Acyclicity is Modular for Orthogonal TRSs

Acyclicity is shown to be a modular property of orthogonal first-order term rewriting systems (TRSs), which answers a question raised by Klop. Here, a rewrite system $\to$ is *acyclic* if it doesn't allow non-empty cycles, where a *cycle on t* is a reduction sequence of the form $t \twoheadrightarrow t$.

We assume knowledge of modularity, standardisation, tracing, and externality in term rewriting. To be specific, we freely make use of the notions and results in Sections 5.7.1, 8.5, 8.6.1, and 9.2.3 of [3]. For more information on acyclicity we refer the reader to [1].

Lemma (Standard Prefix). *In an orthogonal TRS, the collection of standard reductions ending in a head step, is totally ordered by the prefix relation.*

Proof. First note that any standard reduction ending in a head step can be uniquely decomposed into a number of standard reductions such that *only* their final step is a head step. Thus to prove the lemma, it suffices to show that for any term there is *at most one* reduction of the latter type. We claim that a standard reduction of which only the final step is a head step is in fact an external reduction. The result then follows since supposing ρ and θ would be distinct such reductions from t, we may assume w.l.o.g. that they already differ in their first steps, say ϕ and ψ, and by totality of the textual order, we may assume w.l.o.g. that ϕ is to the left of ψ. Therefore, by externality of ϕ and standardness of θ, ϕ must have a unique residual up to the final step of θ. But that final step is a head step, so clearly it nests the residual of ϕ, contradicting externality of ϕ.

We prove the claim that any standard reduction of which only the final step is a head step, is an external reduction by induction on the lengh of the reduction. If the length is 1, it is trivial. Otherwise, we may write the reduction as $\phi \cdot \rho$ for some step ϕ. Since any suffix of a standard reduction is standard, the induction hypothesis yields that ρ is an external reduction.

For a proof by contradiction, suppose that ϕ contracting a redex-pattern at position p were not external. That is, a reduction θ co-initial to ρ would exist, consisting of steps disjoint from p and ending in a term allowing a step ψ at position q nesting p. By standardness of $\phi \cdot \rho$, the position p is in the redex-pattern of the first step above p in ρ (if any). As ρ ends in a head step such a step indeed exists, say it is ϕ' at position p' above p, and let ρ' be the prefix of ρ up to ϕ'.

Now consider the projections $\underline{\theta}$ of θ over ρ', and $\underline{\rho'}$ of ρ' over θ. Since by construction neither θ nor ρ' contracts redex-patterns on the path from the root to p, neither do their projections, hence the common reduct contains unique residuals of both ψ after $\underline{\rho'}$ and of ϕ' after $\underline{\theta}$, respectively at positions q and p' above p. Since the positions above p are totally ordered by the prefix relation, either of q and p' is above the other. We prove that neither is possible.

If q is properly above p', then the reduction $\underline{\theta}$ disjoint from p' and ending in a term containing a redex-pattern nesting p', shows that ϕ' is not external, contradicting the induction hypothesis. If p' is above q, then since q is above p and the redex-pattern at p' overlaps p, the redex-patterns of ϕ' and ψ must have overlap in the common reduct of $\overline{\theta}$ and $\overline{\rho'}$, contradicting orthogonality. ☐

Theorem. *Acyclicity is modular for orthogonal TRSs.*

Proof. Let $\mathcal{R}_b \uplus \mathcal{R}_w$ be the disjoint union of the orthogonal TRSs $\mathcal{R}_b$ and $\mathcal{R}_w$. To prove that acyclicity is a modular property is to prove that the underlying rewrite system $\to_{\mathcal{R}_b \uplus \mathcal{R}_w}$, which we will abbreviate to $\to$, is acyclic if both $\to_{\mathcal{R}_b}$ and $\to_{\mathcal{R}_w}$ are. For a proof by contradiction, assume that $\to_{\mathcal{R}_b \uplus \mathcal{R}_w}$ would allow a non-empty cycle σ on some term t, which we may w.l.o.g. assume to be of minimal rank. Since the rewrite systems $\to_{\mathcal{R}_b}$ and $\to_{\mathcal{R}_w}$ are acyclic by assumption, the rank of t must be positive, say it is $n+1$. By minimality, at least a single step in σ must contract a redex-pattern in the top layer. Finally, w.l.o.g. we may assume t to have a minimal number, say $m+1$, of principal subterms of maximal rank, i.e. of rank n.

Since rewriting does not increase the rank, the fact that σ is a cycle entails that the rank of all the terms along σ must be $n+1$, so none of them has a principal subterm of rank greater than n. Now, let $\vec{p}$ be the vector of positions of principal subterms of maximal rank in t. We claim that for some index i and some positive k, p_i is its own origin when tracing p_i *back* along the k-fold repetition σ^k of σ. The claim holds true by the Pigeon Hole Principle and the fact that a principal subterm of maximal rank has another such subterm as origin.[1] Let s be the principal subterm of maximal rank at the position p_i given by the claim. We show that from t we can obtain a term t' which also allows a non-empty cycle but has at most m principal subterms of rank n, yielding a contradiction. The term t' is obtained from t by replacing a number of principal subterms, the subterm s at position p_i inclusive, by a term s' the rank of which is less than that of s.

The replacement term s' is defined as follows. If s allows some reduction having a destructive step in its top layer, then as standardisation preserves this and a destructive step in the top layer is a head step, the Standard Prefix Lemma yields some standard reduction ρ from s ending in a destructive step, which is least among such in the prefix order, and we let s' be the target of ρ. Otherwise, we let s' be a fresh variable. Either way, the rank of s' is less than the rank of s.

To see which principal subterms, other than s at position p_i, of t are to be replaced by s', we proceed as follows. Consider tracing p_i *forward* along an infinite repetition of σ^k, where we only let a position trace as long as it is the position of a principal subterm.[2] Then we let $\vec{p'}$ be the collection of all descendants which occur in t after some repetititon of σ^k, and let t' be obtained from t by replacing all subterms at positions in $\vec{p'}$ by s'. Note that by the above, p_i itself is among the $\vec{p'}$, and that by construction the subterms of t at positions in $\vec{p'}$ are reachable from s.

Next, we show the non-empty cycle σ^n on t can be *simulated* by a non-empty cycle σ' on t', by simulating each step $\phi : u \to v$ by a *reduction* $\phi' : u' \twoheadrightarrow v'$ depending on the relative positions of the redex-pattern contracted in ϕ and the (pairwise disjoint) descendants of $\vec{p'}$ in u. The invariant is that u' is obtained from u by replacing all subterms at positions of descendants of $\vec{p'}$ by s'.

- If ϕ contracts a redex-pattern *outside*, i.e. in the context of, the descendants of $\vec{p'}$ in u, then we let $\phi' : u' \to v'$ be obtained by contracting the same redex-pattern in u'.
- If ϕ contracts a redex-pattern *inside* some descendant of $\vec{p'}$ and ϕ is not destructive at its top layer, then we let $\phi' : u' \twoheadrightarrow u'$ be the empty reduction.
- If ϕ is a destructive step at a descendant p of $\vec{p'}$, then the subterm $v_{|p}$ is reachable from s as noted above, hence per construction of s' and the Standard Prefix Lemma, there also exists some reduction from s to $v_{|p}$ *via* s', thus using $u' = u'[s']_p$ we may set $\phi' : u'[s']_p \twoheadrightarrow u'[v_{|p}]_p$.

That σ' is non-empty follows from the fact that σ contains at least one step in its top layer, which will be simulated by exactly one step in σ' according to the first item of the simulation. To show that σ' is a cycle, it suffices to show that each position p among $\vec{p'}$ in t traces *back* along σ^k to some position in that set again, which is trivial per construction of the set. □

This complements the results on modularity of acyclicity in [2] based on the distribution of collapsing and duplicating rules. Answering questions of Middeldorp, note the proof method also yields modularity of absence of non-empty *fixed-point* reductions of shape $t \twoheadrightarrow C[t]$ for orthogonal TRSs, and neither non-overlappingness nor left-linearity can be omitted from orthogonality:

Example. *Let $\mathcal{R}_b$ either be the overlapping left-linear TRS with rules $\{g(x, y) \to x, g(x, y) \to y\}$ or the non-overlapping non-left-linear TRS with rules $\{g(x, y, z, z) \to x, g(x, y, z, S(z)) \to y, \infty \to S(\infty)\}$. In either case, $\mathcal{R}_b$ is acyclic since applying a g-rule decreases the number of g-symbols. Combining either with the acyclic orthogonal TRS $\mathcal{R}_w$ $\{f(0, 1, x) \to f(x, x, x)\}$ yields a cyclic combination $\mathcal{R}_b \uplus \mathcal{R}_w$, as can be seen from $f(0, 1, g(0, 1))$ or $f(0, 1, g(0, 1, \infty, \infty))$, respectively.*

References

[1] J. Ketema, J.W. Klop, and V. van Oostrom. Vicious circles in orthogonal term rewriting systems. *Electronic Notes in Theoretical Computer Science*, 124(2):65–77, April 2005. Proceedings WRS 2004.

[2] A. Middeldorp and H. Ohsaki. Type introduction for equational rewriting. *Acta Informatica*, 36(12):1007–1029, 2000.

[3] Terese. *Term Rewriting Systems*. Cambridge University Press, 2003.

[1] Note that the claim need not hold when fixing k to 1. For instance, σ might swap two principal subterms.

[2] Per construction positions trace statically; a redex-pattern overlapping one would be polychrome *quod non*.