Properties of Needed Strategies

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Properties

- Normalisation
- Hyper-normalisation
- Head-normalisation
- Co-finality
- Perpetuality

Properties of strategies

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Typical Result

Theorem

The needed strategy is normalising for combinatory logic.

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\begin{aligned}
(K \cdot x) \cdot y &\longrightarrow x \\
((S \cdot x) \cdot y) \cdot z &\longrightarrow (x \cdot z) \cdot (y \cdot z)
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Example

$$(K \cdot I) \cdot \Omega \longrightarrow I$$

$$
\begin{aligned}
I &= (S \cdot K) \cdot K \\
\Omega &= ((S \cdot I) \cdot I) \cdot ((S \cdot I) \cdot I)
\end{aligned}
$$

Normalisation

Definition

Strategy is normalising if terminating on normalisable objects

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- outermost strategy need not be normalising
  $f(a, a)$ w.r.t. $a \to a$, $f(x, a) \to b$
  Indeed, it's not fair

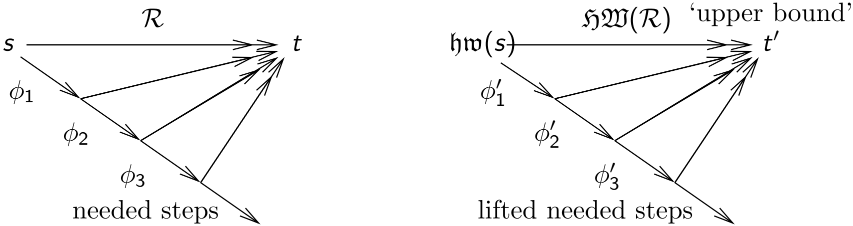
Needed Normalisation

Theorem

Needed strategy is normalising (for ortho fe HRS)

Proof.

all reductions to normal form are permutation equivalent



only finitely many steps contribute to the head □

Needed hyper-normalisation

Definition

Strategy is hyper-normalising if terminating on normalisable objects allowing (finitely sequences) of other steps in-between

Theorem

Needed strategy is hyper-normalising (for ortho fe HRS)

Proof.

Same □

Normalisation of parallel-outermost strategy

Definition

Parallel outermost: contract all outermost redexes

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Parallel-outermost strategy is normalising

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One of outermost redexes is external, so needed □

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Full substitution: contract all redexes
(Gross-Knuth, Complete Development, Takahashi ‘star’)

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eventually they must be eliminated

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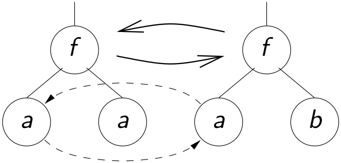
$\{a \to b, f(x,b) \to f(x,a)\}$



Normalisation of outermost-fair strategy

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Outermost fair: treat outermost-redexes fairly
eventually they must be eliminated

Theorem

Outermost-fair strategy is normalising

Proof.

One of outermost redexes is external, so needed
Eventually contracted because of fairness □

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non-erasing if all redexes persist until contracted
(may be duplicated, not erased)

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Example (of non-erasing rewrite systems)

- S-terms
- λI-terms: for $\lambda x.M$, x in free variables of M

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Example (of non-erasing rewrite systems)

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Normalising terms are terminating in non-erasing system

Proof.

In non-erasing system every redex is needed ☐

Head-normalisation

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head normal form if cannot reduce to a redex

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Strategy is head-normalising if eventually reaches head-normal form on head-normalisable objects

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- computing Böhm trees (λ-calculus)

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Example

- computation of infinite list of prime numbers
- computing Böhm trees (λ-calculus)
- ...

Head-needed is head-normalising

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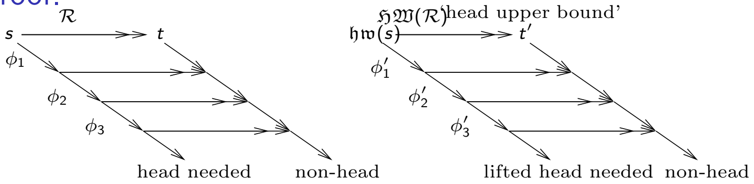
Definition

Head-needed, if needed to reach head-normal form

Theorem

Head-needed strategy is (hyper-)head-normalising (for ortho fe HRS)

Proof.



□

Applications of head-neededness

Lemma

External redex is head-needed, if not in head-normal form

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Parallel-outermost is head-normalising

Applications of head-neededness

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Applications of head-neededness

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External redex is head-needed, if not in head-normal form

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Outermost-fair strategy is head-normalising

Weakly orthogonal systems?

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Some things work some things don't

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Theorem

Outermost-fair reduction is normalising

Weakly orthogonal systems?

Some things work some things don't

Theorem

Outermost-fair reduction is normalising

Outermost-fair reduction is not head-normalising

Example

Term $s = f(g(a,a))$ in

$$\begin{array}{rcl} a & \rightarrow & b \\ f(g(a,x)) & \rightarrow & f(g(b,x)) \\ g(b,x) & \rightarrow & g(x,x) \end{array}$$

Reduction to head-normal form:

$$s \rightarrow f(g(a,b)) \rightarrow f(g(b,b))$$

Outermost-fair reduction not reaching a head-normal form:

$$s \rightarrow f(g(b,a)) \rightarrow s \rightarrow \cdots$$

Co-finality

Definition

Strategy co-final if target of any finite reduction reduces to some object on strategy

Theorem

Full-substitution is co-final

Co-finality

Definition

Strategy co-final if target of any finite reduction reduces to some object on strategy

Theorem

Full-substitution is co-final

Parallel-outermost is not co-final

Example

$f(x) \to f(x)$, $a \to b$

Perpetuality

Definition

Strategy perpetual if non-normalising on non-terminating objects

Theorem

internal needed strategy is perpetual

Summary

- Strategies via intensional ARSs

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- Theorems shown by working with proof terms

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- Strategies via intensional ARSs
- Theorems shown by working with proof terms
- Standard results proven using neededness