94-arithmetic	14.9	–	13.8	?
G94-AC-ring with unit	22.9	7.2	0.1	0.1
MU04-binary arithmetic	2.9	–	3.0	?
MU04-ternary arithmetic	18.1	–	17.3	?
CGA	∞	15.4	15.2	?
CRE	∞	216.7	145.1	?
#successes (F)	10	7	13	4

- completion time in seconds, ∞ is timeout (600 seconds)
- ?: no suitable reduction order for C*i*ME
- (F) 20 problems collected from the literature

Contributions: Completion

H. Sato, S. Winkler, M. Kurihara, and A. Middeldorp.
Multi-completion with Termination Tools (System Description).
In *Proc. IJCAR 2008*, volume 5195 of *LNCS*, pp 306–312, 2008.

S. Winkler, H. Sato, A. Middeldorp, and M. Kurihara.
Optimizing mkbTT (System Description).
In *Proc. RTA 2010*, LIPIcs 13, pp 373–384, 2010.

S. Winkler and A. Middeldorp.
Termination Tools in Ordered Completion.
In *Proc. IJCAR 2010*, volume 6173 of *LNCS*, pp 518–532, 2010.

S. Winkler and A. Middeldorp.
AC Completion with Termination Tools.
In *Proc. CADE-23*, volume 6803 of *LNCS*, pp 492–498, 2011

S. Winkler, H. Sato, A. Middeldorp, and M. Kurihara.
Multi-Completion with Termination Tools.
Journal of Automated Reasoning 50(3):317–354, 2013.

Contributions: Some Other Topics

- C. Sternagel, R. Thiemann, S. Winkler, and H. Zankl.
  CeTA—A Tool for Certified Termination Analysis.
  In *Proc. WST 2009*, pp 84–87, 2009.
- H. Zankl, S. Winkler, and A. Middeldorp.
  Automating Ordinal Interpretations.
  In *Proc. WST 2012*, pp 94–98, 2012.
- S. Winkler, H. Zankl, and A. Middeldorp.
  Ordinals and Knuth-Bendix Orders.
  In *Proc. LPAR 2012*, volume 7180 of *LNCS*, pp 420–434, 2012.

Conclusion

Theory

- inference systems for standard, ordered, and normalized completion combining termination tools with multi-completion
- normalized completion & ordered completion: simplified collapse rules, correctness of critical pair criteria
- normalized completion: new notions for fairness, normalizing pairs

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- inference systems for standard, ordered, and normalized completion combining termination tools with multi-completion
- normalized completion & ordered completion: simplified collapse rules, correctness of critical pair criteria
- normalized completion: new notions for fairness, normalizing pairs

Implementation

- $\mathsf{mkb}_{\mathsf{TT}}$: state-of-the-art completion-based theorem prover
- first fully automatic tool for ordered and normalized completion
- novel convergent systems

Critical Pairs

Let $\ell_1 \to r_1$, $\ell_2 \to r_2$ be variable-disjoint.

Definition

$\langle \ell_1 \to r_1, p, \ell_2 \to r_2 \rangle_\sigma$ is overlap if $p \in \mathcal{P}os_{\mathcal{F}}(\ell_2)$, $\sigma = \text{mgu}(\ell_1, \ell_2|_p)$ and $p \neq \epsilon$ if $\ell_1 \to r_1 \doteq \ell_2 \to r_2$. Then $\ell_2\sigma[r_1\sigma]_p \approx r_2\sigma$ is critical pair

Definition

Let $\mathcal{E}$ set of equations, $\mathcal{R}$ set of rewrite rules, $\succ$ ground-total.
$\langle \ell_1 \approx r_1, p, \ell_2 \approx r_2 \rangle_\sigma$ is extended overlap if $\ell_1 \simeq r_1, \ell_2 \simeq r_2 \in \mathcal{E} \cup \mathcal{R}$, $\sigma = \text{mgu}(\ell_1, \ell_2|_p)$, and $r_i\sigma \not\succ \ell_i\sigma$ for $i \in \{1, 2\}$.
$\ell_2\sigma[r_1\sigma]_p \approx r_2\sigma$ constitutes an extended critical pair

Definition

$\mathcal{T}$-overlap is $\langle \ell_1 \to r_1, p, \ell_2 \to r_2 \rangle_\Sigma$ such that $p \in \mathcal{P}os_{\mathcal{F}}(\ell_2)$ and Σ is complete set of $\mathcal{T}$-unifiers of $\ell_2|_p$ and ℓ_1.
$\ell_2\sigma[r_1\sigma]_p \approx r_2\sigma$ constitutes $\mathcal{T}$-critical pair $\forall \sigma \in \Sigma$

Definition (extended rules)

$\mathcal{R}^e = \mathcal{R} \cup \{f(\ell, x) \to f(r, x) \mid \ell \to r \in \mathcal{R},\ \mathrm{root}(\ell) = f \text{ and } f \in \mathcal{F}_{\mathrm{AC}}$

Definition (normalizing pair)

(Θ, Ψ) constitutes $\mathcal{S}$-normalizing pair for terms u, v if

(i) $\Theta(u, v)$ and $\Psi(u, v)$ are in $\leftrightarrow^*_{\mathcal{E}\cup\mathcal{R}\cup\mathcal{S}\cup\mathcal{T}}$,

(ii) $\Psi(u, v) \subseteq \succ$,

(iii) for $P:\ s \xleftrightarrow[u\approx v]{\epsilon,\sigma} t$ $\quad \exists Q$ in $(\mathcal{S}, \Theta(u, v), \Psi(u, v))$ such that $P \Rightarrow Q$

(iv) for all $\mathcal{R}$, $\ell \to r$ in $\Psi(u, v)$ and $P:\ s\ {}_{\mathcal{S}}\!\leftarrow w \leftrightarrow^*_{\mathrm{AC}} \cdot \to_{\ell\to r} \cdot \to^*_{\mathcal{R}\setminus\mathcal{S}} t$ $\exists Q$ in $(\mathcal{S}, \Theta(u, v), \Psi(u, v) \cup \mathcal{R})$ such that $P \Rightarrow Q$ and terms in Q are smaller than w

Back