Beyond Peano Arithmetic

Automatically proving termination of the Goodstein sequence

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Overview

- Goodstein Sequence
- Ordinal Interpretations
- Automation
- Conclusion

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Example (Hereditary Representation)

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Definitions

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$$(\alpha)_n = n^{(\alpha_k)_n} \cdot a_k + n^{(\alpha_{k-1})_n} \cdot a_{k-1} + \cdots + n^{(\alpha_0)_n} \cdot a_0$$

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$$g_\alpha(0) = \alpha \qquad g_\alpha(i+1) = \begin{cases} (g_\alpha(i))_{i+2}^{i+3} - 1 & \text{if } g_\alpha(i) > 0 \\ 0 & \text{otherwise} \end{cases}$$

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Corollary

g *is not multiple recursive*

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Example

$(1)_2 = 2^0 \cdot 1$ $\quad$ $(3)_2 = 2^1 \cdot 1 + 2^0 \cdot 1$ $\quad$ $(3)_3 = 3^1 \cdot 1$

$[1]_2 = \mathsf{c}(0,0)$ $\quad$ $[3]_2 = \mathsf{c}(0,\mathsf{c}(\mathsf{c}(0,0),0))$ $\quad$ $[3]_3 = \mathsf{c}(\mathsf{c}(0,0),0)$

Definition (TRS $\mathcal{G}$)

- represent natural numbers in hereditary base n with binary c and constant 0
- additional symbols $\bullet$, $\llbracket\,\rrbracket$, $\circ$ (unary) f, h (binary)

$$
\begin{aligned}
\llbracket\,\rrbracket \circ x &\to \circ \llbracket\,\rrbracket\, x & \mathsf{c}(0,x) &\to \circ\, x \\
\bullet \llbracket\,\rrbracket\, x &\to \llbracket\,\rrbracket \bullet \bullet\, x & \bullet\, \mathsf{f}(0,x) &\to \circ\, x \\
\circ\, x &\to \bullet \llbracket\,\rrbracket\, x & \bullet\, \mathsf{h}(x,y) &\to \mathsf{h}(\bullet\, x, \bullet \bullet\, \mathsf{c}(x,y)) \\
\bullet\, x &\to x & \bullet\, \mathsf{f}(x,y) &\to \mathsf{f}(\bullet\, x, y) \\
\circ\, x &\to x & \bullet\, \mathsf{c}(x,y) &\to \mathsf{c}(\bullet\, x, \bullet\, y) \\
\bullet\, \mathsf{c}(\mathsf{c}(x,y),z) &\to \bullet\, \mathsf{f}(\mathsf{c}(x,y),z) & \mathsf{h}(x,y) &\to \circ\, y \\
\bullet\, \mathsf{f}(\mathsf{c}(x,y),z) &\to \mathsf{h}(\bullet\, \mathsf{f}(x,y), \bullet \bullet\, \mathsf{f}(\mathsf{f}(x,y),z)) & &
\end{aligned}
$$

Example

$$(1)_2 = 2^0 \cdot 1 \qquad (3)_2 = 2^1 \cdot 1 + 2^0 \cdot 1 \qquad (3)_3 = 3^1 \cdot 1$$

$$[1]_2 = \mathsf{c}(0,0) \qquad [3]_2 = \mathsf{c}(0, \mathsf{c}(\mathsf{c}(0,0),0)) \qquad [3]_3 = \mathsf{c}(\mathsf{c}(0,0),0)$$

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\bullet x &\to x & \bullet \mathsf{f}(x,y) &\to \mathsf{f}(\bullet x, y) \\
\circ x &\to x & \bullet \mathsf{c}(x,y) &\to \mathsf{c}(\bullet x, \bullet y) \\
\bullet \mathsf{c}(\mathsf{c}(x,y),z) &\to \bullet \mathsf{f}(\mathsf{c}(x,y),z) & \mathsf{h}(x,y) &\to \circ y \\
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$$

Example

$$\bullet [\![[\![\mathsf{c}(0,\mathsf{c}(\mathsf{c}(0,0),0)) \xrightarrow[\mathcal{G}]{+} \bullet [\![[\![[\![\mathsf{c}(\mathsf{c}(0,0),0)$$

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\end{aligned}
$$

Theorem

$\forall\, \alpha > 0\ \forall\, n > 1$

$$\bullet \llbracket^{n} \left[\alpha\right]_{n} \xrightarrow[\mathcal{G}]{+} \bullet \llbracket^{n+1} \left[(\alpha)_{n}^{n+1} - 1\right]_{n+1}$$

Corollary

termination of TRS $\mathcal{G}$ implies Goodstein's theorem

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Claim

TRS $\mathcal{G}$ is terminating

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termination of $\mathcal{G}$ cannot be established by AProVE, μTerm, TTT$_2$ (2012), ...

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termination of $\mathcal{G}$ cannot be established by AProVE, μTerm, $\mathsf{T_TT_2}$ (2012), ...

Conjecture (Schnabl 2012)

derivational complexity of any TRS that can be proved terminating by automatic tool is bounded by multiple recursive function

Battle of Hydra and Hercules

Definition (TRS $\mathcal{H}$) Touzet 1998

$$
\begin{aligned}
\circ\, x &\to \bullet\, [\!]\, x & \mathsf{H}(0,x) &\to \circ\, x \\
\bullet\, [\!]\, x &\to [\!]\, \bullet \bullet\, x & \bullet\, \mathsf{H}(\mathsf{H}(0,y),z) &\to \mathsf{c}^1(y,z) \\
[\!]\, \circ\, x &\to \circ\, [\!]\, x & \bullet\, \mathsf{H}(\mathsf{H}(\mathsf{H}(0,x),y),z) &\to \mathsf{c}^2(x,y,z) \\
\bullet\, x &\to x & \bullet\, \mathsf{c}^1(x,y) &\to \mathsf{c}^1(x,\mathsf{H}(x,y)) \\
\mathsf{c}^1(y,z) &\to \circ\, z & \bullet\, \mathsf{c}^2(x,y,z) &\to \mathsf{c}^2(x,\mathsf{H}(x,y),z) \\
\mathsf{c}^2(x,y,z) &\to \circ\, \mathsf{H}(y,z)
\end{aligned}
$$

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$$\begin{aligned} \circ\, x &\to \bullet\, [\!]\, x \\ \bullet\, [\!]\, x &\to [\!]\, \bullet \bullet\, x \\ [\!]\, \circ\, x &\to \circ\, [\!]\, x \\ \bullet\, x &\to x \\ \mathsf{c}^1(y,z) &\to \circ\, z \\ \mathsf{c}^2(x,y,z) &\to \circ\, \mathsf{H}(y,z) \end{aligned} \qquad \begin{aligned} \mathsf{H}(0,x) &\to \circ\, x \\ \bullet\, \mathsf{H}(\mathsf{H}(0,y),z) &\to \mathsf{c}^1(y,z) \\ \bullet\, \mathsf{H}(\mathsf{H}(\mathsf{H}(0,x),y),z) &\to \mathsf{c}^2(x,y,z) \\ \bullet\, \mathsf{c}^1(x,y) &\to \mathsf{c}^1(x,\mathsf{H}(x,y)) \\ \bullet\, \mathsf{c}^2(x,y,z) &\to \mathsf{c}^2(x,\mathsf{H}(x,y),z) \end{aligned}$$

Remark

$\mathcal{H}$ simulates Hydra battle with specific strategy for hydras up to depth 3

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$$\circ\, x \to \bullet\, [\!]\, x \qquad \mathsf{H}(0,x) \to \circ\, x$$

$$\bullet\, [\!]\, x \to [\!]\, \bullet \bullet\, x \qquad \bullet\, \mathsf{H}(\mathsf{H}(0,y),z) \to \mathsf{c}^1(y,z)$$

$$[\!]\, \circ\, x \to \circ\, [\!]\, x \qquad \bullet\, \mathsf{H}(\mathsf{H}(\mathsf{H}(0,x),y),z) \to \mathsf{c}^2(x,y,z)$$

$$\bullet\, x \to x \qquad \bullet\, \mathsf{c}^1(x,y) \to \mathsf{c}^1(x,\mathsf{H}(x,y))$$

$$\mathsf{c}^1(y,z) \to \circ\, z \qquad \bullet\, \mathsf{c}^2(x,y,z) \to \mathsf{c}^2(x,\mathsf{H}(x,y),z)$$

$$\mathsf{c}^2(x,y,z) \to \circ\, \mathsf{H}(y,z)$$

Remark

$\mathcal{H}$ simulates Hydra battle with specific strategy for hydras up to depth 3

Theorem (Touzet 1998)

TRS $\mathcal{H}$ is terminating

Battle of Hydra and Hercules

Definition (TRS $\mathcal{H}$) Touzet 1998

$$
\begin{aligned}
\circ\, x &\to \bullet\, \square\, x & \mathsf{H}(0,x) &\to \circ\, x \\
\bullet\, \square\, x &\to \square\, \bullet\, \bullet\, x & \bullet\, \mathsf{H}(\mathsf{H}(0,y),z) &\to \mathsf{c}^1(y,z) \\
\square\, \circ\, x &\to \circ\, \square\, x & \bullet\, \mathsf{H}(\mathsf{H}(\mathsf{H}(0,x),y),z) &\to \mathsf{c}^2(x,y,z) \\
\bullet\, x &\to x & \bullet\, \mathsf{c}^1(x,y) &\to \mathsf{c}^1(x,\mathsf{H}(x,y)) \\
\mathsf{c}^1(y,z) &\to \circ\, z & \bullet\, \mathsf{c}^2(x,y,z) &\to \mathsf{c}^2(x,\mathsf{H}(x,y),z) \\
\mathsf{c}^2(x,y,z) &\to \circ\, \mathsf{H}(y,z) & &
\end{aligned}
$$

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Theorem (Touzet 1998)

TRS $\mathcal{H}$ is terminating, with non-multiple recursive derivational complexity

Battle of Hydra and Hercules

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$$
\begin{aligned}
\circ\, x &\to \bullet \,\Box\, x \\
\bullet \,\Box\, x &\to \Box \bullet \bullet\, x \\
\Box \circ x &\to \circ \,\Box\, x \\
\bullet\, x &\to x \\
\mathsf{c}^1(y,z) &\to \circ\, z \\
\mathsf{c}^2(x,y,z) &\to \circ\, \mathsf{H}(y,z)
\end{aligned}
\qquad
\begin{aligned}
\mathsf{H}(0,x) &\to \circ\, x \\
\bullet\, \mathsf{H}(\mathsf{H}(0,y),z) &\to \mathsf{c}^1(y,z) \\
\bullet\, \mathsf{H}(\mathsf{H}(\mathsf{H}(0,x),y),z) &\to \mathsf{c}^2(x,y,z) \\
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\end{aligned}
$$

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although lower than $\mathcal{G}$

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Overview

- Goodstein Sequence
- Ordinal Interpretations
- Automation
- Conclusion

Definitions

algebra $\mathcal{A} = (A, \{f_\mathcal{A} \mid f \in \mathcal{F}\})$ equipped with proper order $>$ is

- weakly monotone if $f_\mathcal{A}(\dots, a_i, \dots) \geqslant f_\mathcal{A}(\dots, b, \dots)$ whenever $a_i > b$

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Theorem (Touzet 1998, Zantema 2001)

TRS compatible with well-founded simple weakly monotone algebra is terminating

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ordinal α is totally ordered transitive set

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Example

$\varnothing \qquad \{\varnothing\} \qquad \{\{\varnothing\}, \varnothing\} \qquad \{\{\{\varnothing\}, \varnothing\}, \{\varnothing\}, \varnothing\} \ \dots$

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0 1 2 3 ...

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Example

$$0 \quad \cup \quad 1 \quad \cup \quad 2 \quad \cup \quad 3 \quad \cup \dots \quad = \omega$$

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$0 \quad 1 \quad 2 \quad 3 \quad \dots \quad \omega \quad \omega + 1$

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$\dots \quad \omega + 4 \quad \dots \quad \omega \cdot 2 \quad \dots \quad \omega^2 + 1 \quad \dots \quad \omega^3 \quad \dots$

$\omega^{10} \quad \dots \quad \omega^{100} \quad \dots \quad \omega^\omega \cdot 4 \quad \dots \quad \omega^{\omega^\omega + 7} \quad \dots$

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Lemma

any ordinal is either

- $\beta = \xi \cup \{\xi\} = \xi + 1$ *successor ordinal*
- $\lambda = \bigcup_{\beta < \lambda} \beta$ *limit ordinal*

Let λ be a non-zero limit ordinal.

Definition (Addition)

1. $\alpha + 0 = \alpha$
2. $\alpha + (\beta + 1) = (\alpha + \beta) + 1$
3. $\alpha + \lambda = \bigcup_{\xi < \lambda} \alpha + \xi$

Definition (Multiplication)

1. $\alpha \cdot 0 = 0$
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Definition (Exponentiation)

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Examples

- $1 + \omega = \bigcup_{\xi < \omega} 1 + \xi = \omega \neq \omega + 1$
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- $2^\omega = \bigcup_{\xi < \omega} 2^\xi = \omega$
- ω^ω

Definition

$\epsilon_0 = \omega^{\omega^{\omega^{\omega^{\cdot^{\cdot^{\cdot}}}}}}$

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Cantor Normal Form (CNF)

every ordinal $\alpha < \epsilon_0$ can be uniquely written as

$$\alpha = \omega^{\alpha_1} + \cdots + \omega^{\alpha_n}$$

such that $\alpha_1 \geqslant \cdots \geqslant \alpha_n$ are in CNF

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$$\omega^{\omega^{\omega}} + \omega^{\omega+\omega+1} + \omega + \omega + \omega + 1 + 1$$

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Definition (Natural Addition)

$\alpha = \omega^{\alpha_1} + \ldots + \omega^{\alpha_n}, \quad \beta = \omega^{\beta_1} + \ldots + \omega^{\beta_m}$ in CNF

$$\alpha \oplus \beta = \omega^{\gamma_1} + \omega^{\gamma_2} + \ldots + \omega^{\gamma_k}$$

such that $\{\gamma_1, \ldots, \gamma_k\} = \{\alpha_1, \ldots, \alpha_n\} \uplus \{\beta_1, \ldots, \beta_m\}$ and $\gamma_1 \geqslant \cdots \geqslant \gamma_k$

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Example

$$1 \oplus \omega = \omega \oplus 1 = \omega + 1$$

Definition

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Remark

addition, natural addition, multiplication, exponentiation are weakly monotone

Theorem

TRS $\mathcal{G}$ is terminating

Proof.

well-founded algebra $\mathcal{A}$ with carrier $\epsilon_0 \times \mathbb{N} \times \mathbb{N}$ and interpretations

$$
\begin{aligned}
0_{\mathcal{A}} &= (0,0,0) & \rrbracket_{\mathcal{A}}(x,m,n) &= (x,2m+2,n)\\
\mathsf{c}_{\mathcal{A}}((x,m,n),(y,k,l)) &= (\omega^x \oplus y+1,0,0) & \circ_{\mathcal{A}}(x,m,n) &= (x,2m+3,n)\\
\mathsf{f}_{\mathcal{A}}((x,m,n),(y,k,l)) &= (\omega^x \oplus y,0,0) & \bullet_{\mathcal{A}}(x,m,n) &= (x,m,n+m+1)\\
\mathsf{h}_{\mathcal{A}}((x,m,n),(y,k,l)) &= (y+\omega^{x+1},0,0)
\end{aligned}
$$

is weakly monotone

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is weakly monotone, simple

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is weakly monotone, simple, and compatible with $\mathcal{G}$:

$$[] \circ x \to \circ [] x$$

translates to

$$(x, 4m+8, n) > (x, 4m+7, n)$$

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is weakly monotone, simple, and compatible with $\mathcal{G}$:

$$\bullet\, c(c(x,y),z) \to \bullet\, f(c(x,y),z)$$

translates to

$$(\omega^{\omega^x \oplus y+1} \oplus z + 1, 0, 1) > (\omega^{\omega^x \oplus y+1} \oplus z, 0, 1)$$

∎

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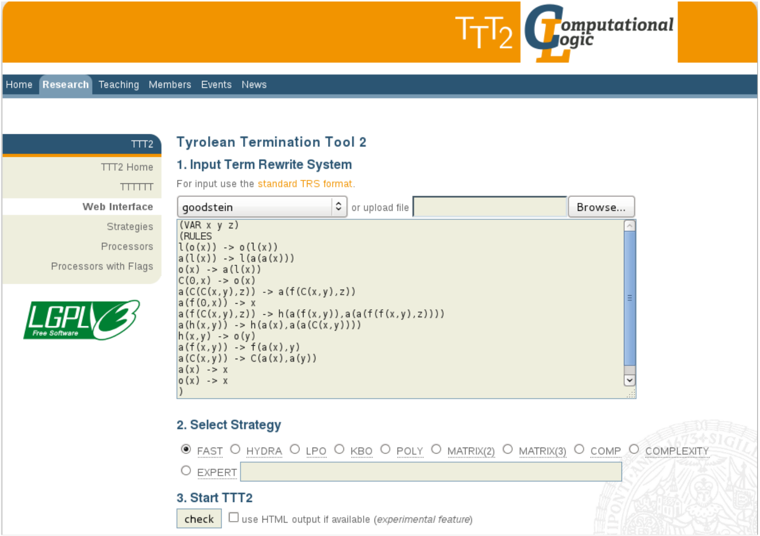
TTT2
Computational Logic
Home
Research
Teaching
Members
Events
News
TTT2
TTT2 Home
TTTTTT
Web Interface
Strategies
Processors
Processors with Flags
LGPLv3
Free Software
Tyrolean Termination Tool 2
1. Input Term Rewrite System
For input use the standard TRS format.
goodstein
or upload file
Browse...
(VAR x y z)
(RULES
l(o(x)) -> o(l(x))
a(l(x)) -> l(a(a(x)))
o(x) -> a(l(x))
C(0,x) -> o(x)
a(C(C(x,y),z)) -> a(f(C(x,y),z))
a(f(0,x)) -> x
a(f(C(x,y),z)) -> h(a(f(x,y)),a(a(f(f(x,y),z))))
a(h(x,y)) -> h(a(x),a(a(C(x,y))))
h(x,y) -> o(y)
a(f(x,y)) -> f(a(x),y)
a(C(x,y)) -> C(a(x),a(y))
a(x) -> x
o(x) -> x
)
2. Select Strategy
FAST
HYDRA
LPO
KBO
POLY
MATRIX(2)
MATRIX(3)
COMP
COMPLEXITY
EXPERT
3. Start TTT2
check
use HTML output if available (experimental feature)

Definition

restricted ordinal expression (ROE) over variables $x_1, \ldots, x_n$

$$f(x_1, \ldots, x_n) \;=\; \left(\sum_{1 \leqslant i \leqslant n} x_i \cdot f_i + \omega^{f'(x_1,\ldots,x_n)} \cdot f_\omega \right) \oplus \bigoplus_{1 \leqslant i \leqslant n} x_i \cdot \hat{f}_i \;\oplus\; f_0$$

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- $f_0, f_1, \ldots, f_n, \hat{f}_1, \ldots, \hat{f}_n, f_\omega$ are (unknowns over) natural numbers

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ROE algebra $\mathcal{O}$ has carrier ϵ_0 and ROEs as interpretation functions

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ROE algebra $\mathcal{O}$ has carrier ϵ_0 and ROEs as interpretation functions

Implementation

- start with parametric ROEs

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- $f_0, f_1, \ldots, f_n, \hat{f}_1, \ldots, \hat{f}_n, f_\omega$ are (unknowns over) natural numbers
- $f'(x_1, \ldots, x_n)$ is ROE over $x_1, \ldots, x_n$

Definition

ROE algebra $\mathcal{O}$ has carrier ϵ_0 and ROEs as interpretation functions

Implementation

- start with parametric ROEs
- encode conditions on coefficients in non-linear integer arithmetic

Definition

restricted ordinal expression (ROE) over variables $x_1, \ldots, x_n$

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Definition

ROE algebra $\mathcal{O}$ has carrier ϵ_0 and ROEs as interpretation functions

Implementation

- start with parametric ROEs
- encode conditions on coefficients in non-linear integer arithmetic
- solve by suitable SMT solver

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restricted ordinal expression (ROE) over variables $x_1, \dots, x_n$

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Major Complications

1. ROE comparison $\qquad f(\vec{x}) > g(\vec{x})$

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Definition

ROE algebra $\mathcal{O}$ has carrier ϵ_0 and ROEs as interpretation functions

Major Complications

1. ROE comparison $f(\vec{x}) > g(\vec{x})$
2. ROEs are not closed under composition $f(g_1(\vec{x}), \ldots, g_n(\vec{x}))$

Complication 1: Comparison of ROEs

Examples

- $\omega^{\omega^x \oplus y+1} \oplus z + 1 > \omega^{\omega^x \oplus y+1} \oplus z$

Complication 1: Comparison of ROEs

Examples

- $\omega^{\omega^x \oplus y+1} \oplus z + 1 > \omega^{\omega^x \oplus y+1} \oplus z$ ✓

Complication 1: Comparison of ROEs

Examples

- $\omega^{\omega^x \oplus y+1} \oplus z + 1 \ > \ \omega^{\omega^x \oplus y+1} \oplus z$ ✓
- $y + \omega^x \ > \ y$

Complication 1: Comparison of ROEs

Examples

- $\omega^{\omega^x \oplus y+1} \oplus z + 1 > \omega^{\omega^x \oplus y+1} \oplus z$ ✓
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- $\omega^{\omega^x \oplus y+1} \oplus z + 1 \; > \; \omega^{\omega^x \oplus y+1} \oplus z$ ✓
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- $\omega^x \oplus y \geqslant y + \omega^x$

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- $\omega^x \oplus y \geqslant y + \omega^x$ ✓
- $x + y \geqslant y + x$

Complication 1: Comparison of ROEs

Examples

- $\omega^{\omega^x \oplus y+1} \oplus z + 1 > \omega^{\omega^x \oplus y+1} \oplus z$ ✓
- $y + \omega^x > y$ ✓
- $1 + \omega \geqslant \omega + 1$ ✗
- $\omega^x \oplus y \geqslant y + \omega^x$ ✓
- $x + y \geqslant y + x$ ✗

skip encoding

$$f(\vec{x}) = \left(\sum_{1 \leqslant i \leqslant n} x_i \cdot f_i + \omega^{f'(\vec{x})} \cdot f_\omega \right) \oplus \bigoplus_{1 \leqslant i \leqslant n} x_i \cdot \hat{f}_i \oplus f_0$$

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Definitions

$$\mathrm{zero}(f(\vec{x})) \;=\; \bigwedge_{0\leqslant i\leqslant n} f_i = 0 \;\wedge\; \bigwedge_{1\leqslant i\leqslant n} \hat{f}_i = 0 \;\wedge\; f_\omega = 0$$

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$$\left[f(\vec{x}) \geqslant_0 g(\vec{x})\right] = \left(\left[f'(\vec{x}) >_0 g'(\vec{x})\right] \wedge f_\omega > 0\right) \vee \left(g_\omega = 0 \wedge f_0 \geqslant g_0\right) \vee \left(\left[f'(\vec{x}) \geqslant_0 g'(\vec{x})\right] \wedge f_\omega \geqslant g_\omega \wedge f_0 \geqslant g_0\right)$$

$$\left[f(\vec{x}) \geqslant_i g(\vec{x})\right] = \left(\neg\, \mathsf{con}(x_i, \omega^{g'(\vec{x})} \cdot g_\omega) \wedge \hat{f}_i \geqslant \hat{g}_i \wedge f_i + \hat{f}_i \geqslant g_i + \hat{g}_i\right) \vee \left(\left(\mathsf{zero}(g'(\vec{x})) \vee g_\omega = 0\right) \wedge f_i + \hat{f}_i \geqslant g_i + \hat{g}_i\right)$$

Complication 2: Composition of ROEs

Example

- $f(x, y) = x + 1$ and $g(x, y) = \omega^y$

Complication 2: Composition of ROEs

Example

- $f(x, y) = x + 1$ and $g(x, y) = \omega^y$

$$f(x, y) + g(x, y) \ = \ x + \omega^y \quad ?$$

Complication 2: Composition of ROEs

Example

- $f(x, y) = x + 1$ and $g(x, y) = \omega^y$

$$\begin{aligned} f(x,y) + g(x,y) &\neq x + \omega^y & \text{if } y = 0 \\ &\neq x + \omega^y + 1 & \text{if } y > 0 \end{aligned}$$

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- $f(x,y) = x + 1$ and $g(x,y) = \omega^y$

$$f(x,y) + g(x,y) \begin{cases} \geqslant x + \omega^y & \text{if } y = 0 \\ \leqslant x + \omega^y + 1 & \text{if } y > 0 \end{cases}$$

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$$f(x,y) + g(x,y) \begin{cases} \neq \omega^{\max(x,y)} & \text{if } x \geqslant y \\ \neq \omega^{\max(x,y)} \cdot 2 & \text{if } x < y \end{cases}$$

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Key Idea

$\forall$ ROE algebra $\mathcal{O}$ $\forall$ term t find ROEs $\mu_{\mathcal{O}}(t)$ and $\nu_{\mathcal{O}}(t)$ such that

$$[\alpha](\mu_{\mathcal{O}}(t)) \leqslant [\alpha](t) \leqslant [\alpha](\nu_{\mathcal{O}}(t))$$

for all assignments α

skip details

$$f(\vec{x}) \; = \; \left(\sum_{1 \leqslant i \leqslant n} x_i \cdot f_i + \omega^{f'(\vec{x})} \cdot f_\omega \right) \oplus \bigoplus_{1 \leqslant i \leqslant n} x_i \cdot \hat{f}_i \; \oplus \; f_0$$

Definition (Bounds for Scalar Multiplication)

if $a = 0$ then $(f \cdot_\mu a)(\vec{x}) = (f \cdot_\nu a)(\vec{x}) = 0$

$$f(\vec{x}) = \left(\sum_{1 \leqslant i \leqslant n} x_i \cdot f_i + \omega^{f'(\vec{x})} \cdot f_\omega\right) \oplus \bigoplus_{1 \leqslant i \leqslant n} x_i \cdot \hat{f}_i \oplus f_0$$

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Example

ROE $f(x,y) = x + y$, $a = 2$

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ROE $f(x, y) = x + y$, $a = 2$

$(f \cdot_\mu 2)(x, y) = x + y$

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Example

ROE $f(x, y) = x + y$, $a = 2$

$(f \cdot_\mu 2)(x, y) = x + y$ $\qquad$ $x \cdot 2 + y \cdot 2 = (f \cdot_\nu 2)(x, y)$

$$f(\vec{x}) \;=\; \left(\sum_{1\leqslant i\leqslant n} x_i \cdot f_i + \omega^{f'(\vec{x})} \cdot f_\omega\right) \oplus \bigoplus_{1\leqslant i\leqslant n} x_i \cdot \hat{f}_i \,\oplus\, f_0$$

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ROE $f(x,y) = x + y$, $a = 2$

$$(f \cdot_\mu 2)(x,y) = x + y \;\leqslant\; (x+y)\cdot 2 \;\leqslant\; x \cdot 2 + y \cdot 2 = (f \cdot_\nu 2)(x,y)$$

Definition (Bounds for Addition)

$$
\begin{aligned}
(f +_\mu g)(\vec{x}) &= \left[\omega^{g'(\vec{x})} \cdot g_\omega > \omega^{f'(\vec{x})} \cdot f_\omega\right] \mathrel{?} g(\vec{x}) : f(\vec{x}) \\
(f +_\nu g)(\vec{x}) &= \left(\left[g'(\vec{x}) > f'(\vec{x})\right] \wedge g_\omega > 0\right) \mathrel{?} \phi_1 : \\
&\qquad \left(\left[\omega^{f'(\vec{x})} \cdot f_\omega > \omega^{g'(\vec{x})} \cdot g_\omega\right] \mathrel{?} \phi_2 : (f \oplus_\nu g)(\vec{x})\right)
\end{aligned}
$$

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\end{aligned}
$$

where

$$
\phi_1 = \left(\sum_{1 \leqslant i \leqslant n} x_i \cdot (f_i a_i b_i + \hat{f}_i b_i + g_i b_i) + \omega^{g'(\vec{x})} \cdot g_\omega\right) \oplus \bigoplus_{1 \leqslant i \leqslant n} x_i \cdot \hat{g}_i \oplus c_0
$$

$$
\phi_2 = \left(\sum_{1 \leqslant i \leqslant n} x_i \cdot f_i a_i + \omega^{f'(\vec{x})} \cdot (f_\omega + 1)\right) \oplus \bigoplus_{1 \leqslant i \leqslant n} x_i \cdot (\hat{f}_i b_i + g_i b_i + \hat{g}_i) \oplus c_0
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$$
\begin{aligned}
\phi_1 &= \left(\sum_{1 \leqslant i \leqslant n} x_i \cdot (f_i a_i b_i + \hat{f}_i b_i + g_i b_i) + \omega^{g'(\vec{x})} \cdot g_\omega\right) \oplus \bigoplus_{1 \leqslant i \leqslant n} x_i \cdot \hat{g}_i \oplus c_0 \\
\phi_2 &= \left(\sum_{1 \leqslant i \leqslant n} x_i \cdot f_i a_i + \omega^{f'(\vec{x})} \cdot (f_\omega + 1)\right) \oplus \bigoplus_{1 \leqslant i \leqslant n} x_i \cdot (\hat{f}_i b_i + g_i b_i + \hat{g}_i) \oplus c_0 \\
a_i &= \neg\,\mathrm{con}(x_i, \omega^{f'(\vec{x})} \cdot f_\omega) \\
b_i &= \neg\,\mathrm{con}(x_i, \omega^{g'(\vec{x})} \cdot g_\omega)
\end{aligned}
$$

Definition (Bounds for Addition)

$$
\begin{aligned}
(f +_\mu g)(\vec{x}) &= \left[\omega^{g'(\vec{x})} \cdot g_\omega > \omega^{f'(\vec{x})} \cdot f_\omega\right] \mathbin{?} g(\vec{x}) : f(\vec{x}) \\
(f +_\nu g)(\vec{x}) &= \left(\left[g'(\vec{x}) > f'(\vec{x})\right] \wedge g_\omega > 0\right) \mathbin{?} \phi_1 : \\
&\qquad \left(\left[\omega^{f'(\vec{x})} \cdot f_\omega > \omega^{g'(\vec{x})} \cdot g_\omega\right] \mathbin{?} \phi_2 : (f \oplus_\nu g)(\vec{x})\right)
\end{aligned}
$$

where

$$
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b_i &= \neg\, \mathrm{con}(x_i, \omega^{g'(\vec{x})} \cdot g_\omega) \\
c_0 &= \left(\left[g'(\vec{x}) > 0\right] \wedge g_\omega > 0\right) \mathbin{?} g_0 : f_0 + g_0
\end{aligned}
$$

Definition (Bounds for Composition)

$$f(\vec{g})_{\mu}(\vec{x}) = \left(\sum_{1 \leqslant i \leqslant n}^{\mu} g_i(\vec{x}) \cdot_{\mu} f_i +_{\mu} \omega^{f'(\vec{g})_{\mu}(\vec{x})} \cdot f_{\omega} \right) \oplus_{\mu} \bigoplus_{1 \leqslant i \leqslant n}^{\mu} g_i(\vec{x}) \cdot_{\mu} \hat{f}_i \oplus_{\mu} f_0$$

$$f(\vec{g})_{\nu}(\vec{x}) = \left(\sum_{1 \leqslant i \leqslant n}^{\nu} g_i(\vec{x}) \cdot_{\nu} f_i +_{\nu} \omega^{f'(\vec{g})_{\nu}(\vec{x})} \cdot f_{\omega} \right) \oplus_{\nu} \bigoplus_{1 \leqslant i \leqslant n}^{\nu} g_i(\vec{x}) \cdot_{\nu} \hat{f}_i \oplus_{\nu} f_0$$

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Definition (Bounds for Term Interpretations)

$\forall$ ROE algebra $\mathcal{O}$ $\forall$ term t

$$\mu_{\mathcal{O}}(t) = \begin{cases} t & \text{if } t \in \mathcal{V} \\ f_{\mathcal{O}}(\mu_{\mathcal{O}}(t_1), \ldots, \mu_{\mathcal{O}}(t_n))_\mu & \text{if } t = f(t_1, \ldots, t_n) \end{cases}$$

$$\nu_{\mathcal{O}}(t) = \begin{cases} t & \text{if } t \in \mathcal{V} \\ f_{\mathcal{O}}(\nu_{\mathcal{O}}(t_1), \ldots, \nu_{\mathcal{O}}(t_n))_\nu & \text{if } t = f(t_1, \ldots, t_n) \end{cases}$$

Key Lemma

$\forall$ ROE algebra $\mathcal{O}$ $\forall$ term t

$$[\alpha](\mu_{\mathcal{O}}(t)) \leqslant [\alpha](t) \leqslant [\alpha](\nu_{\mathcal{O}}(t))$$

for all assignments α

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terms $\ell = \bullet\,\mathsf{f}(\mathsf{c}(x,y),z)$ $r = \mathsf{h}(\bullet\,\mathsf{f}(x,y),\bullet\,\bullet\,\mathsf{f}(\mathsf{f}(x,y),z))$

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bounds	$\mu_{\mathcal{O}}(\ell) = \omega^{\omega^x \oplus y+1} \oplus z$	$\nu_{\mathcal{O}}(r) = z + \omega^{\omega^x \oplus y+1}$
comparison	$\left[\omega^{\omega^x \oplus y+1} \oplus z \geqslant z + \omega^{\omega^x \oplus y+1}\right]$	

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- lexicographic combination with matrix interpretations (ensuring weak monotonicity and simplicity of overall interpretation)
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Termination Proof of $\mathcal{G}$

- initial depth 1, intermediate 2
- 65.000 variables, 217.000 clauses
- 6 seconds

Termination Proof of $\mathcal{H}$

- initial depth 2, intermediate 3
- 117.000 variables, 300.000 clauses
- 12 seconds

Overview

- Goodstein Sequence
- Ordinal Interpretations
- Automation
- Conclusion

Contributions

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- better under- and overapproximations of ROE interpretations

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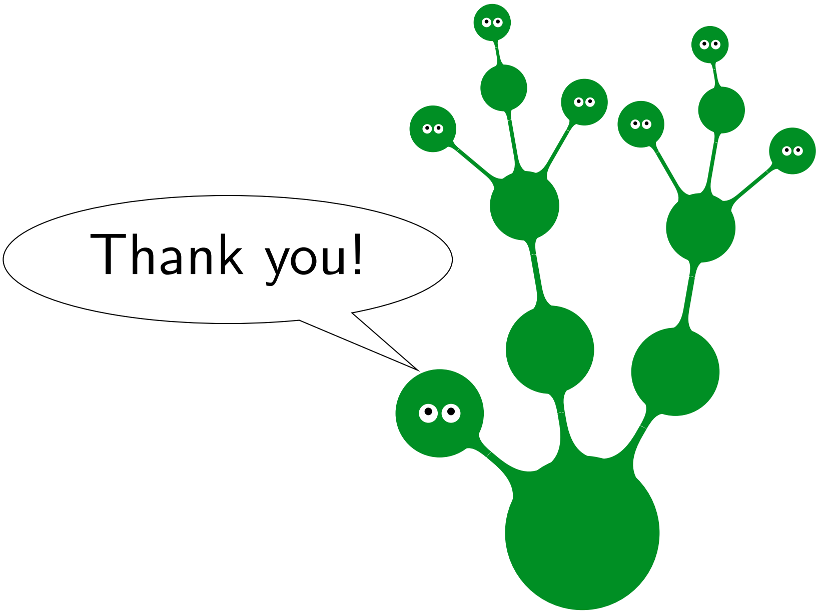
- automation of elementary functions (Lescanne 1995, Lucas 2009)


Thank you!