Normalized Completion Revisited

Sarah Winkler and Aart Middeldorp*

Institute of Computer Science
University of Innsbruck, Austria
`{sarah.winkler, aart.middeldorp}@uibk.ac.at`

Abstract

Normalized completion (Marché 1996) is a widely applicable and efficient technique for completion modulo theories. If successful, a normalized completion procedure computes a rewrite system that allows to decide the validity problem using normalized rewriting. In this paper we consider a slightly simplified inference system for finite normalized completion runs. We prove correctness, show faithfulness of critical pair criteria in our setting, and propose a different notion of normalizing pairs. We then show how normalized completion procedures can benefit from AC-termination tools instead of relying on a fixed AC-compatible reduction order. We outline our implementation of this approach in the completion tool `mkbtt` and present experimental results, including new completions.

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1 Introduction

Since the landmark paper of Knuth and Bendix [12], completion has evolved as a basic deduction method in theorem proving, computer algebra and computational logic. Various generalizations have been proposed to deal efficiently with common algebraic theories. The theory of associativity and commutativity (AC) has been incorporated in [16, 22]. For general theories $\mathcal{T}$ where $\mathcal{T}$-unification is finitary and the subterm ordering modulo $\mathcal{T}$ is well-founded, extensions have been presented in [10, 5]. These limitations on the theory have been partially overcome by constrained completion [11], which allows, e.g., for completion modulo AC with a unit element, but excludes other theories such as abelian groups.

Normalized completion [18, 19] constitutes the last result in this line of research. It has three advantages over earlier methods. (1) It allows completion modulo any theory $\mathcal{T}$ that can be represented as an AC-convergent rewrite system $\mathcal{S}$. (2) Critical pairs need not be computed for the theory $\mathcal{T}$, which may not be finitary or even have a decidable unification problem. Instead, any theory between AC and $\mathcal{T}$ can be used. (3) The AC-compatible reduction order used to establish termination need not be compatible with $\mathcal{T}$. This is beneficial for theories such as AC with a unit element where no $\mathcal{T}$-compatible reduction order can possibly exist.

Normalized completion is thus applicable to many common theories such as AC augmented with axioms for unit elements, idempotency or nilpotency, but also to groups and rings. It

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moreover generalizes Buchberger's algorithm for computing Gröbner bases [20]. Compared to earlier completion techniques, it improves efficiency if the input theory includes a subtheory for which an AC-convergent presentation is known. In computing less critical pairs by focusing on a particular theory, the approach shares advantages with efficient specialized theorem proving techniques with built-in equational theories (e.g. [8, 21]).

The focus of this paper is to transform a given theory into a convergent system, in order to obtain a decision procedure which also allows to (dis)prove equational consequences. In this paper we consider a different proof order for finite normalized completion resulting in a slightly simplified inference system. We incorporate critical pair criteria to limit equational consequences, which has been identified as an issue for future work in [17]. The techniques used to obtain fairness, correctness, and completeness results are similar to the ones for standard completion [6], but the setting of normalized completion involves some subtleties. In contrast to [19], we thus make all AC-steps explicit to enhance clarity. Due to some ambiguities concerning the original definition, we also propose a new notion of normalizing pairs which constitute a key ingredient in normalized completion.

State-of-the-art implementations of normalized completion such as C*i*ME require the input of a suitable AC-compatible reduction order. This parameter is critical for success, but hard to determine in advance. We tackle this problem by applying the by now well-understood combination of two approaches: (1) termination tools replace fixed reduction orders as proposed in [25], and (2) back-tracking is avoided by keeping different orientations of equations. This combined multi-completion approach with termination tools has been investigated for standard completion [29], ordered completion [27] and AC-completion [28]. We present novel convergent systems obtained with our method.

The remainder of this paper is structured as follows. Preliminaries on equational reasoning and rewriting are given in Section 2. In Section 3 we recall normalized completion, present correctness and completeness result based on our proof order, and describe critical pair criteria in the setting of normalized completion. Section 4 describes the extension with termination tools. In Section 5 we give a short description of our implementation in mkbtt, outline the multi-completion approach and some implementation details, and present experimental results. In Section 6 we conclude. Due to a lack of space, some (proof) details can be found in the appendix as well as the first author's PhD thesis [26, Chapter 6].

2 Preliminaries

We assume familiarity with term rewriting and Knuth-Bendix completion [3], and recall only some central notions. We consider term rewrite systems (TRSs) $\mathcal{R}$ over a signature $\mathcal{F}$. If the associated rewrite relation $\to_\mathcal{R}$ is well-founded, we write $s \to^!_\mathcal{R} t$ if s rewrites to a normal form t, and $s{\downarrow_\mathcal{R}}$ to denote some $\mathcal{R}$-normal form of s. We also consider (symmetric) equational systems $\mathcal{E}$ over $\mathcal{F}$ with associated equational theory $=_\mathcal{E}$. If $u \approx v$ is an equation in $\mathcal{E}$ we write $u \simeq v$ to denote $u \approx v$ or $v \approx u$. Let $\mathcal{F}_\mathrm{AC} \subseteq \mathcal{F}$ be a set of binary function symbols. The equational system AC contains equations $x + (y + z) \approx (x + y) + z$ and $x + y \approx y + x$ for all symbols $+ \in \mathcal{F}_\mathrm{AC}$. We denote equivalence modulo AC by $\leftrightarrow^*_\mathrm{AC}$. A term s *rewrites* to t in $\mathcal{R}$ *modulo AC*, denoted by $s \to_{\mathcal{R}/\mathrm{AC}} t$, whenever $s \leftrightarrow^*_\mathrm{AC} \cdot \to_\mathcal{R} \cdot \leftrightarrow^*_\mathrm{AC} t$ holds.

A TRS $\mathcal{R}$ *terminates modulo* AC whenever the relation $\to_{\mathcal{R}/\mathrm{AC}}$ is well-founded. To establish AC-termination we will consider AC-compatible reduction orders $\succ$, i.e., reduction orders that satisfy $\leftrightarrow^*_\mathrm{AC} \cdot \succ \cdot \leftrightarrow^*_\mathrm{AC} \subseteq \succ$. The TRS $\mathcal{R}$ is *convergent modulo AC* if it terminates modulo AC and the relation $\leftrightarrow^*_{\mathrm{AC}\cup\mathcal{R}}$ coincides with $\to^*_{\mathcal{R}/\mathrm{AC}} \cdot \leftrightarrow^*_\mathrm{AC} \cdot \leftarrow^*_{\mathcal{R}/\mathrm{AC}}$.

Let $\mathcal{L}$ be a theory with finitary and decidable unification problem. A substitution σ

constitutes an $\mathcal{L}$-*unifier* of two terms s and t if $s\sigma \leftrightarrow^*_{\mathcal{L}} t\sigma$ holds. An $\mathcal{L}$-*overlap* is a quadruple $\langle \ell_1 \to r_1, p, \ell_2 \to r_2 \rangle_\Sigma$ consisting of rewrite rules $\ell_1 \to r_1$, $\ell_2 \to r_2$, a position $p \in \mathcal{P}\mathrm{os}_{\mathcal{F}}(\ell_2)$, and a complete set Σ of $\mathcal{L}$-unifiers of $\ell_2|_p$ and ℓ_1. Then $\ell_2\sigma[r_1\sigma]_p \leftarrow\!\rtimes\!\to r_2\sigma$ constitutes an $\mathcal{L}$-*critical pair* for every $\sigma \in \Sigma$. We write $s \leftarrow\!\times\!\to t$ if $s \leftarrow\!\rtimes\!\to t$ or $t \leftarrow\!\rtimes\!\to s$ is an $\mathcal{L}$-critical pair. For two sets of rewrite rules $\mathcal{R}_1$ and $\mathcal{R}_2$, we also write $\mathrm{CP}_{\mathcal{L}}(\mathcal{R}_1, \mathcal{R}_2)$ for the set of all $\mathcal{L}$-critical pairs emerging from an overlap where $\ell_1 \to r_1 \in \mathcal{R}_1$ and $\ell_2 \to r_2 \in \mathcal{R}_2$, and $\mathrm{CP}_{\mathcal{L}}(\mathcal{R}_1)$ for the set of all $\mathcal{L}$-critical pairs such that $\ell_1 \to r_1$, $\ell_2 \to r_2 \in \mathcal{R}_1$. A peak $s \;{}_{r\leftarrow\ell}\!\overset{p}{\leftarrow} \cdot \leftrightarrow^*_{\mathcal{T}} \cdot \to^q_{u\to v} t$ is called a *non-overlap* if it is not an instance of an $\mathcal{L}$-overlap.

For a rewrite rule $\ell \to r$ with $+ \in \mathcal{F}_{\mathrm{AC}}$ we write $(\ell \to r)^e$ for the *extended rule* $\ell{+}x \to r{+}x$, where $x \in \mathcal{V}$ is fresh. The TRS $\mathcal{R}^e$ contains all rules in $\mathcal{R}$ plus all extended rules $\ell{+}x \to r{+}x$ such that $\ell \to r \in \mathcal{R}$ [4].

In normalized completion, we consider a fixed rewrite system $\mathcal{S}$ and a pair $(\mathcal{E}, \mathcal{R})$ of equations $\mathcal{E}$ and rewrite rules $\mathcal{R}$. An *equational proof step* $s \leftrightarrow^{p,\sigma}_e t$ in $(\mathcal{S}, \mathcal{E}, \mathcal{R})$ is an *AC-step* (*equality step*) if e or e^{-1} is an equation in AC ($\mathcal{E}$) applied from left to right at position p in s with substitution σ. A proof step $s \leftrightarrow^{p,\sigma}_{\ell\to r} t$ is a *rewrite step* if $s = u[\ell\sigma]_p$ and $t = u[r\sigma]_p$ for some term u with position p and substitution σ and rewrite rule $\ell \to r$ in $\mathcal{R}$ or $\mathcal{S}$. In this case also $t \leftrightarrow^{p,\sigma}_{r\leftarrow\ell} s$ is a rewrite proof step. We call a proof step an $\mathcal{R}$-rewrite ($\mathcal{S}$-rewrite) step if it is a rewrite step using a rule in $\mathcal{R}$ ($\mathcal{S}$).

We sometimes write $s \leftrightarrow t$ to express the existence of some proof step, omitting the position p, substitution σ and equation or rule e. An *equational proof* P of an equation $t_0 \approx t_n$ is a finite sequence

$$t_0 \xleftrightarrow[e_0]{p_0} t_1 \xleftrightarrow[e_1]{p_1} \cdots \xleftrightarrow[e_{n-1}]{p_{n-1}} t_n \qquad (1)$$

of equational proof steps. It has a *subproof* Q, denoted by $P[Q]$, if Q is a sequence $t_i \leftrightarrow \cdots \leftrightarrow t_j$ with $0 \leqslant i \leqslant j \leqslant n$. For a term u with position q, a substitution σ, and a proof P of the shape (1) we write $u[P\sigma]_q$ to denote the sequence

$$u[t_0\sigma]_q \xleftrightarrow[e_0]{qp_0} u[t_1\sigma]_q \xleftrightarrow[e_1]{qp_1} \cdots \xleftrightarrow[e_{n-1}]{qp_{n-1}} u[t_n\sigma]_q$$

which is again an equational proof. A *proof order* $\ggcurly$ is a well-founded order on equational proofs such that (1) $P \ggcurly Q$ implies $u[P\sigma]_p \ggcurly u[Q\sigma]_p$ for all substitutions σ and terms u with position p, and (2) $P \ggcurly P'$ implies $Q[P] \ggcurly Q[P']$ for all proofs P, P' and Q.

In the sequel we will consider a fixed theory $\mathcal{T}$ that is representable as an AC-convergent rewrite system $\mathcal{S}$,[1] so $\leftrightarrow^*_{\mathcal{T}} = \to^!_{\mathcal{S}/\mathrm{AC}} \cdot \leftrightarrow^*_{\mathrm{AC}} \cdot \leftarrow^!_{\mathcal{S}/\mathrm{AC}}$. For example, for the theory ACU consisting of an AC-operator $+$ with unit $\mathsf{0}$, we have $\mathcal{T} = \{x + (y + z) \approx (x + y) + z,\; x + y \approx y + x,\; x + \mathsf{0} \approx x\}$ and $\mathcal{S} = \{x + \mathsf{0} \to x\}$. Note that the representation $\mathcal{S}$ need not be unique.

We define normalized rewriting as in [19] but use a different notation to distinguish it from the by now established notation for rewriting modulo. Two terms s and t admit an $\mathcal{S}$-*normalized* $\mathcal{R}$-*rewrite step* if

$$s \xrightarrow[\mathcal{S}/\mathrm{AC}]{!} s' \xleftrightarrow[\mathrm{AC}]{*} \cdot \xrightarrow[\ell\to r]{p} \cdot \xleftrightarrow[\mathrm{AC}]{*} t \qquad (2)$$

for some rule $\ell \to r$ in $\mathcal{R}$ and position p in s'. We write $s \to^p_{\ell\to r\backslash\mathcal{S}} t$ for (2), and $s \to_{\mathcal{R}\backslash\mathcal{S}} t$ if $s \to^p_{\ell\to r\backslash\mathcal{S}} t$ for a rule $\ell \to r$ in $\mathcal{R}$ and position p.

[1] To avoid confusion we differentiate between the theory and its AC-convergent representation, although both are denoted by S in [19].

deduce	$\dfrac{\mathcal{E}, \mathcal{R}}{\mathcal{E} \cup \{s \approx t\}, \mathcal{R}}$	if $s \approx t \in \mathrm{CP}_{\mathcal{L}}(\mathcal{R})$	simplify	$\dfrac{\mathcal{E} \cup \{s \simeq t\}, \mathcal{R}}{\mathcal{E} \cup \{s \simeq u\}, \mathcal{R}}$	if $t \to_{\mathcal{R}\backslash\mathcal{S}} u$
normalize	$\dfrac{\mathcal{E} \cup \{s \approx t\}, \mathcal{R}}{\mathcal{E} \cup \{s{\downarrow} \approx t{\downarrow}\}, \mathcal{R}}$	if $s \neq s{\downarrow}$ or $t \neq t{\downarrow}$	compose	$\dfrac{\mathcal{E}, \mathcal{R} \cup \{s \to t\}}{\mathcal{E}, \mathcal{R} \cup \{s \to u\}}$	if $t \to_{\mathcal{R}\backslash\mathcal{S}} u$
delete	$\dfrac{\mathcal{E} \cup \{s \approx t\}, \mathcal{R}}{\mathcal{E}, \mathcal{R}}$	if $s \leftrightarrow^*_{\mathrm{AC}} t$	collapse	$\dfrac{\mathcal{E}, \mathcal{R} \cup \{t \to s\}}{\mathcal{E} \cup \{u \approx s\}, \mathcal{R}}$	if $t \to_{\mathcal{R}\backslash\mathcal{S}} u$
orient	$\dfrac{\mathcal{E} \cup \{s \simeq t\}, \mathcal{R}}{\mathcal{E} \cup \Theta(s,t), \mathcal{R} \cup \Psi(s,t)}$	if $s = s{\downarrow}$ and $t = t{\downarrow}$			

Figure 1 $\mathcal{S}$-normalized completion NKB.

Let $\succ$ be an AC-compatible reduction order such that $\mathcal{S} \subseteq \succ$. For any set of rewrite rules $\mathcal{R}$ satisfying $\mathcal{R} \subseteq \succ$ the normalized rewrite relation $\to_{\mathcal{R}\backslash\mathcal{S}}$ is well-founded [18, 19], so we can consider equational proofs of the form $s \xrightarrow[\mathcal{R}\backslash\mathcal{S}]{!} \cdot \xleftrightarrow[\mathcal{T}]{*} \cdot \xleftarrow[\mathcal{R}\backslash\mathcal{S}]{!} t$. These normal form proofs play a special role and are called *normalized rewrite proofs*. Because $\mathcal{S}$ is AC-convergent for $\mathcal{T}$, any such proof can be transformed into a proof $s \downdownarrows_{\mathcal{R}\backslash\mathcal{S}} t$, where $\downdownarrows_{\mathcal{R}\backslash\mathcal{S}}$ abbreviates the relation $\xrightarrow[\mathcal{R}\backslash\mathcal{S}]{!} \cdot \xrightarrow[\mathcal{S}/\mathrm{AC}]{!} \cdot \xleftrightarrow[\mathrm{AC}]{*} \cdot \xleftarrow[\mathcal{S}/\mathrm{AC}]{!} \cdot \xleftarrow[\mathcal{R}\backslash\mathcal{S}]{!}$. A TRS $\mathcal{R}$ is called *$\mathcal{S}$-convergent for* a set of equations $\mathcal{E}$ if $\to_{\mathcal{R}\backslash\mathcal{S}}$ is terminating and the relations $\leftrightarrow^*_{\mathcal{E}\cup\mathcal{T}}$ and $\xrightarrow[\mathcal{R}\backslash\mathcal{S}]{!} \cdot \xleftrightarrow[\mathcal{T}]{*} \cdot \xleftarrow[\mathcal{R}\backslash\mathcal{S}]{!}$ coincide.

3 Normalized Completion

Let $\mathcal{S}$ be AC-convergent for $\mathcal{T}$, and $\succ$ be an AC-compatible reduction order such that $\mathcal{S} \subseteq \succ$. From now on we write $t{\downarrow}$ for $t{\downarrow}_{\mathcal{S}/\mathrm{AC}}$ and $s{\downarrow}_p$ for $s[u{\downarrow}]_p$ where $u = s|_p$. We let $c(s,p,t)$ denote the multiset $\{s\}$ if $s{\downarrow}_p = s$ and $\{s,t\}$ otherwise.

Figure 1 displays the inference system NKB. Note that the collapse rule slightly differs from the version in [19] in that no (strict encompassment) restriction is made on the applied rule in $\mathcal{R}$. This simplification is inspired by the similar modification to standard completion presented in [24] and possible because we restrict to *finite* runs. In the deduce rule, $\mathcal{L}$ denotes some fixed theory such that $\mathrm{AC} \subseteq \mathcal{L} \subseteq \mathcal{T}$.[2] The normalize rule replaces terms in an equation by their normal forms with respect to $\mathcal{S}$, provided that at least one term is not $\mathcal{S}$-normalized. This restriction is missing in [19], but required to ensure progress.

In the orient rule, $\Theta(s,t)$ is a set of equations and $\Psi(s,t)$ is a set of rewrite rules. These functions will be chosen in a way such that (Θ, Ψ) forms a *normalizing pair*. Before giving the definition of this crucial ingredient to normalized completion, we define some properties of inference sequences and our proof order. An inference sequence

$$\gamma\colon (\mathcal{E}_0, \varnothing) \vdash (\mathcal{E}_1, \mathcal{R}_1) \vdash (\mathcal{E}_2, \mathcal{R}_2) \vdash \cdots \vdash (\mathcal{E}_n, \mathcal{R}_n) \tag{3}$$

using the rules in Figure 1 is called a *run* of length n. Throughout this paper we will restrict to *finite* runs,[3] and denote the n-fold composition of the inference relation by $\vdash^n$. A run *fails*

[2] Thus if $\mathcal{T}$ itself is not decidable and finitary with respect to unification, one can simply use AC for $\mathcal{L}$. On the other hand, for example the set of unifiers obtained from ACU or ACUI unification are typically much smaller than those obtained from AC unification.
[3] Normalized completion for infinite runs is discussed in [26].

if $\mathcal{E}_n$ is non-empty, it *succeeds* if $\mathcal{E}_n$ is empty and $\mathcal{R}_n$ is $\mathcal{S}$-convergent for $\mathcal{E}_0$. We assume that all equations and rewrite rules appearing in a run are variable-disjoint, i.e., any equation or rule added in a step $(\mathcal{E}_k, \mathcal{R}_k) \vdash (\mathcal{E}_{k+1}, \mathcal{R}_{k+1})$ is variable-disjoint from $\bigcup_{1 \leqslant i \leqslant k} \mathcal{E}_i \cup \mathcal{R}_i$. We now define a proof reduction relation $\Rightarrow_n$ that depends on the actual run under consideration.

▶ **Definition 3.1.** Consider a run (3) of length n using the reduction order $\succ$, and some $(\mathcal{E}_i, \mathcal{R}_i)$ for $0 \leqslant i \leqslant n$. The *cost* c_n of a proof step in $(\mathcal{T}, \mathcal{E}_i, \mathcal{R}_i)$ is defined as follows:

$$
\begin{aligned}
&c_n(s \xleftrightarrow[u \approx v]{} t) = (\bot, \{s\}, \bot, 0) && \text{if } u \simeq v \in \mathrm{AC} \\
&c_n(s \xleftrightarrow[u \approx v]{p} t) = (\{s{\downarrow_p}, t{\downarrow_p}\}, \{s, t\}, \bot, 0) && \text{if } u \simeq v \in \mathcal{E}_i \\
&c_n(s \xleftrightarrow[\ell \to r]{p} t) = c_n(t \xleftrightarrow[r \leftarrow \ell]{p} s) = (c(s,p,t), \{s\}, (s|_p){\downarrow}, n-k) && \text{if } k \text{ is maximal such that } \ell \to r \in \mathcal{R}_k \\
&c_n(s \xleftrightarrow[\ell \to r]{} t) = c_n(t \xleftrightarrow[r \leftarrow \ell]{} s) = (\bot, \{s\}, \bot, 0) && \text{if } \ell \to r \in \mathcal{S}
\end{aligned}
$$

We compare costs with the lexicographic combination of $(\succ_{\mathsf{mul}}, (\leftrightarrow^*_{\mathrm{AC}})_{\mathsf{mul}})$ for the first two components, $(\rhd_{\mathrm{AC}}, \leftrightarrow^*_{\mathrm{AC}})$, and $(>, =)$ for the standard order $>$ on $\mathbb{N}$. The symbol $\bot$ is considered minimal in the former three orderings. The cost of an equational proof is the multiset consisting of the costs of its steps. The proof order $\ggcurly_n$ is the multiset extension of the order on proof step costs, and $P \Rightarrow_n Q$ holds if and only if $P \ggcurly_n Q$ and P and Q prove the same equation.

As the multiset extension of a lexicographic combination of well-founded orders, the relation $\ggcurly_n$ is well-founded. Hence the following is not difficult to show.

▶ **Lemma 3.2.** *The relation $\Rightarrow_n$ is a proof reduction relation.* ◀

It is easy to see that NKB is sound in that the equational theory is not modified.

Soundness Lemma 3.3. *In any run (3) the relations $\leftrightarrow^*_{\mathcal{E}_0 \cup \mathcal{T}}$ and $\leftrightarrow^*_{\mathcal{E}_n \cup \mathcal{R}_n \cup \mathcal{T}}$ coincide.* ◀

The following lemma links the proof reduction relation $\Rightarrow_n$ to our inference system NKB.

Persistence Lemma 3.4. *Consider a run of the form (3) and let P be an equational proof in $(\mathcal{S}, \mathcal{E}_i, \mathcal{R}_i)$ for $1 \leqslant i \leqslant n$. Then there is a proof Q in $(\mathcal{S}, \mathcal{E}_n, \mathcal{R}_n)$ such that $P \Rightarrow^=_n Q$.* ◀

We next state an AC version of the Extended Critical Pair Lemma [9, 10].

▶ **Lemma 3.5.** *Let $\ell_1 \to r_1$ and $\ell_2 \to r_2$ admit a peak $P\colon s \;{}_{r_1 \leftarrow \ell_1}\!\!\leftarrow \cdot \leftrightarrow^*_{\mathrm{AC}} \cdot \to_{\ell_2 \to r_2} t$. If P does not contain an instance of an AC overlap then $s \to^*_{\ell_2 \to r_2/\mathrm{AC}} \cdot {}_{r_1 \leftarrow \ell_1/\mathrm{AC}}\!\!\overset{*}{\leftarrow} t$. Otherwise, there is a critical pair $u \leftarrow\!\times\!\to v$ in $\mathrm{CP}_{\mathrm{AC}}(\ell_1 \to r_1, \ell_2 \to r_2)$ or $\mathrm{CP}_{\mathrm{AC}}(\ell_1 \to r_1, (\ell_2 \to r_2)^e)$ such that $s \xleftrightarrow[\mathrm{AC}]{*} \cdot \xleftrightarrow[u \approx v]{} \cdot \xleftrightarrow[\mathrm{AC}]{*} t$.* ◀

Note that this implies that any non-joinable peak is an instance of an AC-critical pair between two rules where *at most one* rule is extended, so critical pairs between two extended rules of a rewrite system $\mathcal{R}$ can be ignored. Moreover, it suffices to extend one rule, no matter which one. The following lemma builds upon the previous statement and shows that both joining sequences and critical pairs admit smaller proofs. A proof can be found in [26].

▶ **Lemma 3.6.** *Let $\mathcal{R}$ be a set of rewrite rules such that $\mathcal{R} \subseteq \succ$ and let $n \geqslant 0$.*

(a) If $P\colon s \;{}_{\mathcal{S}}\!\!\leftarrow u \leftrightarrow^*_{\mathrm{AC}} u' \to_{\mathcal{S}} t$ then $P \Rightarrow_n Q$ for some proof $Q\colon s \to^*_{\mathcal{S}/\mathrm{AC}} \cdot {}_{\mathcal{S}/\mathrm{AC}}\!\!\overset{*}{\leftarrow} t$.

(b) *If* $P\colon s\ {}_{\mathcal{R}}\!\leftarrow u \leftrightarrow^*_{\mathrm{AC}} u' \rightarrow_{\mathcal{R}} t$ *then we have* $Q\colon s \rightarrow^*_{\mathcal{R}/\mathrm{AC}} \cdot {}_{\mathcal{R}/\mathrm{AC}}\!\!{}^*\!\!\leftarrow t$ *such that* $P \Rightarrow_n Q$*, or there is some critical pair* $s' \leftarrow\!\times\!\rightarrow t'$ *in* $\mathrm{CP}_{\mathrm{AC}}(\mathcal{R}, \mathcal{R}^e)$ *such that* $P \Rightarrow_n s \leftrightarrow^*_{\mathrm{AC}} \cdot \leftrightarrow_{s'\approx t'} \cdot \leftrightarrow^*_{\mathrm{AC}} t$*.*

(c) *If* $P\colon s\ {}_{\mathcal{R}}\!\leftarrow u \leftrightarrow^*_{\mathrm{AC}} u' \rightarrow_{\mathcal{S}} t$ *then there is a proof* $Q\colon s \rightarrow^*_{\mathcal{S}/\mathrm{AC}} \cdot {}_{\mathcal{R}/\mathrm{AC}}\!\!{}^*\!\!\leftarrow t$ *such that* $P \Rightarrow_n Q$*, or there is a critical pair* $s' \leftarrow\!\times\!\rightarrow t'$ *in* $\mathrm{CP}_{\mathrm{AC}}(\mathcal{R}, \mathcal{S}^e) \cup \mathrm{CP}_{\mathrm{AC}}(\mathcal{S}, \mathcal{R})$ *such that* $P \Rightarrow_n s \leftrightarrow^*_{\mathrm{AC}} \cdot \leftrightarrow_{s'\approx t'} \cdot \leftrightarrow^*_{\mathrm{AC}} t$*.* ◀

We are now ready to define the crucial concept of $\mathcal{S}$-normalizing pairs.[4]

▶ **Definition 3.7.** Let $(\mathcal{E}_i, \mathcal{R}_i)$ occur in a run of the form (3) and u, v be terms such that $u \simeq v \in \mathcal{E}_i$. Let furthermore Θ and Ψ be functions such that $\Theta(u, v)$ is a set of equations and $\Psi(u, v)$ is a set of rewrite rules. Then (Θ, Ψ) constitutes an *$\mathcal{S}$-normalizing pair* for u and v if

(i) $\Theta(u, v)$ and $\Psi(u, v)$ are contained in $\leftrightarrow^*_{\mathcal{E}_i \cup \mathcal{R}_i \cup \mathcal{T}}$, and $\Psi(u, v) \subseteq \succ$,
(ii) for every proof P of the shape $s \xleftrightarrow[u\approx v]{\epsilon,\sigma} t$ there exists a proof Q in $(\mathcal{T}, \Theta(u, v), \Psi(u, v))$ such that $P \Rightarrow_n Q$, and
(iii) for all rules $\ell \rightarrow r$ in $\Psi(u, v)$, all sets of rewrite rules $\mathcal{R}$ and all proofs P of the form $s\ {}_{\mathcal{S}}\!\leftarrow w \leftrightarrow^*_{\mathrm{AC}} \cdot \rightarrow_{\ell\rightarrow r} \cdot \rightarrow^*_{\mathcal{R}\setminus\mathcal{S}} t$ there is a proof Q in $(\mathcal{T}, \Theta(u, v), \Psi(u, v) \cup \mathcal{R})$ such that $P \Rightarrow_n Q$, and all terms in Q are smaller than w.

Here condition (i) ensures that soundness and termination are preserved. Condition (ii) requires that all proofs using the equation $u \approx v$ can be replaced by smaller proofs, which is often achieved by adding the rule $u \rightarrow v$. Condition (iii) takes AC overlaps between rules in $\Psi(u, v)$ and $\mathcal{S}$ into account, but since rules in $\Psi(u, v)$ may at a later stage get composed with other rules, the considered peaks take a more general shape. In the sequel all orient steps in runs will be assumed to apply $\mathcal{S}$-normalizing pairs.

▶ **Example 3.8.** Take the theory ACU where $\mathcal{S} = \{x + 0 \rightarrow x\}$ and consider the $\mathcal{S}$-normalized terms $u = -(x + y)$ and $v = (-x) + (-y)$. Let $\succ$ be an AC-RPO. If the precedence is $- \succ + \succ 0$, we have $u \succ v$. Then $\Theta(u, v) = \{-x \approx (-x) + (-0)\}$ and $\Psi(u, v) = \{u \rightarrow v\}$ form a valid normalizing pair:[5] Condition (i) is clearly satisfied. Condition (ii) holds as any proof using $u \approx v$ can be transformed into a proof using $u \rightarrow v$ which is smaller by Definition 3.1 as $u = u{\downarrow}$. Finally, using Lemma 3.6 it is not hard to see that by adding the AC-critical pair in $\Theta(u, v)$ also condition (iii) holds. If the precedence is $+ \succ - \succ 0$ such that $v \succ u$, one may simply take $\Theta(v, u) = \varnothing$ and $\Psi(v, u) = \{v \rightarrow u\}$.

Marché proposes a *general $\mathcal{S}$-normalizing pair* which is applicable for any choice of the theory $\mathcal{S}$, where $\Psi(u, v)$ consists of the oriented term pair $u \rightarrow v$ and $\Theta(u, v)$ contains AC-critical pairs between $u \rightarrow v$ and a rule in $\mathcal{S}$:

▶ **Definition 3.9** ([19, Definition 3.9])**.** Let u and v be terms in $\mathcal{S}$-normal form such that $u \succ v$. The *general normalizing pair* $(\Theta_{\mathrm{gen}}, \Psi_{\mathrm{gen}})$ is defined by $\Psi_{\mathrm{gen}}(u, v) = \{u \rightarrow v\}$ and $\Theta_{\mathrm{gen}}(u, v) = \mathrm{CP}_{\mathrm{AC}}(u \rightarrow v, \mathcal{S}^e) \cup \mathrm{CP}_{\mathrm{AC}}(\mathcal{S}, u \rightarrow v)$.

We now prove that $(\Theta_{\mathrm{gen}}, \Psi_{\mathrm{gen}})$ is also a normalizing pair according to Definition 3.7.

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[4] The definition of normalizing pairs varies in the literature; the first reference in [17, Definition 4.4] is different from [19, Definition 3.5] and [20, Definition 3.1]. But none of these definitions allowed us to understand and reproduce the correctness proof (cf. the remarks on page 9), thus we use a different notion.

[5] These functions are instances of ACU-normalizing pairs [19].

▶ **Lemma 3.10.** *Let $\mathcal{E}_i$ occur in some run (3) and $u \simeq v \in \mathcal{E}_i$. If u and v are terms in $\mathcal{S}$-normal form such that $u \succ v$ then $(\Theta_{\text{gen}}, \Psi_{\text{gen}})$ forms a normalizing pair.*

Proof. We argue that $\Theta_{\text{gen}}(u,v)$ and $\Psi_{\text{gen}}(u,v)$ satisfy the three requirements demanded in Definition 3.7. Condition (i) is satisfied as due to $u \simeq v \in \mathcal{E}_i$ both $\Theta_{\text{gen}}(u,v)$ and $\Psi_{\text{gen}}(u,v)$ are contained in $\leftrightarrow^*_{\mathcal{E}_i \cup \mathcal{T}}$, and $\Psi_{\text{gen}}(u,v) \subseteq \succ$ as $u \succ v$.

Concerning condition (ii), any proof $P\colon s \leftrightarrow^{\epsilon,\sigma}_{u \approx v} t$ can be transformed into $Q\colon s \leftrightarrow^{\epsilon,\sigma}_{u \to v} t$. We obtain the decrease $u \leftrightarrow^{\epsilon}_{u \approx v} v \Rrightarrow_n u \leftrightarrow^{\epsilon}_{u \to v} v$ because $\{(\{u,v\},\ldots)\} \succ\!\!\succ_n \{(\{u\},\ldots)\}$. As $\Rrightarrow_n$ is a proof reduction relation also $P \Rrightarrow_n Q$ holds.

Finally, consider a proof P of the form $s\ {}_{\mathcal{S}}\!\leftarrow w \leftrightarrow^*_{\text{AC}} w' \to^p_{u \to v} t \to_{\mathcal{R} \setminus \mathcal{S}} \hat{t}$. By Lemma 3.6, there exists a smaller proof of $s \approx t$ (and thus also of $s \approx \hat{t}$) if the peak in P does not constitute a proper overlap. Otherwise P must contain an instance of an AC-critical peak, so $s \leftrightarrow^*_{\text{AC}} C[s'\sigma]$ and $t \leftrightarrow^*_{\text{AC}} C[t'\sigma]$ for some context C, substitution σ, and AC-critical pair $s' \simeq t'$. According to Lemma 3.5 we may assume that one rule comes from $\mathcal{S}^e$ and one rule comes from $\mathcal{R}$. Hence $s' \simeq t' \in \Theta_{\text{gen}}(u,v)$, which gives rise to the proof Q of the form $s \leftrightarrow^*_{\text{AC}} \cdot \leftrightarrow_{s' \approx t'} \cdot \leftrightarrow^*_{\text{AC}} t \to_{\mathcal{R} \setminus \mathcal{S}} \hat{t}$. We have $c_n(P) = \{(\bot, \{w\}, \ldots), (c(w', p, t), \ldots)\} \cup c_{\text{AC}}(P) \cup c_n(P')$ for $P'\colon t \to_{\mathcal{R} \setminus \mathcal{S}} \hat{t}$, whereas $c_n(Q) = \{(\{C[s'\sigma]{\downarrow}, C[t'\sigma]{\downarrow}\}, \ldots)\} \cup c_{\text{AC}}(Q) \cup c_n(P')$. We have $w' \succ t$, and by AC compatibility also $w' \succ s$, $w' \succ C[s'\sigma] \succ C[s'\sigma]{\downarrow}$, and $w' \succ C[t'\sigma] \succ C[t'\sigma]{\downarrow}$. Thus $(c(w', p, t), \ldots) \in c_n(P)$ is greater than all cost tuples in $c_n(Q)$, so $P \Rrightarrow_n Q$. This shows that also condition (iii) is satisfied. ◀

3.1 Fairness and Correctness

Fairness captures the important property of runs that whenever some inference step can achieve progress then progress is eventually made.

▶ **Definition 3.11.** A nonfailing NKB run $(\mathcal{E}_0, \varnothing) \vdash (\mathcal{E}_1, \mathcal{R}_1) \vdash \cdots \vdash (\varnothing, \mathcal{R}_n)$ is *fair with respect to* $\Rrightarrow_n$ if for any proof P in $\mathcal{T} \cup \mathcal{R}_n$ which is not a rewrite proof there is a proof Q in $(\mathcal{T}, \mathcal{E}_i, \mathcal{R}_i)$ for some $0 \leqslant i \leqslant n$ such that $P \Rrightarrow_n Q$.

Note that our definition is less restrictive than the original one, which is essential to incorporate critical pair criteria (see Section 3.2). We show that the original definition [19] constitutes a sufficient criterion for fairness in our sense. Beforehand, we state two technical results about persistent rules and $\mathcal{L}$-critical pairs. Their proofs can be found in [26].

▶ **Lemma 3.12.** *Assume an* NKB *run* $(\mathcal{E}_0, \varnothing) \vdash (\mathcal{E}_1, \mathcal{R}_1) \vdash \cdots \vdash (\mathcal{E}_n, \mathcal{R}_n)$ *has a rule* $\ell \to r \in \mathcal{R}_n$ *giving rise to a peak* $P\colon s\ {}_{\mathcal{S}}\!\leftarrow w \leftrightarrow^*_{\text{AC}} w' \to^p_{\ell \to r \setminus \mathcal{S}} t$. *Then there is a proof P' in $(\mathcal{T}, \mathcal{E}_n, \mathcal{R}_n)$ such that $P \Rrightarrow_n P'$, and for all $(T, \ldots) \in c_n(P')$ the set T contains only terms which are smaller than w.* ◀

▶ **Lemma 3.13.** *Let $\ell_1 \to r_1$ and $\ell_2 \to r_2$ be rewrite rules and $\text{AC} \subseteq \mathcal{L} \subseteq \mathcal{T}$. If $s \simeq t \in \text{CP}_{\text{AC}}(\ell_1 \to r_1, \ell_2 \to r_2)$ then there is some critical pair $s' \leftarrow\!\times\!\to t' \in \text{CP}_{\mathcal{L}}(\ell_1 \to r_1, \ell_2 \to r_2)$ and substitution ρ such that $s \leftrightarrow^*_{\mathcal{T}} s'\rho$ and $t \leftrightarrow^*_{\mathcal{T}} t'\rho$.* ◀

▶ **Lemma 3.14.** *A nonfailing* NKB *run* $(\mathcal{E}_0, \varnothing) \vdash (\mathcal{E}_1, \mathcal{R}_1) \vdash \cdots \vdash (\varnothing, \mathcal{R}_n)$ *satisfying* $\text{CP}_{\mathcal{L}}(\mathcal{R}_n, \mathcal{R}^e_n) \subseteq \bigcup_i \mathcal{E}_i$ *is fair with respect to* $\Rrightarrow_n$.

Proof. We show that every proof in $\mathcal{T} \cup \mathcal{R}_n$ which is minimal with respect to $\Rrightarrow_n$ is a normalized rewrite proof. Assume to the contrary that P minimal but not a rewrite proof. Thus P contains (i) a peak $s\ {}_{\mathcal{R}_n}\!\leftarrow \cdot \leftrightarrow^*_{\text{AC}} \cdot \to_{\mathcal{R}_n} t$, or (ii) a peak $s\ {}_{\mathcal{R}_n/\text{AC}}\!\leftarrow \cdot \to_{\mathcal{S}/\text{AC}} t$ or $s\ {}_{\mathcal{S}/\text{AC}}\!\leftarrow \cdot \to_{\mathcal{R}_n/\text{AC}} t$, or (iii) a subproof $u \to_{\mathcal{R}_n/\text{AC}} t$ such that $u \neq u{\downarrow}$, or (iv) a peak $s\ {}_{\mathcal{S}/\text{AC}}\!\leftarrow \cdot \to_{\mathcal{S}/\text{AC}} t$. For each case we show that a smaller proof contradicts minimality of P.

If a peak of the form (i) originates from a non-overlap then by Lemma 3.6(b) it could be replaced by a smaller proof. Otherwise, by Lemma 3.5 the peak $s \;{}_{\mathcal{R}_n}\!\leftarrow \cdot \leftrightarrow^*_{\mathrm{AC}} \cdot \rightarrow_{\mathcal{R}_n} t$ must satisfy $s \leftrightarrow^*_{\mathrm{AC}} C[s'\sigma]$ and $t \leftrightarrow^*_{\mathrm{AC}} C[t'\sigma]$ for some critical pair $s' \leftarrow\!\times\!\rightarrow t'$ in $\mathrm{CP}_{\mathrm{AC}}(\mathcal{R}_n, \mathcal{R}_n^e)$. Assume $s' \leftarrow\!\times\!\rightarrow t'$ originates from a peak $P'\colon s' \;{}^{\;p}_{\mathcal{R}_n}\!\leftarrow w \leftrightarrow^*_{\mathrm{AC}} w' \rightarrow^q_{\mathcal{R}_n^e} t'$. We show that $\mathcal{T} \cup \mathcal{R}_n$ admits a smaller proof than P', which entails the existence of a smaller proof than P. By Lemma 3.13 there must also be an $\mathcal{L}$-critical pair $s'' \approx t''$ such that $s' \leftrightarrow^*_{\mathcal{T}} s''\rho$ and $t' \leftrightarrow^*_{\mathcal{T}} t''\rho$ for some substitution ρ. As $\mathcal{S}$ is AC convergent for $\mathcal{T}$, s' and $s''\rho$ as well as t' and $t''\rho$ have the same $\mathcal{S}$-normal forms, which we denote by $\hat{s}$ and $\hat{t}$, respectively. We have $c_n(P') = \{(c(w,p,s'),\ldots),(c(w',q,t'),\ldots)\} \cup c_{\mathrm{AC}}(P')$ while the proof $Q\colon s' \leftrightarrow^*_{\mathcal{S}\cup\mathrm{AC}} s''\rho \leftrightarrow_{s''\approx t''} t''\rho \leftrightarrow^*_{\mathcal{S}\cup\mathrm{AC}} t'$ has cost $c_n(Q) = \{(\{\hat{s},\hat{t}\},\ldots)\} \cup c_{\mathcal{S}\cup\mathrm{AC}}(Q)$, so $P' \Rrightarrow_n Q$ holds because $w \succ s' \succ \hat{s}$ and $w' \succ t' \succ \hat{t}$. As $\mathrm{CP}_{\mathcal{L}}(\mathcal{R}_n, \mathcal{R}_n^e) \subseteq \bigcup_i \mathcal{E}_i$ the proof Q actually exists in some $(\mathcal{T}, \mathcal{E}_i, \mathcal{R}_i)$. By the Persistence Lemma 3.4 there is also a proof Q' in $\mathcal{T} \cup \mathcal{R}_n$ such that $P' \Rrightarrow_n Q \Rrightarrow^=_n Q'$.

Next, assume P contains a peak of the form (ii). If such a pattern originates from a non-overlap then by Lemma 3.6(c) it could be replaced by a smaller proof. Otherwise, by Lemma 3.5, the proof P must contain a proof corresponding to an AC-critical pair $s' \leftarrow\!\times\!\rightarrow t'$ in $\mathrm{CP}_{\mathrm{AC}}(\mathcal{R}_n, \mathcal{S}^e) \cup \mathrm{CP}_{\mathrm{AC}}(\mathcal{S}, \mathcal{R}_n)$. Then $s' \leftarrow\!\times\!\rightarrow t'$ must originate from an AC-critical peak Q of the form $s' \;{}_{r\leftarrow\ell}\!\leftarrow \cdot \leftrightarrow^*_{\mathrm{AC}} \cdot \rightarrow_{u\rightarrow v} t'$ between rules $\ell \rightarrow r \in \mathcal{R}_n$ and $u \rightarrow v \in \mathcal{S}$, and a proof Q' in $\mathcal{T} \cup \mathcal{R}_n$ satisfying $Q \Rrightarrow_n Q'$ exists according to Lemma 3.12. This implies $P = P[Q] \Rrightarrow_n P[Q']$.

If P contains a subproof Q of the form (iii) we have $c_n(Q) = \{(\{u,t\},\ldots)\} \cup c_{\mathrm{AC}}(Q)$. Since $u \neq u{\downarrow}$ there is some step $u \rightarrow_{\mathcal{S}/\mathrm{AC}} s$, and thus a peak $P'\colon s \;{}_{\mathcal{S}/\mathrm{AC}}\!\leftarrow u \rightarrow_{\mathcal{R}_n/\mathrm{AC}} t$. If P' does not constitute a proper overlap then there exists a rewrite proof Q' of $s \approx t$ which contains only terms smaller than u. For $Q''\colon u \rightarrow_{\mathcal{S}/\mathrm{AC}} s$ the proof $Q''Q'$ is thus smaller than Q as $(\{u,t\},\ldots) \in c_n(Q)$ dominates all cost tuples in $c_n(Q''Q')$. If P' constitutes a critical peak then by Lemma 3.12 there exists a proof Q' of $s \approx t$ such that $P' \Rrightarrow_n Q'$ and for $(T,\ldots) \in c_n(Q')$ all terms in T are smaller than u. Again $Q \Rrightarrow_n Q''Q'$ holds.

Finally, if P contains a subproof of the form (iv) then AC convergence of $\mathcal{S}$ yields a smaller proof according to Lemma 3.6(a). ◀

Correctness Theorem 3.15. *A fair and nonfailing* NKB *run succeeds.*

Proof. Let $(\mathcal{E}_0, \varnothing) \vdash (\mathcal{E}_1, \mathcal{R}_1) \vdash \cdots \vdash (\varnothing, \mathcal{R}_n)$ be the run under consideration. We show that $\leftrightarrow^*_{\mathcal{E}_0\cup\mathcal{T}} \subseteq \rightarrow^*_{\mathcal{R}\backslash\mathcal{S}} \cdot \leftrightarrow^*_{\mathcal{T}} \cdot \leftarrow^*_{\mathcal{R}\backslash\mathcal{S}}$. According to the Persistence Lemma 3.4, any pair of terms in $\leftrightarrow^*_{\mathcal{E}_0\cup\mathcal{T}}$ has a proof in $\mathcal{T} \cup \mathcal{R}_n$. Let P be such a proof which is minimal with respect to $\Rrightarrow_n$, and assume it is not a normalized rewrite proof. By fairness there exists a proof Q in $(\mathcal{T}, \mathcal{E}_i, \mathcal{R}_i)$ for some $0 \leq i \leq n$ such that $P \Rrightarrow_n Q$. According to persistence $Q \Rrightarrow^=_n Q'$ for some Q' in $\mathcal{T} \cup \mathcal{R}_n$, contradicting the minimality of P. By the Soundness Lemma 3.3 the relations $\leftrightarrow^*_{\mathcal{T}\cup\mathcal{R}_n}$ and $\leftrightarrow^*_{\mathcal{T}\cup\mathcal{E}_0}$ coincide, so $\mathcal{R}_n$ is $\mathcal{S}$-convergent for $\mathcal{E}_0$. ◀

▶ **Example 3.16.** Consider an Abelian group with AC operator $\cdot$ and an endomorphism f as described by the following set of equations:

$$\mathsf{e} \cdot x \approx x \qquad \mathsf{i}(x) \cdot x \approx \mathsf{e} \qquad \mathsf{f}(x \cdot y) \approx \mathsf{f}(x) \cdot \mathsf{f}(y)$$

together with LPO with precedence $\mathsf{f} \succ \mathsf{i} \succ \cdot \succ \mathsf{e}$. We can obviously apply normalized completion with respect to AC, so $\mathcal{S} = \varnothing$. This results in the AC-convergent TRS $\mathcal{R}_{\mathrm{AC}}$:

$$\begin{array}{lll} \mathsf{e} \cdot x \rightarrow x & \mathsf{i}(x) \cdot x \rightarrow \mathsf{e} & \mathsf{i}(\mathsf{e}) \rightarrow \mathsf{e} \\ \mathsf{i}(\mathsf{i}(x)) \rightarrow x & \mathsf{i}(x \cdot y) \rightarrow \mathsf{i}(x) \cdot \mathsf{i}(y) & \mathsf{f}(x \cdot y) \rightarrow \mathsf{f}(x) \cdot \mathsf{f}(y) \\ \mathsf{f}(\mathsf{e}) \rightarrow \mathsf{e} & \mathsf{f}(\mathsf{i}(x)) \rightarrow \mathsf{i}(\mathsf{f}(x)) & \end{array}$$

Alternatively, we can consider $\mathcal{S}_{\mathrm{G}} = \{\mathsf{e} \cdot x \to x,\ \mathsf{i}(x) \cdot x \to \mathsf{e},\ \mathsf{i}(\mathsf{e}) \to \mathsf{e},\ \mathsf{i}(\mathsf{i}(x)) \to x,\ \mathsf{i}(x \cdot y) \to \mathsf{i}(x) \cdot \mathsf{i}(y)\}$ which is known to be an AC-convergent representation of Abelian groups [4]. Note that $\mathcal{S}_{\mathrm{G}} \subseteq \succ$. An NKB run with respect to $\mathcal{S}_{\mathrm{G}}$ results in the TRS $\mathcal{R}_{\mathrm{G}}$:

$$\mathsf{f}(x \cdot y) \to \mathsf{f}(x) \cdot \mathsf{f}(y) \qquad \mathsf{f}(\mathsf{e}) \to \mathsf{e} \qquad \mathsf{f}(\mathsf{i}(x)) \to \mathsf{i}(\mathsf{f}(x))$$

A TRS $\mathcal{R}$ is called $\mathcal{S}$-*reduced* if for all rules $\ell \to r$ in $\mathcal{R}$ the term r is in normal form with respect to $\to_{\mathcal{S}/\mathrm{AC}}$ and $\to_{\mathcal{R}\setminus\mathcal{S}}$, and ℓ is in normal form with respect to $\to_{\mathcal{S}/\mathrm{AC}}$ and $\to_{\ell' \to r' \setminus \mathcal{S}}$ for every rule $\ell' \to r'$ in $\mathcal{R}$ different from $\ell \to r$. A TRS $\mathcal{R}$ is $\mathcal{S}$-*canonical for* $\mathcal{E}$ if it is both $\mathcal{S}$-reduced and $\mathcal{S}$-convergent for $\mathcal{E}$, and a run is *simplifying* if simplify, compose and collapse are applied exhaustively. As two TRSs that are $\mathcal{S}$-canonical for $\mathcal{E}$ and contained in the same AC-compatible reduction order $\succ$ are equal up to variable renaming and AC equivalence [17], correctness implies the following completeness result.

▶ **Corollary 3.17.** *Assume $\mathcal{R}$ is a finite $\mathcal{S}$-canonical system for $\mathcal{E}$ and let $\succ$ be an AC-compatible reduction order that contains $\mathcal{R}$ and $\mathcal{S}$. Then any fair, nonfailing, and simplifying run from $\mathcal{E}$ using $\succ$ will produce an $\mathcal{S}$-canonical system $\mathcal{R}'$ such that $\mathcal{R}$ and $\mathcal{R}'$ are equal up to variable renaming and AC equivalence.*

For infinite runs one can show the stronger completeness result that whenever a finite $\mathcal{S}$-canonical system for some theory exists, any nonfailing run applying a corresponding reduction order succeeds in *finitely many steps* [26].

We conclude this section by commenting on the definition of normalizing pairs. In [19, Definition 3.5] and [20, Definition 3.1], normalizing pairs are defined as follows. Given terms u and v such that $u = u{\downarrow}$, $v = v{\downarrow}$, and $u \succ v$, the functions (Θ, Ψ) form an $\mathcal{S}$-normalizing pair if and only if

(i) for any single-step proof $s \leftrightarrow_{u \approx v} t$ there is a proof P in $(\mathcal{T}, \Theta(u,v), \Psi(u,v))$ such that $s \leftrightarrow_{u \approx v} t \Rightarrow P$, and
(ii) for all $\ell \to r \in \Psi(u,v)$, all sets of rules $\mathcal{R}$ and all r' such that $r \to^*_{\mathcal{R}\setminus\mathcal{S}} r'$ and any single-step irreducible[6] proof $s \to_{\ell \to r'} t$ there is a proof P in $(\mathcal{T}, \Theta(u,v), \Psi(u,v) \cup \mathcal{R})$ such that $s \to_{\ell \to r'} t \Rightarrow P$.

In our understanding four issues arise with this definition.

(a) It does not require $\Theta(u,v)$ and $\Psi(u,v)$ to be part of the equational theory.
(b) It does not guarantee termination of $\Psi(u,v)$ together with previously oriented rules.
(c) Joinability of AC-critical pairs between $\mathcal{S}$ and $\Psi(u,v)$ is not ensured: Consider the simple example where the theory $\mathcal{E}_0 = \{x + \mathsf{a} \approx \mathsf{a}\}$ is to be completed with respect to $\mathcal{S} = \{y + \mathsf{b} \to \mathsf{b}\}$. We can choose $\Theta(x + \mathsf{a}, \mathsf{a}) = \varnothing$ and $\Psi(x + \mathsf{a}, \mathsf{a}) = \{x + \mathsf{a} \to \mathsf{a}\}$, satisfying (i) and (ii). We obtain the run $(\{x + \mathsf{a} \approx \mathsf{a}\}, \varnothing) \vdash (\varnothing, \{x + \mathsf{a} \to \mathsf{a}\})$ which is obviously fair. But $\{x + \mathsf{a} \to \mathsf{a}\}$ is not $\mathcal{S}$-convergent as the AC-critical pair $\mathsf{a} \leftarrow \rtimes \to \mathsf{b}$ between $\mathcal{S}$ and $x + \mathsf{a} \to \mathsf{a}$ is not considered.
(d) The general normalizing pair [19, Definition 3.9] does not match this definition: Assume we orient $x + \mathsf{a} \approx \mathsf{a}$ as $x + \mathsf{a} \to \mathsf{a}$. The general normalizing pair sets $\Theta(x + \mathsf{a}, \mathsf{a}) = \mathrm{CP}_{\mathrm{AC}}(\mathcal{S}, x + \mathsf{a} \to \mathsf{a}) \cup \mathrm{CP}_{\mathrm{AC}}(x + \mathsf{a} \to \mathsf{a}, \mathcal{S}^e)$ and $\Psi(x + \mathsf{a}, \mathsf{a}) = \{x + \mathsf{a} \to \mathsf{a}\}$. Then property (ii) is not satisfied: for $\mathcal{R} = \varnothing$ and $r = r' = \mathsf{a}$ there exists no smaller proof than $x + \mathsf{a} \to^{\epsilon}_{x+\mathsf{a}\to\mathsf{a}} \mathsf{a}$ (and there is also no reason why such a proof should be necessary).

With the earlier definition in [17, Definition 4.4] similar issues arise, cf. [26]. Due to these ambiguities the notion of normalizing pairs was modified according to Definition 3.7.

[6] A proof is irreducible if it is minimal with respect to the proof reduction relation $\Rightarrow$.

3.2 Critical Pair Criteria

Critical pair criteria constitute a means to filter out critical pairs that can be ignored without compromising completeness. Let $\mathcal{L}$ be a theory between AC and $\mathcal{T}$. A critical pair criterion CPC maps $(\mathcal{E}, \mathcal{R})$ to a set of equations such that $\mathrm{CPC}(\mathcal{E}, \mathcal{R})$ is a subset of $\mathrm{CP}_{\mathcal{L}}(\mathcal{R}, \mathcal{R}^e)$. As for standard completion, the *compositeness criterion* serves as a general condition.

▶ **Definition 3.18.** Let $\mathcal{E}$ be a set of equations and $\mathcal{R}$ be a set of rewrite rules. An equational proof P that has the form of a peak $s \leftarrow \cdot \leftrightarrow^*_{\mathcal{L}} \cdot \rightarrow t$ is *composite* in $(\mathcal{T}, \mathcal{E}, \mathcal{R})$ if there exist terms $u_0, \ldots, u_{k+1}$ where $s = u_0$, $t = u_{k+1}$ and $u \succ u_j$ for all $0 \leqslant j \leqslant k+1$, and proofs $P_0, \ldots, P_k$ in $(\mathcal{T}, \mathcal{E}, \mathcal{R})$ such that P_j proves $u_j \approx u_{j+1}$ and $P \ggcurly_n P_j$ for all $1 \leqslant j \leqslant k$, and any $n \geqslant 0$. The *compositeness criterion* $\mathrm{CCP}_{\mathcal{L}}(\mathcal{E}, \mathcal{R})$ returns all $\mathcal{L}$-critical pairs among rules in $\mathcal{R}$ for which the associated overlaps are composite.

We now relax Lemma 3.14 by proving that composite critical pairs can safely be ignored.

▶ **Lemma 3.19.** *Consider a nonfailing* NKB *run* $\gamma\colon (\mathcal{E}_0, \varnothing) \vdash (\mathcal{E}_1, \mathcal{R}_1) \vdash \cdots \vdash (\varnothing, \mathcal{R}_n)$ *and let* $\mathcal{C}$ *be a subset of* $\bigcup_i \mathrm{CCP}(\mathcal{E}_i, \mathcal{R}_i)$. *If* $\mathrm{CP}_{\mathcal{L}}(\mathcal{R}_n, \mathcal{R}^e_n) \setminus \mathcal{C} \subseteq \bigcup_i \mathcal{E}_i$ *then* γ *is fair.* ◀

Although the compositeness criterion is very general, several special cases can be checked efficiently. Consider an overlap $\langle \ell_1 \to r_1, p, \ell_2 \to r_2 \rangle_\Sigma$ giving rise to the set of critical peaks

$$P\colon s \xleftarrow[r_1 \leftarrow \ell_1]{p,\sigma} u \xleftrightarrow[\mathcal{L}]{*} u' \xrightarrow[\ell_2 \to r_2]{\epsilon,\sigma} t \tag{4}$$

such that $\sigma \in \Sigma$. If $u \neq u{\downarrow}$ or $u' \neq u'{\downarrow}$ then the $\mathcal{L}$-critical pair $s \leftarrow \rtimes \rightarrow t$ is *$\mathcal{S}$-reducible.*

Let us now assume that both u and u' in a peak (4) are in normal form with respect to $\rightarrow_{\mathcal{S}/\mathrm{AC}}$. By AC convergence of $\mathcal{S}$ and $\mathcal{L} \subseteq \mathcal{S}$ we thus have $u \leftrightarrow^*_{\mathrm{AC}} u'$. Now assume there is a rewrite step $u \leftrightarrow^*_{\mathrm{AC}} \cdot \rightarrow_{\mathcal{R}} v$ using a rule $\ell_3 \to r_3$ at position q, such that $(\ell_3 \to r_3, q)$ is different from $(\ell_1 \to r_1, p)$ and $(\ell_2 \to r_2, \epsilon)$. Thus there are proofs

$$P_1\colon s \xleftarrow[r_1 \leftarrow \ell_1]{p} u \xleftrightarrow[\mathrm{AC}]{*} v' \xrightarrow[\ell_3 \to r_3]{q} v \qquad\qquad P_2\colon v \xleftarrow[r_3 \leftarrow \ell_3]{q} v' \xleftrightarrow[\mathrm{AC}]{*} u' \xrightarrow[\ell_1 \to r_1]{\epsilon} t \tag{5}$$

such that $P_1 P_2$ proves $s \approx t$. An AC-critical pair (4) is *not prime* if $u|_p \rhd_{\mathrm{AC}} v'|_q$.

▶ **Lemma 3.20.** *Any $\mathcal{L}$-critical pair which is $\mathcal{S}$-reducible or non-prime is composite.* ◀

It can be shown that also the *connectedness criterion* proposed for standard completion [14] is applicable in normalized completion, and all these critical pair criteria are also compatible with a proof order based upon [19] and hence applicable in *infinite* runs, cf. [26].

4 Normalized Completion with Termination Tools

Classical Knuth-Bendix completion requires a fixed reduction order as input. To avoid fixing this critical parameter from the very beginning and obtain a greater variety of usable orders, Wehrman *et al.* [25] proposed *completion with termination tools*. In this section we take a similar approach to normalized completion.

The inference rules in Figure 2 describe normalized completion with termination tools (NKBtt). In the orient rule, (Θ, Ψ) is again assumed to form an $\mathcal{S}$-normalizing pair for the terms s and t. A sequence $(\mathcal{E}_0, \varnothing, \varnothing) \vdash (\mathcal{E}_1, \mathcal{R}_1, \mathcal{C}_1) \vdash (\mathcal{E}_2, \mathcal{R}_2, \mathcal{C}_2) \vdash \cdots$ of NKBtt inference steps is called a *run.* Before giving a correctness proof we illustrate NKBtt on an example.

orient	$\dfrac{\mathcal{E} \uplus \{s \simeq t\}, \mathcal{R}, \mathcal{C}}{\mathcal{E} \cup \Theta(s,t), \mathcal{R} \cup \Psi(s,t), \mathcal{C}'}$	if $s = s{\downarrow}$, $t = t{\downarrow}$ and $\mathcal{C}' \cup \mathcal{S}$ is AC terminating for $\mathcal{C}' = \mathcal{C} \cup \Psi(s,t)$
deduce	$\dfrac{\mathcal{E}, \mathcal{R}, \mathcal{C}}{\mathcal{E} \cup \{s \approx t\}, \mathcal{R}, \mathcal{C}}$	if $s \approx t \in \mathrm{CP}_{\mathcal{L}}(\mathcal{R}, \mathcal{R}^e)$
delete	$\dfrac{\mathcal{E} \uplus \{s \approx t\}, \mathcal{R}, \mathcal{C}}{\mathcal{E}, \mathcal{R}, \mathcal{C}}$	if $s \leftrightarrow^*_{\mathrm{AC}} t$
normalize	$\dfrac{\mathcal{E} \uplus \{s \approx t\}, \mathcal{R}, \mathcal{C}}{\mathcal{E} \cup \{s{\downarrow} \approx t{\downarrow}\}, \mathcal{R}, \mathcal{C}}$	if $s \neq s{\downarrow}$ or $t \neq t{\downarrow}$
simplify	$\dfrac{\mathcal{E} \uplus \{s \simeq t\}, \mathcal{R}, \mathcal{C}}{\mathcal{E} \cup \{s \simeq u\}, \mathcal{R}, \mathcal{C}}$	if $t \to_{\mathcal{R} \backslash \mathcal{S}} u$
compose	$\dfrac{\mathcal{E}, \mathcal{R} \uplus \{s \to t\}, \mathcal{C}}{\mathcal{E}, \mathcal{R} \cup \{s \to u\}, \mathcal{C}}$	if $t \to_{\mathcal{R} \backslash \mathcal{S}} u$
collapse	$\dfrac{\mathcal{E}, \mathcal{R} \uplus \{t \to s\}, \mathcal{C}}{\mathcal{E} \cup \{u \approx s\}, \mathcal{R}, \mathcal{C}}$	if $t \to_{\mathcal{R} \backslash \mathcal{S}} u$

Figure 2 $\mathcal{S}$-normalized completion with termination tools (NKBtt).

▶ **Example 4.1.** Consider the initial set of equations $\mathcal{E}_0 = \{\mathsf{a} + x \approx \mathsf{b} + \mathsf{g}(\mathsf{a})\}$ where $+$ is an AC symbol with unit $\mathsf{0}$, such that the theory $\mathcal{T}$ can be represented by $\mathcal{S} = \{x + \mathsf{0} \to x\}$. Note that the given equation cannot be oriented with an AC-compatible simplification order. Thus any completion tool restricted to orders such as AC-RPO or AC-KBO [13] fails immediately. But termination tools can verify AC termination of the rule $\mathsf{a} + x \to \mathsf{b} + \mathsf{g}(\mathsf{a})$ using e.g. AC dependency pairs [2]. Hence the equation $\mathsf{a} + x \approx \mathsf{b} + \mathsf{g}(\mathsf{a})$ can be oriented in an NKBtt run. When using ACU-normalizing pairs [19], this results in the state

$$\mathcal{E}_1: \quad \mathsf{a} + \mathsf{0} \approx \mathsf{b} + \mathsf{g}(\mathsf{a}) \qquad \mathcal{R}_1: \quad \mathsf{a} + x \to \mathsf{b} + \mathsf{g}(\mathsf{a}) \qquad \mathcal{C}_1: \quad \mathsf{a} + x \to \mathsf{b} + \mathsf{g}(\mathsf{a})$$

After normalizing $\mathsf{a} + \mathsf{0}$ to a, we have

$$\mathcal{E}_2: \quad \mathsf{a} \approx \mathsf{b} + \mathsf{g}(\mathsf{a}) \qquad \mathcal{R}_2: \quad \mathsf{a} + x \to \mathsf{b} + \mathsf{g}(\mathsf{a}) \qquad \mathcal{C}_2: \quad \mathsf{a} + x \to \mathsf{b} + \mathsf{g}(\mathsf{a})$$

Since $\mathcal{C}_2 \cup \{\mathsf{b} + \mathsf{g}(\mathsf{a}) \to \mathsf{a}\}$ is AC terminating, we may perform an orient step:

$$\begin{array}{llllll} \mathcal{E}_3: & & \mathcal{R}_3: & \mathsf{a} + x \to \mathsf{b} + \mathsf{g}(\mathsf{a}) & \mathcal{C}_3: & \mathsf{a} + x \to \mathsf{b} + \mathsf{g}(\mathsf{a}) \\ & & & \mathsf{b} + \mathsf{g}(\mathsf{a}) \to \mathsf{a} & & \mathsf{b} + \mathsf{g}(\mathsf{a}) \to \mathsf{a} \end{array}$$

In a compose step, the new rule can be used to replace $\mathsf{a} + x \to \mathsf{b} + \mathsf{g}(\mathsf{a})$ by $\mathsf{a} + x \to \mathsf{a}$. Three subsequent applications of deduce yield the state

$$\begin{array}{llllll} \mathcal{E}_7: & \mathsf{a} + \mathsf{g}(\mathsf{a}) \approx \mathsf{a} + \mathsf{a} & \mathcal{R}_7: & \mathsf{a} + x \to \mathsf{a} & \mathcal{C}_7: & \mathsf{a} + x \to \mathsf{b} + \mathsf{g}(\mathsf{a}) \\ & \mathsf{a} + \mathsf{a} \approx \mathsf{a} + \mathsf{b} & & \mathsf{b} + \mathsf{g}(\mathsf{a}) \to \mathsf{a} & & \mathsf{b} + \mathsf{g}(\mathsf{a}) \to \mathsf{a} \\ & \mathsf{a} + \mathsf{a} \approx \mathsf{a} & & & & \end{array}$$

All terms in $\mathcal{E}_7$ simplify to a, so the resulting trivial equations can be deleted. As all critical pairs among rules in $\mathcal{R}_7$ were already deduced the run is fair, so $\mathcal{R}_7$ is $\mathcal{S}$-convergent for $\mathcal{E}_0$.

The proof of the following correctness result can be found in the appendix.

Correctness Theorem 4.2. *Any finite nonfailing and fair* NKBtt *run succeeds.* ◀

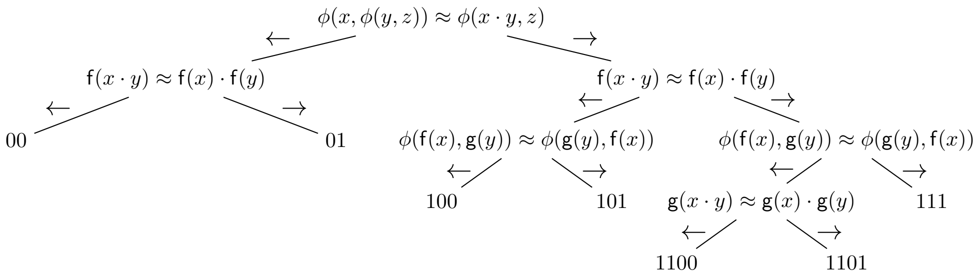

Figure 3 Part of the process tree developed in a run on CGA where process 1101 succeeds.

5 Implementation Details and Experimental Results

5.1 Multi-Completion

In completion with termination tools, the orient rule leaves a choice if the considered equation can be oriented in both directions. As the appropriate orientation of an equation is hard to predict, it is beneficial to keep track of multiple orientations. Thus, in our tool mkbtt we implemented a *multi-completion* variant of normalized completion with termination tools, following the approach suggested for completion with multiple reduction orders [15]. The basic idea is to simulate multiple NKBtt processes in parallel, but share common inferences to gain efficiency. Here a process corresponds to a sequence of decisions on how to orient equations. In our implementation, we model a process as a bit string. The initial process is denoted by ϵ. A formal description of this approach can be found in [26]. Here we content ourselves with giving an example.

▶ **Example 5.1.** We consider the system CGA describing an abelian group with a group action ϕ on itself such that two endomorphisms f and g commute with respect to ϕ:

$$x \cdot x^{-1} \approx \mathsf{e} \qquad \mathsf{f}(x \cdot y) \approx \mathsf{f}(x) \cdot \mathsf{f}(y) \qquad \mathsf{g}(x \cdot y) \approx \mathsf{g}(x) \cdot \mathsf{g}(y)$$
$$\phi(\mathsf{e}, x) \approx x \qquad \phi(x, \phi(y, z)) \approx \phi(x \cdot y, z) \qquad \phi(\mathsf{f}(x), \mathsf{g}(y)) \approx \phi(\mathsf{g}(y), \mathsf{f}(x))$$

together with the theory ACU, so $\mathcal{T} = \{x \cdot y \approx y \cdot x,\ (x \cdot y) \cdot z \approx x \cdot (y \cdot z),\ x \cdot \mathsf{e} \approx x\}$. Several equations are orientable in both directions. A multi-completion run thus gives rise to a process tree, where each branch corresponds to a possible sequence of orientations. Part of the process tree developed in a run on CGA run is shown in Figure 3. Note that the equation $\phi(\mathsf{f}(x), \mathsf{g}(y)) \approx \phi(\mathsf{g}(y), \mathsf{f}(x))$ cannot be oriented with AC-RPO or AC-compatible polynomial interpretations. Hence e.g. C*i*ME[7] cannot succeed, but by using muterm [1] for termination checks, mkbtt can produce an ACU-convergent system.

5.2 Implementation

We extended our tool mkbtt [28] to handle normalized multi-completion with termination tools. While the basic control loop remained the same, some changes had to be made to apply normalized completion. First of all, an AC-convergent TRS $\mathcal{S}$ representing the theory $\mathcal{T}$ is fixed and all terms are kept in $\mathcal{S}$-normalized form. The TRS $\mathcal{S}$ can be supplied by the user, otherwise mkbtt detects an applicable theory automatically (currently ACU, groups and

[7] We compared with C*i*ME 3.0.2, see `http://cime.lri.fr` and [7].

	mkbtt								CiME[3]
	AC		ACU		AG		auto		
	(1)	(2)	(1)	(2)	(1)	(2)	(1)	(2)	(1)
abelian groups (AG)	1.6	77	2.4	61	0.1	5	0.1	5	0.05
AG + homomorphism	181.7	928	173.5	993	4.8	104	4.8	104	0.05
G0	1.9	82	1.9	70	0.1	8	0.1	8	?
G2	∞		∞		12.4	49	12.5	49	?
arithmetic	14.9	503	15.1	483	–		13.8	483	?
AC ring with unit	22.9	501	28.5	466	7.2	301	0.1	9	0.1
binary arithmetic	2.9	199	2.8	185	–		3.0	185	?
ternary arithmetic	18.1	816	17.3	781	–		17.3	781	?
Example 4.1	0.3	26	0.2	17	–		0.3	26	?
Example 5.1	∞		∞		15.4	486	15.2	486	?
Example 5.2	∞		∞		216.7	457	145.1	400	?
semiring	3.3	209	3.6	192	–		3.5	193	0.1
sum	1.4	4	1.5	5	–		1.4	4	?
completed systems	10		10		7		13		4

Table 1 Comparison of mkbtt using different theories.

rings are supported, besides AC). We use general normalizing pairs, thus the orient inference had to be changed to add equations in the Θ component. Currently we always compute AC-critical pairs. In order to limit the number of nodes, the critical pair criteria described in Section 3.2 were implemented. Termination checks required in orient inference steps may be performed by an external termination tool supporting AC termination. Alternatively, mkbtt can also apply AC-RPO [23] or AC-KBO [13] internally. Further details can be found in [28] or obtained from the mkbtt website.[8]

5.3 Experiments

To evaluate our approach we ran mkbtt on problems collected from a number of different sources. All of the following tests were performed on an Intel Core Duo running at a clock rate of 1.4 GHz with 2.8 GB of main memory. Termination checks were done with muterm, and the primality critical pair criterion was used. The global timeout and the timeout for each termination check were set to 300 and 2 seconds, respectively.

In Table 1 we compare the results obtained with mkbtt applying different theories $\mathcal{T}$ (AC, AC with unit (ACU) and the theory of abelian groups (AG)) as well as automatic theory detection. The examples were collected from the literature, and some additional problems were added by the authors. The test set can be obtained from the website, where also the problems' sources are indicated. Columns (1) list the total time in seconds while columns (2) give the number of nodes created during the run. The symbol ∞ marks a timeout, and – indicates that the theory is not applicable. In line with [19], we observed that completion with respect to larger theories $\mathcal{T}$ is typically faster. Only in some cases such as the ring problem ACU-normalized completion is slower than AC-normalized completion, due to an unfortunate selection sequence. As expected, CiME is much faster if an appropriate reduction order is supplied as input. But as already mentioned, such a reduction order is hard to

[8] `http://cl-informatik.uibk.ac.at/software/mkbtt`

determine in advance, and in some cases no usable AC-RPO or polynomial interpretation exists. This is e.g. the case for Example 5.1, where mkbtt is able to find an ACU convergent system in a bit more than one hour, and for the example given below. When comparing AC-RPO with AC-KBO, there are some problems which can only be completed with the latter (e.g. binary arithmetic), but overall AC-RPO is more useful.

Concerning critical pair criteria, we found that the primality criterion decreased the total number of nodes by nearly 40%, which reduces the computation time by about 25%. $\mathcal{S}$-reducibility does not filter out any critical pairs if completion modulo ACU is performed. For normalized completion modulo group theory, very few redundant critical pairs are detected. The connectedness criterion was found to be comparatively expensive, and also the combined criterion could not achieve the same performance gain as the simpler primality criterion due to the additional effort of testing the criterion. Complete tables and more details on experimental results can be obtained from the website.

▶ **Example 5.2.** Consider ring theory with two commuting multiplicative mappings as defined by AC axioms for + together with the equations

$$\begin{array}{rcl} x + 0 \approx x & \mathsf{f}(1) \approx 1 & x \cdot (y+z) \approx (x \cdot y) + (x \cdot z) \\ x + (-x) \approx 0 & \mathsf{g}(1) \approx 1 & (x+y) \cdot z \approx (x \cdot z) + (x \cdot z) \\ 1 \cdot x \approx x & \mathsf{f}(x \cdot y) \approx \mathsf{f}(x) \cdot \mathsf{f}(y) & (x \cdot y) \cdot z \approx x \cdot (y \cdot z) \\ x \cdot 1 \approx x & \mathsf{g}(x \cdot y) \approx \mathsf{g}(x) \cdot \mathsf{g}(y) & \mathsf{f}(x) \cdot \mathsf{g}(y) \approx \mathsf{g}(y) \cdot \mathsf{f}(x) \end{array}$$

Our tool computes a convergent system using normalized completion modulo group theory/ring theory in 216.7/145.1 seconds producing 457/400 nodes, respectively. Normalized completion modulo AC and ACU yields a timeout. Due to the permutative equation $\mathsf{f}(x) \cdot \mathsf{g}(y) \approx \mathsf{g}(y) \cdot \mathsf{f}(x)$ no suitable input for C*i*ME is known.

6 Conclusion

We considered finite normalized completion runs, and give correctness, completeness and uniqueness results using a slightly simpler proof order. Critical pair criteria for this setting were presented and proved correct using a relaxed notion of fairness. In order to tackle the limitation of a fixed reduction order, we proposed the use of automatic termination tools supporting AC-termination. Thus a user does not need to fix an AC-compatible reduction order in advance, a suitable ordering is instead found automatically. We implemented $\mathcal{S}$-normalized multi-completion with termination tools in mkbtt to evaluate our approach, which led to the construction of new convergent systems.

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A Proofs of Section 3.2

Proof of Lemma 3.19. Induction on $\succ\!\!\succ_n$ shows that any proof in $\mathcal{T} \cup \mathcal{R}_n$ can be transformed into a normalized rewrite proof. Any non-rewrite proof must contain (i) a peak $s \;{}_{\mathcal{R}_n/\mathrm{AC}}\!\!\leftarrow \cdot \rightarrow_{\mathcal{R}_n/\mathrm{AC}} t$, or (ii) a peak $s \;{}_{\mathcal{R}_n/\mathrm{AC}}\!\!\leftarrow \cdot \rightarrow_{\mathcal{S}/\mathrm{AC}} t$, or (iii) a subproof $u \rightarrow_{\mathcal{R}_n/\mathrm{AC}} t$ such that $u \neq u{\downarrow}$, or (iv) a peak $s \;{}_{\mathcal{S}/\mathrm{AC}}\!\!\leftarrow \cdot \rightarrow_{\mathcal{S}/\mathrm{AC}} t$. In the latter three cases existence of a smaller proof can be argued as in Lemma 3.14. This also holds for (i) if the peak is a non-overlap, or if it is a proper overlap and the respective critical pair occurs in $\bigcup_i \mathcal{E}_i$. In all these cases this smaller proof can thus be transformed into a rewrite proof by the induction hypothesis. It remains to consider the subcase of (i) where there are a proof $P\colon s \;{}_{\mathcal{R}_n/\mathrm{AC}}\!\!\leftarrow u \rightarrow_{\mathcal{R}_n/\mathrm{AC}} t$ and a critical pair $\ell \simeq r \in \mathrm{CP}_{\mathcal{L}}(\mathcal{R}_n, \mathcal{R}_n^e)$ such that $s \leftrightarrow_{\mathcal{L}}^* C[\ell\sigma] \leftrightarrow_{\ell \approx r} C[r\sigma] \leftrightarrow_{\mathcal{L}}^* t$ but $\ell \simeq r$ does not occur in any set $\mathcal{E}_i$. Hence we must have $\ell \simeq r \in \mathrm{CCP}(\mathcal{E}_i, \mathcal{R}_i)$ for some i. Let the corresponding critical overlap be $P'\colon \ell \leftarrow v \leftrightarrow_{\mathcal{L}}^* v' \rightarrow r$, so $P = P[C[P'\sigma]]$. By definition, there are terms $v_0, \ldots, v_{k+1}$ such that $\ell = v_0$, $r = v_{k+1}$ and $v \succ v_j$ for all $0 \leqslant j \leqslant k+1$, and $(\mathcal{E}_i, \mathcal{R}_i)$ admits proofs P_j of $v_j \approx v_{j+1}$ which are smaller than P'. By the Persistence Lemma 3.4 there are respective proofs P_j' in $\mathcal{T} \cup \mathcal{R}_n$ such that $P_j \Rightarrow_n^= P_j'$. By the induction hypothesis all these proofs P_j' can be transformed into normalized rewrite proofs Q_j in $\mathcal{T} \cup \mathcal{R}_n$. Consequently all terms in the combined proof $Q\colon Q_0 \cdots Q_k$ of $\ell \approx r$ must be smaller than v, so $P' \Rightarrow_n Q$ and hence $P = P[C[P'\sigma]] \Rightarrow_n P[C[Q\sigma]]$. Hence, as P can be transformed into a smaller proof it can be transformed into a normalized rewrite proof by the induction hypothesis. ◀

Proof of Lemma 3.20. First we consider the case of an $\mathcal{L}$-critical pair. Let P be a peak of the form (4), and assume $u \rightarrow_{\mathcal{S}/\mathrm{AC}} v$. We thus also have another equational proof $P_1 P_2$ of $s \approx t$, with

$$P_1\colon s \xleftarrow[r_1 \leftarrow \ell_1]{p,\sigma} u \xrightarrow[\mathcal{S}/\mathrm{AC}]{} v \qquad\qquad P_2\colon v \xleftarrow[\mathcal{S}/\mathrm{AC}]{} u \leftrightarrow_{\mathcal{L}}^* u' \xrightarrow[\ell_2 \rightarrow r_2]{\epsilon,\sigma} t$$

As u is $\mathcal{S}/\mathrm{AC}$-reducible we have $c(u', \epsilon, t) = \{u', t\}$, such that the proof costs amount to $c_n(P) = \{(c(u,p,s), \ldots), (\{u',t\}, \ldots)\} \cup c_{\mathcal{L}}(P)$, $c_n(P_1) = \{(c(u,p,s), \ldots), (\bot, \ldots)\} \cup c_{\mathrm{AC}}(P_1)$, and $c_n(P_2) = \{(\bot, \ldots), (\{u', t\}, \ldots)\} \cup c_{\mathrm{AC}}(P_2) \cup c_{\mathcal{L}}(P)$ where $c_{\mathcal{L}}(P)$ refers to the cost of the subproof $u \leftrightarrow_{\mathcal{L}}^* u'$ and $c_{\mathrm{AC}}(P_i)$ corresponds to the complexities of possibly required AC-steps in $u \rightarrow_{\mathcal{S}/\mathrm{AC}} v$. Note that the complexities of AC steps are smaller than the first two cost tuples in $c_n(P)$. We have $P \succ\!\!\succ_n P_1$ and $P \succ\!\!\succ_n P_2$, so the AC-critical pair is composite for NKB. A symmetric argument shows compositeness of any critical pair where u' is $\mathcal{S}/\mathrm{AC}$-reducible.

Let us now consider the case of a non-prime critical pair. As u, u', and v' are in $\mathcal{S}$-normal form, proof costs have the following shape, independent of n:

$$\begin{aligned}
c_n(P) &= \{(\{u\}, \{u\}, u|_p, \ldots), (\{u'\}, \{u'\}, u', \ldots)\} \cup c_{\mathrm{AC}}(P)\\
c_n(P_1) &= \{(\{u\}, \{u\}, u|_p, \ldots), (\{v'\}, \{v'\}, v'|_q, \ldots)\} \cup c_{\mathrm{AC}}(P_1)\\
c_n(P_2) &= \{(\{u'\}, \{u'\}, u', \ldots), (\{v'\}, \{v'\}, v'|_q, \ldots)\} \cup c_{\mathrm{AC}}(P_2)
\end{aligned}$$

From $u' \leftrightarrow_{\mathrm{AC}}^* v'$ we obtain $\{u'\} \succeq_{\mathsf{mul}} \{v'\}$. Therefore $u' \leftrightarrow_{\mathrm{AC}}^* u \trianglerighteq u|_p \;\rhd_{\mathrm{AC}}\; v'|_q$ and thus $u' \;\rhd_{\mathrm{AC}}\; v'|_q$, so we have $P \succ\!\!\succ_n P_1$ and $P \succ\!\!\succ_n P_1$. Furthermore, as $u|_p \;\rhd_{\mathrm{AC}}\; v'|_q$ we have $P \succ\!\!\succ_n P_2$ and $P \succ\!\!\succ_n P_2$ for any $n \geqslant 0$. It follows that P is composite. ◀

B Proofs of Section 4

The proof of Correctness Theorem 4.2 requires the fact that NKBtt simulates NKB runs and vice versa [26].

▶ **Lemma 2.1.**

(1) Any NKBtt *run* $(\mathcal{E}_0, \varnothing, \varnothing) \vdash^n (\mathcal{E}_n, \mathcal{R}_n, \mathcal{C}_n)$ *admits an* NKB *run* $(\mathcal{E}_0, \varnothing) \vdash^n (\mathcal{E}_n, \mathcal{R}_n)$ *using the AC-compatible reduction order* $\rightarrow^+_{(\mathcal{C}_n \cup \mathcal{S})/\mathrm{AC}}$*.*

(2) If $(\mathcal{E}_0, \varnothing) \vdash^n (\mathcal{E}_n, \mathcal{R}_n)$ *is a valid* NKB *run using an AC-compatible reduction order* $\succ$ *then there is also a valid* NKBtt *run* $(\mathcal{E}_0, \varnothing, \varnothing) \vdash^n (\mathcal{E}_n, \mathcal{R}_n, \mathcal{C}_n)$ *such that* $\mathcal{C}_n \subseteq \succ$*.*

Proof.

(1) Note that all TRSs $\mathcal{C}_i \cup \mathcal{S}$ are AC terminating. The relations $\rightarrow^+_{(\mathcal{C}_i \cup \mathcal{S})/\mathrm{AC}}$ are thus AC-compatible reduction orders, which we abbreviate by $\succ_i$. We prove the claim by induction on n, which is trivial for $n = 0$. For an NKBtt run $(\mathcal{E}_0, \varnothing, \varnothing) \vdash^* (\mathcal{E}_n, \mathcal{R}_n, \mathcal{C}_n) \vdash (\mathcal{E}_{n+1}, \mathcal{R}_{n+1}, \mathcal{C}_{n+1})$, the induction hypothesis yields a normalized completion run $(\mathcal{E}_0, \mathcal{R}_0) \vdash^* (\mathcal{E}_n, \mathcal{R}_n)$ using the reduction order $\succ_n$. Since constraint rules are never removed we have $\mathcal{C}_n \subseteq \mathcal{C}_{n+1}$, so the same run can be obtained with $\succ_{n+1}$. Case distinction on the applied NKBtt rule shows that a step $(\mathcal{E}_n, \mathcal{R}_n) \vdash (\mathcal{E}_{n+1}, \mathcal{R}_{n+1})$ using $\succ_{n+1}$ is possible in NKB: If orient is applied to $s \simeq t$ then $\Psi(s,t) \subseteq \succ_{n+1}$ by definition, so NKB can apply orient as well. In all remaining cases the step can obviously be simulated by the corresponding NKB rule as no conditions on the order are involved.

(2) By induction on n. For $n = 0$ the claim is trivially satisfied by setting $\mathcal{C}_0 = \varnothing$. So suppose $(\mathcal{E}_0, \varnothing) \vdash^n_{\mathsf{NKB}} (\mathcal{E}_n, \mathcal{R}_n) \vdash (\mathcal{E}_{n+1}, \mathcal{R}_{n+1})$. The induction hypothesis yields an NKBtt run $(\mathcal{E}_0, \varnothing, \varnothing) \vdash^* (\mathcal{E}_n, \mathcal{R}_n, \mathcal{C}_n)$ such that $\mathcal{C}_n \subseteq \succ$. An easy case distinction on the last inference step $(\mathcal{E}_n, \mathcal{R}_n) \vdash (\mathcal{E}_{n+1}, \mathcal{R}_{n+1})$ shows that using $\succ$ for AC termination checks allows for a corresponding NKBtt step. If the applied inference rule is orient we have $\mathcal{E}_n = \mathcal{E}_{n+1} \cup \{s \simeq t\}$ and $\mathcal{R}_{n+1} = \mathcal{R}_n \cup \Psi(s,t)$ such that $\Psi(s,t) \subseteq \succ$ as (Θ, Ψ) constitutes a normalizing pair. Thus for $\mathcal{C}' = \mathcal{C}_n \cup \Psi(s,t)$ also $\mathcal{C}' \subseteq \succ$ is satisfied, ensuring AC termination of the system $\mathcal{C}' \cup \mathcal{S}$ because $\mathcal{S} \subseteq \succ$ by assumption. Hence the NKBtt inference rule orient can be applied to obtain $(\mathcal{E}_n, \mathcal{R}_n, \mathcal{C}_n) \vdash (\mathcal{E}_n \setminus \{s \simeq t\}, \mathcal{R}_n \cup \Psi(s,t), \mathcal{C}')$. In the remaining cases one can set $\mathcal{C}_{n+1} = \mathcal{C}_n$ and replace the applied rule by its NKBtt counterpart since no conditions on the order are involved. ◀

Proof of Correctness Theorem 4.2. Let $(\mathcal{E}_0, \varnothing, \varnothing) \vdash^n (\varnothing, \mathcal{R}_n, \mathcal{C}_n)$ be a finite and fair run. According to Lemma 2.1(1) the same TRS $\mathcal{R}_n$ can be derived in a fair and nonfailing NKB run using the reduction order $\rightarrow^+_{(\mathcal{C}_n \cup \mathcal{S})/\mathrm{AC}}$. By Theorem 3.15 the TRS $\mathcal{R}_n$ is $\mathcal{S}$-convergent for $\mathcal{E}_0$. ◀