Automating Ordinal Interpretations

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Abstract

In this note we study weakly monotone interpretations for direct proofs of termination which is sound if the interpretation functions are "simple". This is e.g. the case for standard addition and multiplication of ordinal numbers. We compare the power of such interpretations to polynomial interpretations over the natural numbers and report on preliminary experimental results.

1 Introduction

Polynomial interpretations [9] are a well-established termination technique. By now powerful techniques are known for their automation [1,5]. Recently it has been shown that allowing different domains, (e.g., $\mathbb{N}$, $\mathbb{Q}$, $\mathbb{R}$) results in incomparable termination criteria [11,14]. Matrix interpretations consider linear interpretations over vectors or matrices of numbers (in $\mathbb{N}$, $\mathbb{Q}$, $\mathbb{R}$) and have been shown to be powerful in theory and practice [2,4,6,18]. However, for other extensions (e.g., elementary functions [10,12] or interpretations into ordinal numbers [16]) practical implementations remain an open problem.

In this note we revisit polynomial interpretations using ordinals as carrier [16]. Based on recent results [17], we present an implementation for string rewrite systems (with interpretation functions of a special shape), which is the first one to our knowledge. Our efforts could be seen as a first step towards automatically proving the battle of Hercules and Hydra [16] terminating. However—for the encoding of the battle from [3]—Moser [13, Section 7] anticipates that an extension of polynomial interpretations into ordinal domains is not sufficient. In the remainder of this introductory section we recall preliminaries.

Ordinals: We assume basic knowledge of ordinals [8]. By O we denote the set of ordinal numbers strictly less than ϵ_0. Every ordinal $\alpha \in \mathsf{O}$ has a unique representation in Cantor Normal Form (CNF): $\alpha = \sum_{1 \leqslant i \leqslant n} \omega^{\alpha_i} \cdot a_i$, where $a_1, \ldots, a_n \in \mathbb{N} \setminus \{0\}$ and $\alpha_1, \ldots, \alpha_n \in \mathsf{O}$ are also in CNF, with $\alpha > \alpha_1 > \cdots > \alpha_n$. We denote standard addition and multiplication on O (and hence also on $\mathbb{N}$) by $+$ and $\cdot$. We furthermore drop $\cdot$ whenever convenient.

Term Rewriting: We assume familiarity with term rewriting and termination [15]. Let $>$ be a relation and $\geqslant$ its reflexive closure. A function f is monotone if $a > b$ implies $f(\ldots, a, \ldots) > f(\ldots, b, \ldots)$ and weakly monotone if $a > b$ implies $f(\ldots, a, \ldots) \geqslant f(\ldots, b, \ldots)$. A function f is simple if $f(\ldots, a, \ldots) \geqslant a$.

An $\mathcal{F}$-*algebra* $\mathcal{A}$ consists of a carrier set A and an interpretation function $f_{\mathcal{A}} \colon A^k \to A$ for each k-ary function symbol $f \in \mathcal{F}$. By $[\alpha]_{\mathcal{A}}(\cdot)$ we denote the usual evaluation function of $\mathcal{A}$ according to an assignment α which maps variables to values in A. An $\mathcal{F}$-algebra together with a well-founded order $>$ on A is called a (well-founded) $\mathcal{F}$-algebra $(\mathcal{A}, >)$. Often we denote $(\mathcal{A}, >)$ by $\mathcal{A}$ if $>$ is clear from the context. The order $>$ induces a well-founded order on terms: $s >_{\mathcal{A}} t$ if and only if $[\alpha]_{\mathcal{A}}(s) > [\alpha]_{\mathcal{A}}(t)$ for all assignments α. A TRS $\mathcal{R}$ and an algebra $\mathcal{A}$ are *compatible* if $\ell >_{\mathcal{A}} r$ for all $\ell \to r \in \mathcal{R}$. A well-founded algebra $(\mathcal{A}, >)$ is a *monotone (weakly monotone / simple) algebra* if for every function symbol $f \in \mathcal{F}$ the interpretation function $f_{\mathcal{A}}$ is monotone (weakly monotone / simple) in all arguments. By $\mathcal{O}$ ($\mathcal{N}$) we denote well-founded algebras with the carrier O ($\mathbb{N}$) and the standard order $>$.

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For (direct) termination proofs one typically exploits the following theorem.

▶ **Theorem 1.1.** *A TRS is terminating if and only if it is compatible with a well-founded monotone algebra.* ◀

However, monotonicity can be replaced by weak monotonicity, provided the interpretation functions are simple. This result is less known.

▶ **Theorem 1.2** ([19, Proposition 12])**.** *A TRS is terminating if it is compatible with a well-founded weakly monotone simple algebra.* ◀

2 Ordinal Interpretations

Although standard addition, multiplication and exponentiation on ordinals are in general only weakly monotone, Theorem 1.2 nevertheless constitutes a way to use interpretations into the ordinals with these operations in termination proofs.

The next example shows that (fairly small) ordinals add power to linear interpretations.

▶ **Example 2.1.** Consider the SRS $\mathcal{R}$ consisting of the rule $\mathsf{a}(\mathsf{b}(x)) \to \mathsf{b}(\mathsf{a}(\mathsf{a}(x)))$. The linear ordinal interpretation

$$\mathsf{a}_{\mathcal{O}}(x) = x + 1 \qquad\qquad \mathsf{b}_{\mathcal{O}}(x) = x + \omega$$

is simple and proves termination of $\mathcal{R}$ since $x + \omega + 1 >_{\mathcal{O}} x + 1 + 1 + \omega = x + \omega$. Linear interpretations with coefficients in $\mathbb{N}$ are not sufficient. Assuming abstract interpretations $\mathsf{a}_{\mathcal{N}}(x) = a_1 x + a_0$ and $\mathsf{b}_{\mathcal{N}}(x) = b_1 x + b_0$, we obtain the constraints

$$a_1 b_1 \geqslant b_1 a_1 a_1 \qquad\qquad a_1 b_0 + a_0 > b_1 a_1 a_0 + b_1 a_0 + b_0$$

Since $\mathsf{a}_{\mathcal{N}}$ and $\mathsf{b}_{\mathcal{N}}$ must be simple (or monotone) $a_1, b_1 \geqslant 1$. From the first constraint we conclude $a_1 = 1$, which makes the second one unsatisfiable.

In the rest of this note we consider ordinal interpretations (for SRSs) of the following shape

$$f_{\mathcal{O}}(x) = x \cdot f' + \omega^d \cdot f_d + \cdots + \omega^1 \cdot f_1 + f_0 \tag{1}$$

where $f', f_d, \ldots, f_0 \in \mathbb{N}$. Interpretations of the shape (1) will be called *linear ordinal interpretations (of degree d)*. Note that interpretations of the shape (1) are weakly monotone and simple if $f' \geqslant 1$. To show the power of linear ordinal interpretations (with respect to the derivational complexity) we define the parametrized SRS $\mathcal{R}_m$.

▶ **Definition 2.2.** For any $m \in \mathbb{N}$ the SRS $\mathcal{R}_m$ consists of the rules

$$\mathsf{a}_i(\mathsf{a}_{i+1}(x)) \to \mathsf{a}_{i+1}(\mathsf{a}_i(\mathsf{a}_i(x))) \qquad\qquad \mathsf{a}_{i+1}(x) \to x$$

for each $0 \leqslant i < m$.

Note that $\mathcal{R}_0$ is empty. We have the following properties.

▶ **Lemma 2.3.** *For any $\mathcal{R}_m$ and $i \leqslant m$ we have $\mathsf{a}_i(\mathsf{a}_{i+1}^n(x)) \to^{2^n-1} \mathsf{a}_{i+1}^n(\mathsf{a}_i^{2^n}(x))$.*

Proof. By induction on n. In the base case $n = 0$ and $\mathsf{a}_i(x) \to^0 \mathsf{a}_i(x)$. In the step case

$$\begin{aligned}\mathsf{a}_i(\mathsf{a}_{i+1}^{n+1}(x)) &\to \mathsf{a}_{i+1}(\mathsf{a}_i(\mathsf{a}_i(\mathsf{a}_{i+1}^n(x)))) \to^{2^n-1} \mathsf{a}_{i+1}(\mathsf{a}_i(\mathsf{a}_{i+1}^n(\mathsf{a}_i^{2^n}(x))))\\ &\to^{2^n-1} \mathsf{a}_{i+1}(\mathsf{a}_{i+1}^n(\mathsf{a}_i^{2^n}(\mathsf{a}_i^{2^n}(x)))) = \mathsf{a}_{i+1}^{n+1}(\mathsf{a}_i^{2^{n+1}}(x))\end{aligned}$$

◀

shape	yes	time (avg.)	timeout (60s)
linear interpretations	19	0.8	1
linear ordinal interpretations (degree 1)	40	2.5	1
linear ordinal interpretations (degree 2)	40	3.8	6
linear ordinal interpretations (degree 3)	38	2.1	21
$\sum$	40	–	–

Table 1 Evaluation on 720 SRSs of TPDB 7.0.2

▶ **Lemma 2.4.** *We have* $\mathsf{a}_0(\mathsf{a}_1(\cdots(\mathsf{a}_{m-1}(\mathsf{a}_m^n(x))))) \to^*_{\mathcal{R}_m} \mathsf{a}_0{}^{2^{2^{\cdot^{\cdot^{\cdot^{2^n}}}}}}(x)$ *where the tower of* 2*'s has height* m*.*

Proof. By induction on m. In the base case $m = 0$ and the claim trivially holds. In the step case we have

$$\mathsf{a}_0(\cdots(\mathsf{a}_m(\mathsf{a}_{m+1}^n(x)))) \to^{2^n-1} \mathsf{a}_0(\cdots(\mathsf{a}_{m+1}^n(\mathsf{a}_m^{2^n}(x)))) \to^n \mathsf{a}_0(\cdots(\mathsf{a}_m^{2^n}(x))) \to^* \mathsf{a}_0{}^{2^{2^{\cdot^{\cdot^{\cdot^{2^n}}}}}}(x)$$

where Lemma 2.3 is applied in the first step and the induction hypothesis in the last step. ◀

As a consequence of Lemma 2.4 we get that $\mathsf{dc}_{\mathcal{R}_m}(n) = \Omega(2^{2^{\cdot^{\cdot^{\cdot^{2^n}}}}})$ where the tower of 2's has height m.

▶ **Lemma 2.5.** *For every* $\mathcal{R}_m$ *with* $m \in \mathbb{N}$ *there exists a compatible linear ordinal interpretation of degree* m *but not of degree* $m - 1$*.*

Proof. To show the first item we take $(\mathsf{a}_i)_{\mathcal{O}}(x) = x+\omega^i$. Then $\mathsf{a}_i(\mathsf{a}_{i+1}(x)) >_{\mathcal{O}} \mathsf{a}_{i+1}(\mathsf{a}_i(\mathsf{a}_i(x)))$ because of $x + \omega^{i+1} + \omega^i > x + \omega^i + \omega^i + \omega^{i+1} = x + \omega^{i+1}$ and $\mathsf{a}_{i+1}(x) >_{\mathcal{O}} x$ because of $x + \omega^{i+1} > x$ for all $x \in \mathsf{O}$. The second item follows from the claim that for any linear ordinal interpretation compatible with $\mathcal{R}_m$ we have that at least ω^i occurs in $(\mathsf{a}_i)_{\mathcal{O}}(x)$. The claim is proved by induction on m. ◀

From Lemma 2.5 we infer that allowing larger degrees increases the power of linear ordinal interpretations and in connection with Lemma 2.4 it shows that linear ordinal interpretations can prove SRSs terminating whose derivational complexity is multiple exponential.

3 Implementation and Evaluation

We implemented linear ordinal interpretations for SRSs of the shape (1). As illustration, we abstractly encode the rule $\mathsf{a}(\mathsf{b}(x)) \to \mathsf{b}(\mathsf{a}(\mathsf{a}(x)))$ with $d = 1$. For the left-hand side we get

$$x \cdot b' \cdot a' + \omega^1 \cdot b_1 \cdot a' + b_0 \cdot a' + \omega^1 \cdot a_1 + a_0$$

which can be written in the canonical form

$$x \cdot b' \cdot a' + \omega^1 \cdot (b_1 \cdot a' + a_1) + (a_1 > 0 \mathbin{?} 0 : b_0 \cdot a') + a_0$$

where the $(\cdot \mathbin{?} \cdot : \cdot)$ operator implements if-then-else, i.e., if a_1 is greater than zero then the summand $b_0 \cdot a'$ vanishes. To determine whether

$$x \cdot l' + \omega^1 \cdot l_1 + l_0 \geqslant x \cdot r' + \omega^1 \cdot r_1 + r_0$$

for all values of x, we use the criterion $l' \geqslant r' \wedge (l_1 > r_1 \vee (l_1 = r_1 \wedge l_0 \geqslant r_0))$. Finally, $f' \geqslant 1$ ensures that the interpretation $f_{\mathcal{O}}$ is simple while the interpretation functions are then weakly monotone for free. Hence the search for suitable coefficients can be encoded in non-linear integer arithmetic.

The results[1] are given in Table 1 where 4 bits are used to represent the coefficients $f_0, \ldots, f_n, f'$ and 8 bits are allowed for intermediate calculations. The column labeled "yes" indicates how many systems the given method could show terminating. Times are given in seconds.

4 Conclusion

We conclude this note with a short discussion on the relationship of linear ordinal interpretations with matrix interpretations [4]. In contrast to the latter the induced ordering is still total which makes it valuable for ordered completion. Secondly as Lemmata 2.4 and 2.5 show interpretations of the shape (1) allow to prove termination of SRSs whose derivational complexity is beyond exponential while matrix interpretations are restricted to an exponential upper bound.

Concerning future work we want to investigate if and how Theorem 1.2 could make automated termination and complexity tools more powerful.

For matrix interpretations over $\mathbb{N}$ (as defined in [4]) the answer is that Theorem 1.2 does not increase the power of the method. The reason is that the condition for a function to be simple ($M_{ii} \geqslant 1$ for all $1 \leqslant i \leqslant d$ where d is the dimension of the matrices) is a stronger requirement than monotonicity demanding $M_{11} \geqslant 1$.

However, if one considers matrix interpretations over O then additional termination proofs can be obtained (note that any linear ordinal interpretation corresponds to a matrix interpretation over O).

Another natural question is whether Theorem 1.2 helps arctic interpretations. Because of monotonicity requirements, direct proofs with arctic matrices are currently limited to dummy systems (SRSs augmented with constants).

Finally we recall that Theorem 1.2 allows direct proofs with polynomial interpretations augmented with "max". This has already been observed in [19, example on p. 13] but seems to have been forgotten. A similar statement holds for quasi-periodic functions [20].

As future work we want to consider linear ordinal interpretations for TRSs. The problem for TRSs is that for comparisons of polynomials the absolute positiveness approach [7] might not apply. To see this note that $f_1 \geqslant g_1$ and $f_2 \geqslant g_2$ does not imply $x \cdot f_1 + y \cdot f_2 \geqslant y \cdot g_2 + x \cdot g_1$ for all values of x and y if $f_1, f_2, g_1, g_2 \in \mathbb{N}$ and $x, y \in \mathsf{O}$. To also cope with such cases we propose a combination of *standard* and *natural* operations on ordinals, as illustrated in the following example, where $\oplus$ denotes natural addition on O.

▶ **Example 4.1** (Adapted from [17, Example 17]). Consider the TRS $\mathcal{R}$ consisting of the single rule $\mathsf{s}(\mathsf{f}(x, y)) \rightarrow \mathsf{f}(\mathsf{s}(y), \mathsf{s}(\mathsf{s}(x)))$. The weakly monotone interpretation $\mathsf{f}_{\mathcal{O}}(x, y) = (x \oplus y) + \omega$ and $\mathsf{s}_{\mathcal{O}}(x) = x + 1$ is simple and induces a strict decrease between left- and right-hand side:

$$(x \oplus y) + \omega + 1 >_{\mathcal{O}} ((y + 1) \oplus (x + 2)) + \omega = (x \oplus y) + 3 + \omega = (x \oplus y) + \omega$$

[1] Details are available from `http://colo6-c703.uibk.ac.at/ttt2/tpoly/`.

Hence $\mathcal{R}$ can be oriented by a linear ordinal interpretation. Again, linear interpretations with coefficients in $\mathbb{N}$ are not sufficient. Assuming abstract interpretations $\mathsf{f}_{\mathcal{N}}(x, y) = f_1 x + f_2 y + f_0$ and $\mathsf{s}_{\mathcal{N}}(x) = s_1 x + s_0$, we get the constraints

$$s_1 f_1 \geqslant f_2 s_1 s_1 \qquad s_1 f_2 \geqslant f_1 s_1 \qquad s_1 f_0 + s_0 > f_1 s_0 + f_2(s_0 + s_1 s_0) + f_0$$

Since $\mathsf{s}_{\mathcal{N}}$ and $\mathsf{f}_{\mathcal{N}}$ must be simple (or monotone) $s_1, f_1, f_2 \geqslant 1$. From the first two constraints we conclude $s_1 = 1$, such that the third simplifies to $f_0 + s_0 > f_0 + (f_1 + 2f_2)s_0$. This contradicts f_1 and f_2 being positive.

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