Preponement

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Abstract We refine the analysis of [1] for rewrite relations $\to = \triangleright \cup \blacktriangleright$.

Preponement Theorem. Repeatedly replacing in a $\to$-reduction sequence the first occurrence of a subsequence of shape $a \triangleright \cdot \blacktriangleright b$ for some a, b, by either

(i) $a \triangleright b$; or

(ii) $a \blacktriangleright \cdot \twoheadrightarrow b$,

yields a coinitial $\to$-reduction sequence in which every object $\to$-reduces to an object in the original sequence, and in which $\blacktriangleright$-steps precede $\triangleright$-steps, except possibly for an infinite tail of $\blacktriangleright$-steps in case transformation (i) was applied infinitely often. The transformation preserves infiniteness and, in case only transformation (ii) was applied, the number of $\blacktriangleright$-steps (it may increase).

Proof. Clearly, both infiniteness and the property that every object in the transformed sequence $\to$-reduces to some object in the original one are preserved by each transformation step. We show that an ever growing prefix remains stable, is such that $\blacktriangleright$-steps precede $\triangleright$-steps, and in case only transformation (ii) was applied includes an ever growing number of the $\blacktriangleright$-steps, initially taking the empty prefix. Suppose from some stage on a prefix of length n remains stable in the sense that no subsequence involving a step to the left of the nth object is replaced in subsequent transformation steps. Clearly, $\blacktriangleright$-steps must precede $\triangleright$-steps in it. Distinguish cases on the $\to$-step to its right.

- if there is no such step, the process stops as desired.
- if the first step to the right of the prefix is a $\triangleright$-step, then either it is not involved in any subsequent transformation step and adjoining it to the right of the prefix yields a prefix of length $n+1$ stable from that stage on, or it is at some later stage the left step of a transformed subsequence of shape $a \triangleright b \blacktriangleright b'$ and we distinguish case on the transformation step:

  (i) then $a \triangleright b'$ and note that as $b \blacktriangleright b'$ if we infinitely often end up in this case we obtain a reduction comprising the prefix and $a \triangleright b \blacktriangleright b' \blacktriangleright \ldots$, from which we conclude. Otherwise at some stage either the $\triangleright$-step from a becomes stable or the next item applies;

  (ii) then $a \blacktriangleright \cdot \twoheadrightarrow b'$ and in the next stage the next item applies.

  If the suffix contains $\blacktriangleright$-steps and only transformation (ii) is used, then case (ii) applies causing an increase in the number of $\blacktriangleright$-steps in the prefix.

- if the first step to the right of the prefix is a $\blacktriangleright$-step, then by the stability assumption, the prefix does not end in a $\triangleright$-step, so in fact comprises only $\blacktriangleright$-steps (possibly 0), and adjoining the $\blacktriangleright$-step to its right yields a prefix of length $n+1$ stable from that stage on.

If the stable prefix ever increases in length then the property that $\blacktriangleright$-steps precede $\triangleright$-steps in it, will also hold for the final $\to$-reduction sequence. $\square$

Observe that under the conditions of the Preponement Theorem an infinite $\to$-reduction sequence is transformed into one having an infinite $\triangleright$- or $\blacktriangleright$-suffix.

Remark. If we require instead that $a \triangleright \cdot \blacktriangleright b$ implies either $a \triangleright b$ or $a \blacktriangleright b'$, where b' allows some infinite $\to$-reduction sequence, the result goes through except that objects may not $\to$-reduce on the original sequence and the number of $\blacktriangleright$-steps may not be preserved.

Corollary ([1])**.** If $\triangleright \cdot \blacktriangleright \subseteq \triangleright \cup (\blacktriangleright \cdot \twoheadrightarrow)$, then $\to$ is terminating if $\triangleright$ and $\blacktriangleright$ are.

Proof. By the Preponement Theorem an infinite $\to$-reduction sequence would yield one with an infinite $\triangleright$- or $\blacktriangleright$-suffix, contradicting termination of $\triangleright$ or $\blacktriangleright$. $\square$

Corollary (Geser, Exercise 1.3.20 in [3])**.** If $\to$ is transitive, then $\to$ is terminating iff $\triangleright$ and $\blacktriangleright$ are.

Proof. By the previous corollary using that $\to \cdot \to \subseteq \to$, entails $\triangleright \cdot \blacktriangleright \subseteq \triangleright \cup \blacktriangleright$. $\square$

Corollary ([2] Lemma 51)**.** If $\triangleright \cdot \blacktriangleright \subseteq \blacktriangleright \cdot \triangleright$, then if a has an infinite $\to$-reduction but not an infinite $\blacktriangleright$-reduction, then a has an infinite $\to$-reduction comprising a finite $\blacktriangleright$-prefix and an infinite $\triangleright$-suffix.

Proof. By the Preponement Theorem, if a has an infinite $\to$-reduction then it has an infinite $\to$-reduction in which the $\blacktriangleright$-steps precede the $\triangleright$-steps, from which we conclude by the assumption that a does not have an infinite $\blacktriangleright$-reduction. $\square$

Corollary (Bachmair and Dershowitz, Exercise 1.3.19 in [3])**.** If $\triangleright \cdot \blacktriangleright \subseteq \blacktriangleright \cdot \twoheadrightarrow$, then $\triangleright^* \cdot \blacktriangleright \cdot \triangleright^*$ is terminating iff $\blacktriangleright$ is.

Proof. The only-if direction being trivial, the Preponement Theorem yields that if there is an infinite $\triangleright^* \cdot \blacktriangleright \cdot \triangleright^*$-reduction from a there is an infinite $\to$-reduction from a in which the $\blacktriangleright$-steps precede the $\triangleright$-steps, having infinitely many $\blacktriangleright$-steps, i.e. an infinite $\blacktriangleright$-reduction sequence from a, contradicting termination of $\blacktriangleright$. $\square$

References

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[2] Alfons Geser, Dieter Hofbauer, Johannes Waldmann, and Hans Zantema. On tree automata that certify termination of left-linear term rewriting systems. *Information and Computation*, 205(4):512–534, April 2007. `doi:10.1016/j.ic.2006.08.007`.

[3] Terese. *Term Rewriting Systems*, volume 55 of *Cambridge Tracts in Theoretical Computer Science*. Cambridge University Press, 2003. `doi:10.2277/0521391156`.