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UNIQUE NORMAL FORMS IN INFINITARY WEAKLY ORTHOGONAL TERM REWRITING

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Abstract. We present some contributions to the theory of infinitary rewriting for weakly orthogonal term rewrite systems, in which critical pairs may occur provided they are trivial.

We show that the infinitary unique normal form property (UN^∞) fails by a simple example of a weakly orthogonal TRS with two collapsing rules. By translating this example, we show that UN^∞ also fails for the infinitary $\lambda\beta\eta$-calculus.

As positive results we obtain the following: Infinitary confluence, and hence UN^∞, holds for weakly orthogonal TRSs that do not contain collapsing rules. To this end we refine the compression lemma. Furthermore, we consider the triangle and diamond properties for infinitary multi-steps (complete developments) in weakly orthogonal TRSs, by refining an earlier cluster-analysis for the finite case.

1. Introduction

While the theory of infinitary term rewriting is well-developed for orthogonal rewrite systems, much less is known about infinitary rewriting in non-orthogonal systems, in which critical pairs between rules may occur. In this paper we consider the simplest such systems, namely weakly orthogonal ones, in which all critical pairs are trivial. Conceptually, weakly orthogonal systems deviate little from orthogonal systems. But for the development of their rewrite theory specific notions and techniques had to be developed [5].

We show that the infinitary rewrite theory known for orthogonal systems fails dramatically in the case of weakly orthogonal systems. In Section 2, we give a simple counterexample to the infinitary unique normal form property UN^∞. Moreover, by a straightforward translation we obtain a counterexample to UN^∞ in the infinitary $\lambda\beta\eta$-calculus (Section 3), the paradigmatic example of a weakly orthogonal higher-order rewrite system.

In the remaining sections we show that, under simple restrictions, much of the theory of infinitary rewriting in orthogonal systems can be regained: we establish the diamond

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property, and consider the triangle property (Section 6) for weakly orthogonal TRSs without collapsing rules. An important ingredient in their proofs is a refinement of the compression lemma (Section 4).

For a general introduction to infinitary rewriting, as well as for notations used in this paper, we refer to [9, Ch.12], [6, 3].

2. A Counterexample to UN^∞ for Weakly Orthogonal Systems

In [3] it is shown that every orthogonal TRS exhibits the infinitary unique normal forms (UN^∞) property, see also [6]. In strong contrast, we will now give a counterexample showing that the UN^∞ property does *not* generalize to weakly orthogonal TRSs. The counterexample is very simple: its signature consists of the unary symbols P and S with the reduction rules: $\mathsf{P}(\mathsf{S}(x)) \to x$ and $\mathsf{S}(\mathsf{P}(x)) \to x$. Clearly this TRS is weakly orthogonal. In the sequel we consider the corresponding string rewrite system (SRS):

$$\mathsf{PS} \to \varepsilon \qquad\qquad \mathsf{SP} \to \varepsilon$$

where ε is the empty word. If w is a finite word, we write w^ω for the infinite word $www\ldots$. Using S and P we have infinite words such as $\zeta = (\mathsf{PS})^\omega$. Note that S^ω and P^ω are the only infinite normal forms, and that ζ only reduces to itself.

Given an infinite PS-word w we can plot in a graph the surplus number of S's of w when stepping through the word w from left to right, see e.g. Figure 1. The graph is obtained by counting S for $+1$ and P for -1. We define $\mathrm{sum}(w,n)$ as the result of this counting up to depth n in the word w (if w is finite we define $\mathrm{sum}(w) = \mathrm{sum}(w,|w|)$):

$$\mathrm{sum}(w,0) = 0 \qquad \mathrm{sum}(\mathsf{S}w, n+1) = \mathrm{sum}(w,n) + 1 \qquad \mathrm{sum}(\mathsf{P}w, n+1) = \mathrm{sum}(w,n) - 1$$

For $w = (\mathsf{SP})^\omega$ the graph takes values, consecutively, $1, 0, 1, 0, \ldots$, for $w = \mathsf{S}^\omega$ it takes $1, 2, 3, \ldots$, and for $w = \mathsf{P}^\omega$ we have $-1, -2, -3, \ldots$.

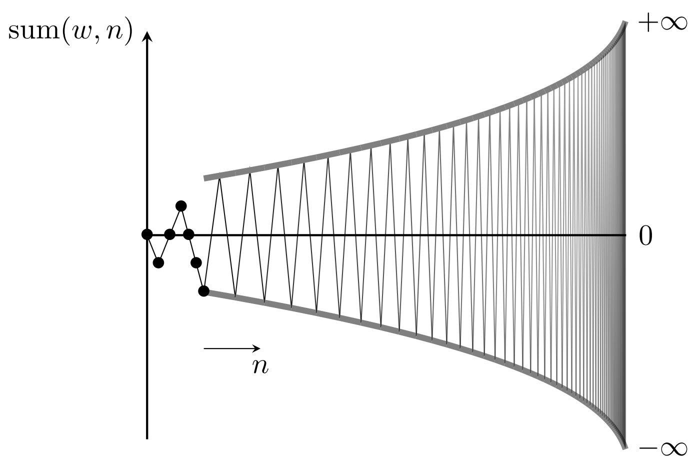


Figure 1: *Graph for the oscillating* PS*-word* $\psi = \mathsf{P}^1\,\mathsf{S}^2\,\mathsf{P}^3\,\ldots$.

We define the S-*norm* $\|w\|_{\mathsf{S}}$ and P-*norm* $\|w\|_{\mathsf{P}}$ of w:

$$\|w\|_{\mathsf{S}} = \sup_{n\in\mathbb{N}} \mathrm{sum}(w,n) \qquad\qquad \|w\|_{\mathsf{P}} = \sup_{n\in\mathbb{N}} (-\mathrm{sum}(w,n)) \tag{2.1}$$

So the S-norm (P-norm) of $(\mathsf{SP})^\omega$ is 1 (0), of S^ω it is ∞ (0), and of P^ω it is 0 (∞).

Lemma 2.1. *Let w be a finite PS-word, and let $n = \mathrm{sum}(w)$. If $n \geq 0$ then $w \twoheadrightarrow \mathsf{S}^n$, and $w \twoheadrightarrow \mathsf{P}^{-n}$, otherwise.*

Proof. For finite words u, v we have that $u \to v$ implies $\mathrm{sum}(u) = \mathrm{sum}(v)$. Moreover, $\to$ is normalising, and the only normal forms are of the form S^k and P^k for $k \geq 0$. ∎

Proposition 2.2.

(i) $w \twoheadrightarrow\!\!\!\!\twoheadrightarrow \mathsf{S}^\omega$ if and only if $\|w\|_\mathsf{S} = \infty$,
(ii) $w \twoheadrightarrow\!\!\!\!\twoheadrightarrow \mathsf{P}^\omega$ if and only if $\|w\|_\mathsf{P} = \infty$.

Proof. We consider only (i) as case (ii) can be treated analogously. From $\|w\|_\mathsf{S} = \infty$ it follows that $w = w_1 w_2 \ldots$ with finite words $w_1, w_2, \ldots$ such that $\mathrm{sum}(w_i) = 1$ for all $i \in \mathbb{N}$. Then $w_i \twoheadrightarrow \mathsf{S}$ for all $i \in \mathbb{N}$ by Lemma 2.1 and hence $w \twoheadrightarrow\!\!\!\!\twoheadrightarrow S^\omega$. ∎

Note that in Proposition 2.2 $w \twoheadrightarrow\!\!\!\!\twoheadrightarrow \mathsf{S}^\omega$ can always be achieved using the rule $\mathsf{PS} \to \varepsilon$ only. And likewise the rule $\mathsf{SP} \to \varepsilon$ for $w \twoheadrightarrow\!\!\!\!\twoheadrightarrow \mathsf{P}^\omega$.

Now let us take a term ψ with $\|\psi\|_\mathsf{S} = \infty$ *and* $\|\psi\|_\mathsf{P} = \infty$! Then by the previous proposition ψ reduces to both S^ω and P^ω, both normal forms. Hence UN^∞ fails. Indeed, such a term ψ can be found:

$$\psi = \mathsf{P}\,\mathsf{SS}\,\mathsf{PPP}\,\mathsf{SSSS}\,\mathsf{PPPPP}\,\mathsf{SSSSSS}\,\ldots$$

The graph for this term is displayed in Figure 1. If we only apply rule $\mathsf{PS} \to \varepsilon$ the P-blocks are absorbed by the larger S-blocks to their right, leaving the normal form S^ω. Likewise, applying only $\mathsf{SP} \to \varepsilon$ yields P^ω.

We find that $\psi \twoheadrightarrow\!\!\!\!\twoheadrightarrow w$ for every infinite PS-word w, and more generally:

Proposition 2.3. *Every PS-word that reduces to both S^ω and P^ω reduces to any infinite PS-word.*

Proof. Let w be a PS-word such that $\mathsf{P}^\omega \twoheadleftarrow\!\!\!\!\twoheadleftarrow w \twoheadrightarrow\!\!\!\!\twoheadrightarrow \mathsf{S}^\omega$. And let u be the infinite PS-word we want to obtain. Then, by Proposition 2.2 we have that $\|w\|_\mathsf{P} = \|w\|_\mathsf{S} = \infty$. From this it follows that $w = w_1 w_2 \ldots$ with w_i finite PS-words such that $\mathrm{sum}(w_i) = 1$ if $u(i) = \mathsf{S}$ and $\mathrm{sum}(w_i) = -1$ if $u(i) = \mathsf{P}$. By Lemma 2.1, we get that $w_i \twoheadrightarrow u(i)$, and hence $w \twoheadrightarrow\!\!\!\!\twoheadrightarrow u$. ∎

Hence, not only is ψ a counterexample to UN^∞ for weakly orthogonal rewrite systems. But also, ψ rewrites to $(\mathsf{PS})^\omega$, a word which has no normal form. Thus, in contrast to orthogonal systems, weak normalisation is not preserved under infinite rewriting.

Figure 2 shows a more detailed analysis of various classes of PS-words. By Proposition 2.2 an infinite word w reduces to S^ω iff $\|w\|_\mathsf{S} = \infty$, and to P^ω iff $\|w\|_\mathsf{P} = \infty$. The shaded non-empty intersection ($\|w\|_\mathsf{S} = \|w\|_\mathsf{P} = \infty$) contains the counterexample term ψ mentioned above. All terms in this intersection are root-active (RA), that is, every $\twoheadrightarrow$-reduct can be reduced to a redex (at the root). However, there are also other root-active terms. For example $\xi = \mathsf{S}\,\mathsf{P}\,\mathsf{S}^2\,\mathsf{P}^2\,\mathsf{S}^3\,\mathsf{P}^3 \ldots$ is a root-active term which reduces to S^ω but not to P^ω (i.e., $\|\xi\|_\mathsf{P} = 0 < \infty$ and $\|\xi\|_\mathsf{S} = \infty$). The term $\xi' = \mathsf{S}\,\xi$ (a reduct of ξ) is not root-active but still not SN^∞, yet it reduces to S^ω. An example of a root-active term which reduces only to itself (implying that $\|\xi\|_\mathsf{S}$ and $\|\xi\|_\mathsf{P}$ are finite) is $\zeta = (\mathsf{P}\,\mathsf{S})^\omega$. The dotted part consists of terms with the property of infinitary strong normalization (SN^∞,[6]), normalizing to S^ω, or P^ω, respectively. For instance $(\mathsf{S}\,\mathsf{S}\,\mathsf{P})^\omega$ is in the left dotted triangle.

The root-active terms can be characterized as follows.

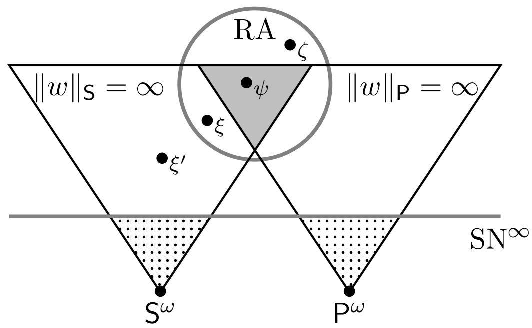


Figure 2: *Venn diagram of infinite* PS*-words.*

Proposition 2.4. *A* PS*-word* w *is root-active if and only if* w *is the concatenation of infinitely many finite 'zero-words'* $w_1, w_2, w_3, \ldots$*, that is, words* w_i *with* $\mathrm{sum}(w_i) = 0$*.* ∎

As a consequence of this proposition, an infinite PS-word w is root-active if and only if $\mathrm{sum}(w, n) = 0$ for infinitely many n, and hence, if $((\liminf)_{n\to\infty}|\mathrm{sum}(w, n)|) = 0$.

Corollary 2.5. *For an infinite* PS*-word* w *we have* $\mathrm{SN}^\infty(w)$ *if and only if each value* $\mathrm{sum}(w, n)$ *for* $n = 0, 1 \ldots$ *occurs only finitely often.* ∎

It follows that $\mathrm{SN}^\infty(w)$ holds if and only if $((\liminf)_{n\to\infty}|\mathrm{sum}(w, n)|) = \infty$, and hence, if $\lim_{n\to\infty} \mathrm{sum}(w, n) \in \{\infty, -\infty\}$.

3. A Counterexample to UN^∞ of the Infinitary $\lambda\beta\eta$-Calculus

We give a straightforward translation of the word $\psi = \mathsf{P}^1\,\mathsf{S}^2\,\mathsf{P}^3 \ldots$ from the previous section into an infinite λ-term which then forms a counterexample to the infinitary unique normal form property UN^∞ for $\lambda^\infty\beta\eta$, the infinitary $\lambda\beta\eta$-calculus. The infinitary $\lambda\beta\eta$-calculus [7, 8] is a well-known example of a weakly orthogonal higher-order term rewrite system.

The set $\mathit{Ter}^\infty(\lambda)$ of (potentially) infinite λ-terms is coinductively defined by:

$$M ::= x \mid MM \mid \lambda x.M \qquad (\mathit{Ter}^\infty(\lambda))$$

The rewrite rules of $\lambda^\infty\beta\eta$ are:

$$\lambda x.MN \to M[x{:=}N] \qquad (\beta)$$

$$\lambda x.Mx \to M \qquad \text{if } x \text{ is not free in } M \qquad (\eta)$$

where $M[x{:=}N]$ denotes the result of substituting N for all free occurrences of x in M. The $\lambda^\infty\beta\eta$-calculus allows for two critical pairs[1]:

$$Mx \xleftarrow{\beta} (\lambda x.Mx)x \xrightarrow{\eta} Mx \qquad\qquad \lambda x.M[y{:=}x] \xleftarrow{\beta} \lambda x.(\lambda y.M)x \xrightarrow{\eta} \lambda y.M$$

As we have that $\lambda x.M[y{:=}x]$ and $\lambda y.M$ are equal modulo renaming of bound variables, both of these critical pairs are trivial. Hence $\lambda^\infty\beta\eta$ is weakly orthogonal.

We translate infinite PS-words to λ-terms.

[1]We use the notation of infinitary λ-calculus, but we view the rule schemes (β) and (η) as rules of a second-order HRS, thereby obtaining a formal notion of critical pairs ([9, Def. 11.6.10]). Likewise, CRSs can be viewed as second-order HRSs.

Definition 3.1. We define $(\!|\_|\!) : \{\mathsf{P},\mathsf{S}\}^\omega \to \mathit{Ter}^\infty(\lambda)$ by $(\!|w|\!) = (\!|w|\!)_0$, for all $w \in \{\mathsf{P},\mathsf{S}\}^\omega$, where $(\!|w|\!)_i$ is defined coinductively, for all $i \in \mathbb{Z}$, as follows:

$$(\!|\mathsf{P}w|\!)_i = (\!|w|\!)_{i-1}\, x_i \qquad\qquad (\!|\mathsf{S}w|\!)_i = \lambda x_{i+1}.(\!|w|\!)_{i+1}$$

The translation of ψ is the λ-term $(\!|\psi|\!)$, displayed in the middle of Figure 3. This term has

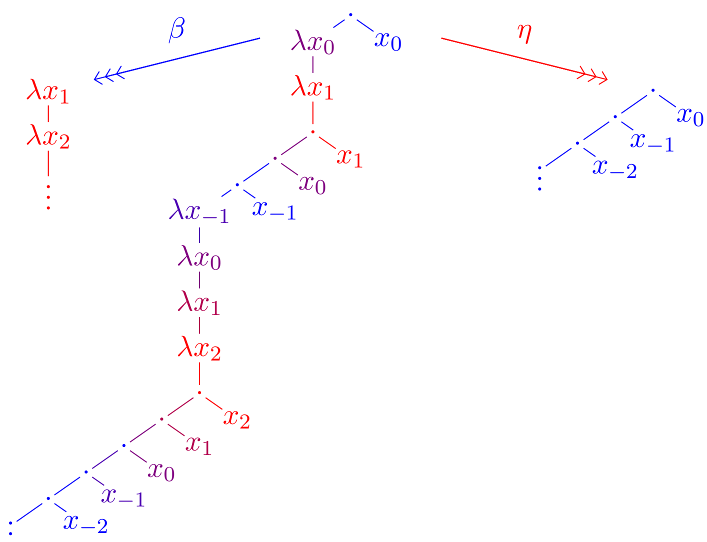


Figure 3: *Counterexample to unique normal forms in $\lambda^\infty\beta\eta$.*

two normal forms (corresponding to S^ω and P^ω), as indicated in the figure.

While $(\!|\psi|\!)$ cannot be generated from a finite λ-term (it has infinitely many free variables), the finite term WWI where $W = \lambda wf.f(ww(\lambda abc.f(abc))x_0)$ and $I = \lambda a.a$ exhibits a similar behaviour, reducing both to $A = \lambda x.A$ and $B = Bx_0$. This can be seen as follows: Let $V_n = \lambda v_1 \dots v_n.(v_1 \dots v_n)$. First note that $WWI \to_\beta^2 I(WW(\lambda abc.I(abc))x_0) \to_\beta^2 WWV_3x_0$. Then we get:

$$\begin{aligned} WWV_3x_0 &\to_\beta^2 V_3(WW(\lambda abc.V_3(abc))x_0)x_0 \to_\beta^3 \lambda v_3.WWV_5x_0x_0v_3 \\ &\to_\beta^6 \lambda v_3v_5.WWV_7x_0x_0x_0v_3v_5 \twoheadrightarrow\!\!\!\!\!\rightarrow_\beta \lambda v_3v_5v_7\dots. =_\alpha A \\ WWV_3x_0 &\to_\eta^2 (WWI)x_0 \twoheadrightarrow\!\!\!\!\!\rightarrow_{\beta\eta} B \end{aligned}$$

Note that the number of bound variables needed along the reduction from $WW(\lambda a.a)$ to A is unbounded, but that A can be written using only a single one. We conjecture that it holds for every counterexample to UN^∞ in the infinitary $\lambda\beta\eta$-calculus that during the rewrite process to one of the normal forms unboundedly many variables are needed.

The translation given in Definition 3.1 lifts $\mathsf{PS} \to \varepsilon$ to β, and $\mathsf{SP} \to \varepsilon$ to η.

Lemma 3.2. *An application of the rule $\mathsf{PS} \to \varepsilon$ at depth k in an infinite PS-word w corresponds to a β-step in $\lambda^\infty\beta\eta$ at depth k in $(\!|w|\!)_i$. Similarly so for the rule $\mathsf{SP} \to \varepsilon$ and the η-rule. These correspondences are indicated in the following diagrams:*

$$
\begin{tikzcd}
\mathsf{PS}w \arrow[r, "\llparenthesis \_ \rrparenthesis_i"] \arrow[d, "\mathsf{PS}"'] & (\lambda x_i . \llparenthesis w \rrparenthesis_i)\, x_i \arrow[d, "\beta"'] \\
w \arrow[r, "\llparenthesis \_ \rrparenthesis_i"] & \llparenthesis w \rrparenthesis_i
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathsf{SP}w \arrow[r, "\llparenthesis \_ \rrparenthesis_i"] \arrow[d, "\mathsf{SP}"'] & \lambda x_{i+1} . \llparenthesis w \rrparenthesis_i\, x_{i+1} \arrow[d, "\eta"'] \\
w \arrow[r, "\llparenthesis \_ \rrparenthesis_i"] & \llparenthesis w \rrparenthesis_i
\end{tikzcd}
$$

The counterexample to the infinitary unique normal form property UN^∞ for infinitary $\lambda\beta\eta$-calculus ($\lambda^\infty\beta\eta$) establishes a striking contrast to the situation for infinitary $\lambda\beta$-calculus ($\lambda^\infty\beta$). In the latter, infinitary confluence breaks down, but infinitary normal forms stay unique. Therefore $\lambda^\infty\beta$ clearly is of importance in the model theory of λ-calculus; for several models the equality is captured by convertibility in $\lambda^\infty\beta$. E.g. Böhm Trees, Lévy–Longo trees and Berarducci trees are unique normal forms in this rewrite system, when suitable $\perp$-normalization rules are added. (See [1, 2] and [9, Ch.12]). However, when the η-rule is added, and the infinitary perspective is maintained, then 'everything' breaks down dramatically: not only infinitary confluence, but also unique infinitary normal forms.

From the perspective of combinatory reduction systems (CRSs, see [9]) the η-rule has many undesirable properties: (i) it is undecidable whether an infinite term is an η-redex, since it is undecidable whether an infinite term contains a variable freely; (ii) single-step η-reduction is not lower semi-continuous: if t η-reduces to u, then for a given $\epsilon > 0$ we cannot always find a $\delta > 0$ such that anything within δ-distance of t η-reduces to something within ϵ-distance of u; (iii) the η-rule is not fully-extended, and various existing results for orthogonal infinite CRSs require fully-extendedness, see [4].

4. A Refinement of the Compression Lemma

As a preparation for Section 5 we will prove the following lemma, which is a refined version of the Compression Lemma in left-linear TRSs. In its original formulation (e.g. see Theorem 12.7.1 on page 689 in [9]), it states that strongly convergent rewrite sequences in left-linear TRSs can be compressed to length less or equal to ω. We recall that a rewrite sequence of ordinal length α is strongly convergent if for each limit ordinal $\lambda \leq \alpha$ the depth of the contracted redexes tends to infinity.

Lemma 4.1 (Refined Compression Lemma)**.** *Let R be a left-linear iTRS. Let $\kappa : s \to_R^\alpha t$ be a rewrite sequence, d the minimal depth of a step in κ, and n the number of steps at depth d in κ. Then there exists a rewrite sequence $\kappa' : s \to_R^{\leq\omega} t$ in which all steps take place at depth $\geq d$, and where precisely n steps contract redexes at depth d.*

Proof. We proceed by transfinite induction on the ordinal length α of rewrite sequences $\kappa : s \to_R^\alpha t$ with d the minimal depth of a step in κ, and n the number of steps at depth d in κ.

In case that $\alpha = 0$ nothing needs to be shown.

Suppose α is a successor ordinal. Then $\alpha = \beta + 1$ for some ordinal β, and κ is of the form $s \to^\beta s' \to t$. Applying the induction hypothesis to $s \to^\beta s'$ yields a rewrite sequence $s \to^\gamma s'$ of length $\gamma \leq \omega$ that contains the same number of steps at depth d, and no steps at depth less than d.

If $\gamma < \omega$, then $s \to^\gamma s' \to t$ is a rewrite sequence of length $\gamma + 1 < \omega$, in which all steps take place at depth $\geq d$ and precisely n steps at depth d.

If $\gamma = \omega$, we obtain a rewrite sequence of the form $s \equiv s_0 \to s_1 \to \ldots \to^\omega s_\omega \to t$. Let $\ell \to r \in R$ be the rule applied in the final step $s_\omega \to t$, that is, $s_\omega \equiv C[\ell\sigma] \to C[r\sigma] \equiv t$ for some context C and substitution σ. Moreover, let d_h be the depth of the hole in C, and d_p the depth of the pattern of ℓ. Since the reduction $s_0 \to^{\omega} s_\omega$ is strongly convergent, there exists $n \in \mathbb{N}$ such that all rewrite steps in $\xi : s_n \to^{\omega} s_\omega$ have depth $> d_h + d_p$, and hence are below the pattern of the redex contracted in the last step $s_\omega \to t$. As a consequence, there exists a context D and a substitution τ such that $s_n \equiv D[\ell\tau]$. Since the rewrite sequence $\xi : s_n \equiv D[\ell\tau] \to^{\omega} C[\ell\sigma] \equiv s_\omega$ consists only of steps at depth $> d_h + d_p$, it follows that:

- there exists a rewrite sequence $\vartheta : D[\Box] \to^{\leq\omega} C[\Box]$ at depth $> d_h + d_p$, and
- there exist rewrite sequences $\vartheta_x : \tau(x) \to^{\leq\omega} \sigma(x)$ for all $x \in \mathit{Var}(\ell)$.

We now prepend the final step $s_\omega \to t$ to s_n, that is: $s_n \equiv D[\ell\tau] \to D[r\tau]$. Even if the term r is infinite, this creates at most ω-many copies of subterms $\tau(x)$ with reduction sequences $\vartheta_x : \tau(x) \to^{\leq\omega} \sigma(x)$ of length $\leq \omega$. Since the rewrite sequences ϑ and ϑ_x for $x \in \mathit{Var}(\ell)$ are in disjoint (parallel) subterms, there exists an interleaving $D[r\tau] \to^{\leq\omega} C[r\sigma]$ of length at most ω (the idea is similar to establishing countability of ω^2 by dovetailing). We obtain a rewrite sequence $\kappa' : s \to^{\leq\omega} t$, since $s \to^{n} s_n \equiv D[\ell\tau] \to D[r\tau] \to^{\leq\omega} C[r\sigma] \equiv t$.

It remains to be shown that κ' contains only steps at depth $\geq d$, and that it has the same number of steps as the original sequence κ at depth d. This follows from the induction hypothesis and the fact that all steps in $s_n \to^{\omega} s_\omega$ have depth $> d_h + d_p$ and thus also all steps of the interleaving $D[r\tau] \to^{\leq\omega} C[r\sigma]$ have depth $> d_h + d_p - d_p = d_h \geq d$ (the application of $\ell \to r$ can lift steps at most by the pattern depth d_p of ℓ).

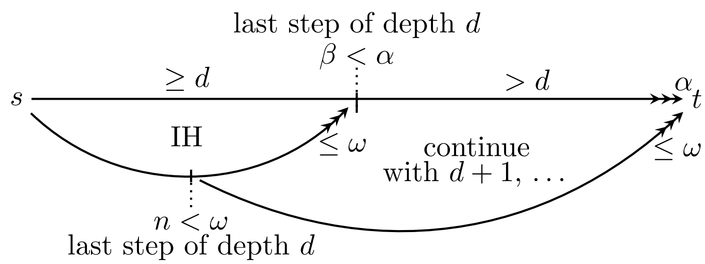


Figure 4: *Compression Lemma, in case α is a limit ordinal.*

Finally, suppose that α is a limit ordinal $> \omega$. We refer to Figure 4 for a sketch of the proof. Since κ is strongly convergent, only a finite number of steps take place at depth d. Hence there exists $\beta < \alpha$ such that s_β is the target of the last step at depth d in κ. We have $s \to^{\beta} s_\beta \to^{\leq\alpha} t$ and all rewrite steps in $s_\beta \to^{\leq\alpha} t$ are at depth $> d$. By induction hypothesis there exists a rewrite sequence $\xi : s \to^{\leq\omega} s_\beta$ containing an equal amount of steps at depth d as $s \to^{\beta} s_\beta$. Consider the last step of depth d in ξ. This step has a finite index $n < \omega$. Thus we have $s \to^{*} s_n \to^{\leq\alpha} t$, and all steps in $s_n \to^{\leq\alpha} t$ are at depth $> d$. By successively applying this argument to $s_n \to^{\leq\alpha} t$ we construct finite initial segments $s \to^{*} s_n$ with strictly increasing minimal rewrite depth d. Concatenating these finite initial segments yields a reduction $s \to^{\leq\omega} t$ containing as many steps at depth d as the original sequence. ∎

With this refined compression lemma we now prove that also divergent rewrite sequences can be compressed to length less or equal to ω.

Theorem 4.2. *Let R be a left-linear iTRS. For every divergent rewrite sequence $\kappa : s \to_R^{\alpha}$ of length α there exists a divergent rewrite sequence $\kappa' : s \to_R^{\leq\omega}$ of length less or equal to ω.*

Proof. Let $\kappa : s \to_R^\alpha$ be a divergent rewrite sequence. Then there exist $k \in \mathbb{N}$ such that infinitely many steps in κ take place at depth k. Let d be the minimum of all numbers k with that property. Let β be the index of the last step above depth d in κ, $\kappa : s \to^\beta s_\beta \to^{\leq\alpha}$. Then by Lemma 4.1 the rewrite sequence $s \to^\beta s_\beta$ can be compressed to a rewrite sequence $s \to^{\leq\omega} s_\beta$ such that $s_\beta \to^{\leq\alpha}$ consists only of steps at depth $\geq d$, among which infinitely many steps are at depth d. Let n be the index of the last step of depth $\leq d$ in the rewrite sequence $s \to^{\leq\omega} s_\beta$. Then $s \to^* s_n \to^{\leq\omega} s_\beta \to^{\leq\alpha}$, and $s_n \to^{\leq\omega} s_\beta \to^{\leq\alpha}$ contains only steps at depth $\geq d$. Thus all steps with depth less than d occur in the finite prefix $s \to^* s_n$.

Now consider the rewrite sequence $\kappa_1 : s_n \to^{\leq\omega} \cdot \to^{\leq\alpha}$, say $\kappa_1 : s_n \to^\gamma$ for short, containing infinitely many steps at depth d. Let γ' be the index of the first step at depth d in κ_1. Then $\kappa_1 : s_n \to^{\gamma'} u \to^{\leq\gamma}$ for some term u and $s_n \to^{\gamma'} u$ can be compressed to $s_n \to^{\leq\omega} u$ containing exactly one step at depth d. Now let m be the index of this step, then $s_n \to^m u' \to^{\leq\omega} u \to^{\leq\gamma}$ where $s_n \to^m u'$ contains one step at depth d. Repeatedly applying this construction to $u' \to^{\leq\omega} u \to^{\leq\gamma}$ we obtain a rewrite sequence $\kappa' : s \to^* s_n \to^* u' \to^* u'' \to \ldots$ that contains infinitely many steps at depth d, and hence is divergent. ∎

5. Infinitary Confluence

In Section 2 we have seen that the property UN^∞ fails for weakly orthogonal TRSs when collapsing rules are present, and hence also CR^∞. Now we show that weakly orthogonal TRSs without collapsing rules are infinitary confluent (CR^∞), and as a consequence also have the property UN^∞.

We adapt the projection of parallel steps in weakly orthogonal TRSs from [9, Section 8.8.4.] to infinite terms. The basic idea is to orthogonalize the parallel steps, and then project the orthogonalized steps. The orthogonalization uses that overlapping redexes have the same effect and hence can be replaced by each other. In case of overlaps we replace the outermost redex by the innermost one. This is possible since the maximal nesting depth of the union of two infinite parallel steps is at most 2, that is, there can not be infinite chains of overlapping nested redexes in such a union (see Example 6.3). For a treatment of infinitary multi-steps where such chains can occur, we refer to Section 6. See further [9, Proposition 8.8.23] for orthogonalization in the finitary case.

Definition 5.1. Let R be a TRS, and $t \in \mathit{Ter}^\infty(\Sigma)$ a term.

A *redex* in t is a pair consisting of a position p and a rule $\ell \to r$, such that $t|_p = \ell^\sigma$ for some substitution σ. We call p and $\ell \to r$ the *root* and *rule* of the redex, respectively. The pattern of a redex $\langle p, \ell \to r\rangle$ is the set of all positions pq such that $\ell(q)$ is a function symbol.

Two sets of positions are *overlapping* if they have a non-empty intersection. For redexes u and v in t we say that u *and* v *overlap*, denoted by $u \leftrightsquigarrow v$, if the patterns of u and v overlap. A set U of redexes is called *non-overlapping* if, for all $u, v \in U$ with $u \neq v$, u does not overlap with v.

For a study of developments we refer to [9, Sec. 4.5.2] and [10]. Here, we briefly introduce developments and multi-steps via labelling (underlining).

Definition 5.2. Let R be a weakly orthogonal TRS over Σ. For symbols $f \in \Sigma$ and $\rho \in R$ we write f^ρ for f labelled with ρ. For labelled terms t, we write $\lfloor t \rfloor$ to denote the term obtained from t by dropping all labels.

We define the TRS $R^\triangleright$ to consist of all rules $\ell^\rho \to r$ for $\rho : \ell \to r \in R$ where ℓ^ρ is the term obtained from ℓ by labelling the root-symbol of ℓ with ρ.

Let $t, t' \in \mathit{Ter}^{\infty}(\Sigma)$ be terms, and U a set of non-overlapping redexes in t. Let t^U be the term obtained from t by labelling for each redex $\langle p, \rho\rangle \in U$ the symbol at position p in t with ρ. A *development of* U in t is a rewrite sequence $t \twoheadrightarrow_R t'$ (in R) that can be lifted to a reduction $t^U \twoheadrightarrow_{R^{\triangleright}} t''$ (in $R^{\triangleright}$) such that t'' arises from t' by adding some labels; the development is called *complete* if $t' \equiv t''$. A *multi-step with respect to* U is a step $t \mathrel{-\!\!\circ\!\!\!\rightarrow}_U t'$ such that there exists a reduction $t^U \twoheadrightarrow_{R^{\triangleright}} t'$.

In non-collapsing, weakly orthogonal TRSs, every set U of non-overlapping redexes has a complete development, and every complete development of U ends in the same term [9]. Multi-steps arise from complete developments, and are uniquely determined by their starting term and redex set.

Definition 5.3. Let R be a TRS, $t \in \mathit{Ter}^{\infty}(\Sigma)$ a term, and let U and V be sets of redexes in t. We call U and V *orthogonal (to each other)* if $U \cup V$ is a non-overlapping set of redexes.

Definition 5.4. Let R be a non-collapsing, weakly orthogonal TRS, and let U and V be orthogonal sets of redexes in a term t. For multi-steps $\phi : t \mathrel{-\!\!\circ\!\!\!\rightarrow}_U t'$ and $\psi : t \mathrel{-\!\!\circ\!\!\!\rightarrow}_V t''$ with respect to U and V we define the projection ϕ/ψ as the multi-step $t'' \mathrel{-\!\!\circ\!\!\!\rightarrow}_{U'} s$ with respect to the set of residuals $U' = U/\psi$ as defined in [9].[1] In the sequel we frequently write $\mathrel{-\!\!\circ\!\!\!\rightarrow}$ for the multi-step relation, suppressing the set of redexes U that induces the multi-step $\mathrel{-\!\!\circ\!\!\!\rightarrow}_U$.

Definition 5.5. An *orthogonalization* of a pair $\langle \phi, \psi\rangle$ of multi-steps $\phi : s \mathrel{-\!\!\circ\!\!\!\rightarrow}_U t_1$ and $\psi : s \mathrel{-\!\!\circ\!\!\!\rightarrow}_V t_2$ with respect to sets U and V of redexes in s is a pair $\langle \phi', \psi'\rangle$ of multi-steps $\phi' : s \mathrel{-\!\!\circ\!\!\!\rightarrow}_{U'} t_1$ and $\psi' : s \mathrel{-\!\!\circ\!\!\!\rightarrow}_{V'} t_2$ with respect to orthogonal sets U' and V' of redexes in s.

A parallel step $\phi : s \mathrel{-\!\!\!\!+\!\!\!\!+\!\!\!\rightarrow} t$ is a multi-step $\phi : s \mathrel{-\!\!\circ\!\!\!\rightarrow}_U t$ with respect to a set U of parallel redexes, that is, redexes at pairwise disjoint positions.

Proposition 5.6. *Let $\phi : s \mathrel{-\!\!\!\!+\!\!\!\!+\!\!\!\rightarrow} t_1$ and $\psi : s \mathrel{-\!\!\!\!+\!\!\!\!+\!\!\!\rightarrow} t_2$ be parallel steps in a weakly orthogonal TRS. Then there exists an orthogonalization $\langle \phi', \psi'\rangle$ of ϕ and ψ with the special property that $\phi' : s \mathrel{-\!\!\!\!+\!\!\!\!+\!\!\!\rightarrow} t_1$ and $\psi' : s \mathrel{-\!\!\!\!+\!\!\!\!+\!\!\!\rightarrow} t_2$.*

Proof. In case of overlaps between U and V, then for every overlap we replace the outermost redex by the innermost one (if there are multiple inner redexes overlapping, then we choose the left-most among the top-most redexes). If there are two redexes at the same position but with respect to different rules, then we replace the redex in V with the one in U. See also Figure 5. ∎

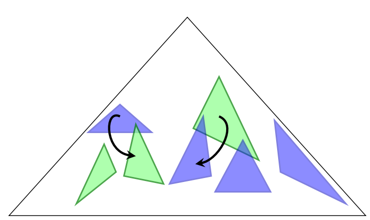


Figure 5: *Orthogonalization of parallel steps; the arrow indicates replacement.*

Definition 5.7. Let $\phi : s \mathrel{-\!\!\!\!+\!\!\!\!+\!\!\!\rightarrow} t_1$, $\psi : s \mathrel{-\!\!\!\!+\!\!\!\!+\!\!\!\rightarrow} t_2$ be parallel steps in a weakly orthogonal TRS. The *weakly orthogonal projection ϕ/ψ of ϕ over ψ* is defined as the orthogonal projection

[1] We refer to Def. 12.5.3 in [9], and note that the definition not only applies in orthogonal TRSs, but also to every non-overlapping set U of redexes versus a multistep ϕ w.r.t. a redex set V that is orthogonal to U.

ϕ'/ψ' where $\langle \phi', \psi' \rangle$ is the orthogonalization of ϕ and ψ given in the proof of Proposition 5.6.

Remark 5.8. The weakly orthogonal projection does not give rise to a residual system in the sense of [9]. The projection fulfils the three identities $\phi/\phi \approx 1$, $\phi/1 \approx \phi$, and $1/\phi \approx 1$, but not the *cube identity* $(\phi/\psi)/(\chi/\psi) \approx (\phi/\chi)/(\psi/\chi)$.

Lemma 5.9. *Let $\phi : s \mathrel{-\!\!\|\!\!\to} t_1$, $\psi : s \mathrel{-\!\!\|\!\!\to} t_2$ be parallel steps in a weakly orthogonal TRS R. Let d_ϕ and d_ψ be the minimal depth of a step in ϕ and ψ, respectively. Then the minimal depth of the weakly orthogonal projections ϕ/ψ and ψ/ϕ is greater or equal $\min(d_\phi, d_\psi)$. If R contains no collapsing rules then the minimal depth of ϕ/ψ and ψ/ϕ is greater or equal $\min(d_\phi, d_\psi + 1)$ and $\min(d_\psi, d_\phi + 1)$, respectively.*

Proof. Immediate from the definition of the orthogonalization (for overlaps the innermost redex is chosen) and the fact that in the orthogonal projection a non-collapsing rule applied at depth d can lift nested redexes at most to depth $d + 1$ (but not above). ∎

Lemma 5.10 (Parallel Moves Lemma)**.** *Let R be a weakly orthogonal TRS, $\kappa : s \to^\alpha t_1$ a rewrite sequence, and $\phi : s \mathrel{-\!\!\|\!\!\to} t_2$ a parallel rewrite step. Let d_κ and d_ϕ be the minimal depth of a step in κ and ϕ, respectively. Then there exist a term u, a rewrite sequence $\xi : t_2 \to^{\leq\omega} u$ and a parallel step $\psi : t_1 \mathrel{-\!\!\|\!\!\to} u$ such that the minimal depth of the rewrite steps in ξ and ψ is $\min(d_\kappa, d_\xi)$; see Figure 6 (left).*

If additionally R contains no collapsing rules, then the minimal depth of a step in ξ and ψ is $\min(d_\kappa, d_\xi + 1)$ and $\min(d_\xi, d_\kappa + 1)$, respectively. See also Figure 6 (right).

$$\begin{tikzcd}
s \arrow[r, twoheadrightarrow, "\geq d_\kappa"] \arrow[d, "\geq d_\xi"'] & t_1 \arrow[d, "\geq \min(d_\kappa, d_\xi)"'] \\
t_2 \arrow[r, dashed, twoheadrightarrow, "\geq \min(d_\kappa, d_\xi)"'] & u
\end{tikzcd}
\qquad
\begin{tikzcd}
s \arrow[r, twoheadrightarrow, "\geq d_\kappa"] \arrow[d, "\geq d_\xi"'] & t_1 \arrow[d, "\geq \min(d_\xi, d_\kappa + 1)"'] \\
t_2 \arrow[r, dashed, twoheadrightarrow, "\geq \min(d_\kappa, d_\xi + 1)"'] & u
\end{tikzcd}$$

Figure 6: *Parallel Moves Lemma; with (left) and without (right) collapsing rules.*

Proof. By compression we may assume $\alpha \leq \omega$ in $\kappa : s \to^{\leq\omega} t_1$ (note that, the minimal depth d is preserved by compression). Let $\kappa : s \equiv s_0 \to s_1 \to s_2 \to \ldots$, and define $\psi_0 = \psi$. Furthermore, let $\kappa_{\leq n}$ denote the prefix of κ of length n, that is, $s_0 \to \ldots \to s_n$ and let $\kappa_{\geq n}$ denote the suffix $s_n \to s_{n+1} \to \ldots$ of κ. We employ the projection of parallel steps to close the elementary diagrams with top $s_n \to s_{n+1}$ and left $\psi_n : s_n \mathrel{-\!\!\|\!\!\to} s'_n$, that is, we construct the projections $\psi_{i+1} = \psi_i/(s_i \to s_{i+1})$ (right) and $(s_i \to s_{i+1})/\psi_i$ (bottom). Then by induction on n using Lemma 5.9 there exists for every $1 \leq n \leq \alpha$ a term s'_n, and parallel steps $\phi_n : s_n \mathrel{-\!\!\|\!\!\to} s'_n$ and $s'_{n-1} \mathrel{-\!\!\|\!\!\to} s'_n$. See Figure 7 for an overview.

We show that the rewrite sequence constructed at the bottom $s'_0 \mathrel{-\!\!\|\!\!\to} s'_1 \mathrel{-\!\!\|\!\!\to} \ldots$ of Figure 7 is strongly convergent, and that the parallel steps ϕ_i have a limit for $i \to \infty$ (parallel steps are always strongly convergent).

Let $d \in \mathbb{N}$ be arbitrary. By strong convergence of κ there exists $n_0 \in \mathbb{N}$ such that all steps in $\kappa_{\geq n_0}$ are at depth $\geq d$. Since ϕ_{n_0} is a parallel step there are only finitely many redexes $\phi_{n_0,<d} \subseteq \phi_{n_0}$ in ϕ_{n_0} rooted above depth d. By projection of ϕ_{n_0} along $\kappa_{\geq n_0}$ no fresh redexes above depth d can be created. The steps in $\phi_{n_0,<d}$ may be cancelled out due to overlaps, nevertheless, for all $m \geq n_0$ the set of steps above depth d in ϕ_m is a subset of $\phi_{n_0,<d}$.

Let p be the maximal depth of a left-hand side of a rule applied in $\phi_{n_0,<d}$. By strong

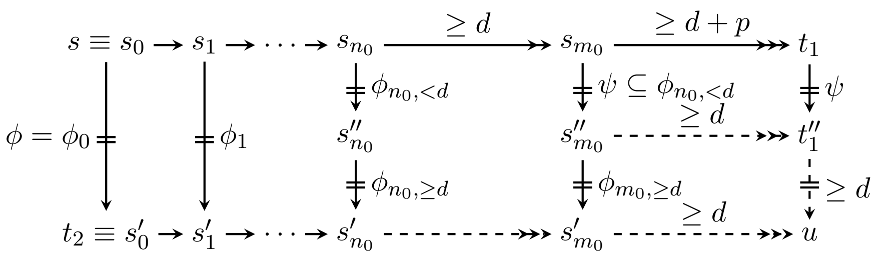


Figure 7: *Parallel Moves Lemma, proof overview.*

convergence of κ there exists $m_0 \geq n_0 \in \mathbb{N}$ such that all steps in $\kappa_{\geq n_0}$ are at depth $\geq d+p$. As a consequence the steps ψ in ϕ_{m_0} rooted above depth d will stay fixed throughout the remainder of the projection. Then for all $m \geq m_0$ the parallel step ϕ_m can be split into $\phi_m = s_m \mathrel{+\!\!\!\!\!\twoheadrightarrow}_{\psi} s''_m \mathrel{+\!\!\!\!\!\twoheadrightarrow}_{\phi_{m,\geq d}} s'_m$ where $\phi_{m,\geq d}$ consists of the steps of ϕ_m at depth $\geq d$. Since d was arbitrary, it follows that projection of ϕ over κ has a limit. Moreover the steps of the projection of $\kappa_{\geq m_0}$ over ϕ_{m_0} are at depth $\geq d+p-p = d$ since rules with pattern depth $\leq p$ can lift steps by at most by p. Again, since d was arbitrary, it follows that the projection of κ over ϕ is strongly convergent.

Finally, both constructed rewrite sequences (bottom and right) converge towards the same limit u since all terms $\{s'_m, s''_m \mid m \geq m_0\}$ coincide up to depth $d-1$ (the terms $\{s_m \mid m \geq m_0\}$ coincide up to depth $d+p-1$ and the lifting effect of the steps ϕ_m is limited by p). ∎

Theorem 5.11. *Every weakly orthogonal TRS without collapsing rules is infinitary confluent.*

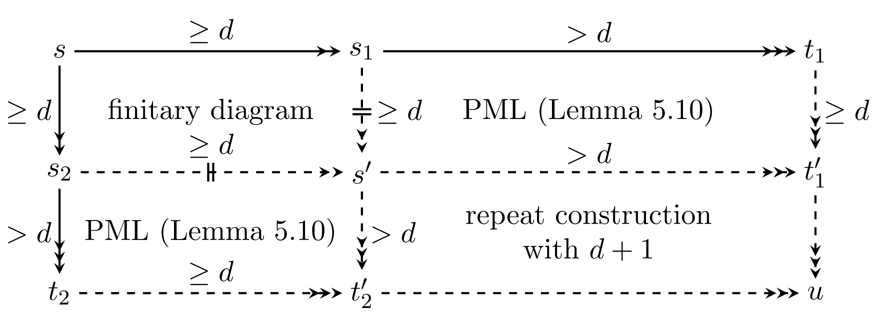


Figure 8: *Infinitary confluence.*

Proof. An overview of the proof is given in Figure 8. Let $\kappa : s \to^{\alpha} t_1$ and $\xi : s \to^{\beta} t_2$ be two rewrite sequences. By compression we may assume $\alpha \leq \omega$ and $\beta \leq \omega$. Let d be the minimal depth of any rewrite step in κ and ξ. Then κ and ξ are of the form $\kappa : s \to^* s_1 \to^{\leq\omega} t_1$ and $\xi : s \to^* s_2 \to^{\leq\omega} t_2$ such that all steps in $s_1 \to^{\leq\omega} t_1$ and $s_2 \to^{\leq\omega} t_2$ at depth $> d$.

Then $s \to^* s_1$ and $s \to^* s_2$ can be joined by finitary diagram completion employing the diamond property for parallel steps. If follows that there exists a term s' and finite sequences of (possibly infinite) parallel steps $s_1 \mathrel{+\!\!\!\!\!\twoheadrightarrow}^* s'$ and $s_2 \mathrel{+\!\!\!\!\!\twoheadrightarrow}^* s'$ all steps of which are at depth $\geq d$ (Lemma 5.9). We project $s_1 \to^{\leq\omega} t_1$ over $s_1 \mathrel{+\!\!\!\!\!\twoheadrightarrow}^* s'$, $s_2 \to^{\leq\omega} t_2$ over $s_2 \mathrel{+\!\!\!\!\!\twoheadrightarrow}^* s'$ by repeated application of the Lemma 5.10, obtaining rewrite sequences $t_1 \twoheadrightarrow t'_1$, $s' \twoheadrightarrow t'_1$, $t_2 \twoheadrightarrow t'_2$, and $s' \twoheadrightarrow t'_2$ with depth $\geq d$, $> d$, $\geq d$, and $> d$, respectively. As a consequence we have t_1', s' and t_2' coincide up to (including) depth d. Recursively applying the construction to the rewrite sequences $s' \twoheadrightarrow t_1'$ and $s' \twoheadrightarrow t_2'$ yields strongly convergent rewrite sequences $t_2 \twoheadrightarrow t_2' \twoheadrightarrow t_2'' \twoheadrightarrow \ldots$ and $t_1 \twoheadrightarrow t_1' \twoheadrightarrow t_1'' \twoheadrightarrow \ldots$ where the terms $t_1^{(n)}$ and $t_2^{(n)}$ coincide up to depth $d + n - 1$. Thus these rewrite sequences converge towards the same limit u. ∎

We consider an example to illustrate that the absence of collapsing rules is a necessary condition for Theorem 5.11.

Example 5.12. Let R be a TRS over the signature $\{f, a, b\}$ consisting of the collapsing rule: $f(x, y) \to x$ Then, using a self-explaining recursive notation, the term $s = f(f(s, b), a)$ rewrites in ω many steps to $t_1 = f(t_1, a)$ as well as $t_2 = f(t_2, b)$ which have no common reduct. The TRS R is weakly orthogonal (even orthogonal) but not confluent. The same phenomenon occurs in the infinitary version of combinatory logic, due to the rule $Kxy \to x$.

6. The Diamond and Triangle Property for Multi-Steps

We prove that infinitary multi-steps in weakly orthogonal TRSs without collapsing rules have the diamond property. For all TRSs in this section we assume that are weakly orthogonal and do not contain collapsing rules.

Definition 6.1. A binary relation $\to$ on A is said to have:

- the *diamond property* if $\leftarrow \cdot \to \subseteq \to \cdot \leftarrow$, and
- the *triangle property* if $\forall a \in A.\, \exists a' \in A.\, a \to a' \wedge (\forall b \in A.\, a \to b \Rightarrow b \to a')$.

We develop an orthogonalization algorithm that, given two co-initial multisteps, makes them orthogonal to each other by eliminating overlaps. Since overlapping steps in weakly orthogonal TRSs have the same targets, we can replace one by the other. The challenge is to do this in such a way that no new overlaps are created.

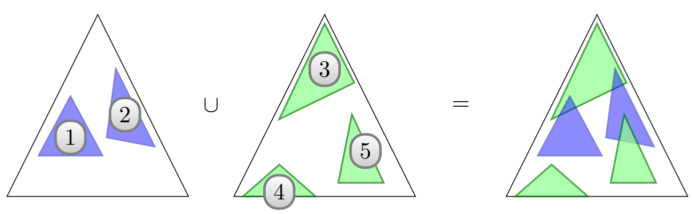


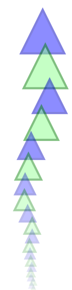
Figure 9: *Orthogonalization in a weakly orthogonal TRS.*

Consider for example Figure 9, where the redexes 2 and 3 overlap with each other. When trying to solve this overlap, we have to be careful since replacing the redex 2 by 3 as well as 3 by 2 creates new conflicts.

The case of finitary weakly orthogonal rewriting is treated in [9, Theorem 8.8.23]. There an inside-out algorithm is employed, consisting of inductively extending an orthogonalization of the subtrees to the whole tree. The basic observation is that one overcomes the difficulties pointed out above by starting at the bottom of the tree and solving overlaps by choosing the deeper (innermost) redex.

Example 6.2. We consider Figure 9 and apply the orthogonalization algorithm from [9, Theorem 8.8.23]. We start at the bottom of the tree. The first overlap we find is between the redexes 2 and 5; this is removed by replacing 2 with 5. Then the overlap between 2 and 3 has also disappeared. The only remaining overlap is between the redexes 3 and 1. Hence we replace 3 by 1. As result we obtain two orthogonal multi-steps $\{1, 5\}$ and $\{1, 4, 5\}$.

Note that the above algorithm does not carry over to the case of infinitary multi-steps since we may have infinite chains of overlapping redexes and thus have no bottom to start at. This is illustrated on the right.



Example 6.3. As an example where such an infinite chain of overlaps arises we consider the TRS R consisting of the rule:

$$A(A(A(x))) \to A(x)$$

together with two multi-steps of blue and green redexes in the term A^ω:

$$A(A(A(A(A(A(A(A(A(A(A(A(A(A(A(\ldots)\ldots)$$

The blue redexes are marked by overlining, the green redexes by underlining.

Definition 6.4. A *cluster* is a non-empty set of redexes which forms a connected component with respect to $\leftrightsquigarrow$. The *pattern* of a cluster is the union of the patterns of its redexes. A cluster is a Y-*cluster* if it contains a pair of redexes at parallel positions (Figure 10, cases (ii) and (iv)); otherwise it is an I-*cluster* (Figure 10, cases (i) and (iii)). A Y-*redex* is a redex in a Y-cluster.

At first sight one might expect that Y-redexes are due to trivial rules of the form $\ell \to r$ with $\ell \equiv r$. However, the following example illustrates that, in general, this is not the case.

Example 6.5. Let R consist of the following (non-trivial) rules:

$$f(g(x, y), z, g(a, a)) \to f(g(y, x), z, g(a, a)) \qquad (\rho_1)$$

$$f(g(a, a), z, g(x, y)) \to f(g(a, a), z, g(y, x)) \qquad (\rho_2)$$

$$g(x, y) \to g(y, x) \qquad (\rho_3)$$

We consider the term $f(g(a, a), t, g(a, a))$ which contains both a ρ_1-redex and a ρ_2-redex at the root, a ρ_3-redex at disjoint positions 1 and 3. These redexes form a Y-cluster.

Notwithstanding the above example, it is always safe to drop Y-redexes from multi-steps without changing the outcome of the multi-step. This result is implicit in [5]. In particular in [5, Remark 4.38] it is mentioned that Y-clusters are a generalisation of Takahashi-configurations.

Lemma 6.6. *Let Y be a term in which the non-variable positions form the pattern of a Y-cluster, σ a substitution, and $Y\sigma \to s$ a step in Y. Then $s \equiv Y\sigma$ and subterms outside of Y have not been affected. (Note that subterms fully contained in the pattern of a Y-cluster* can *be affected.)*

Proof. By weak orthogonality redexes in a cluster have the same effect. Since Y-clusters have redexes at disjoint positions, it follows that contraction of any redex in a Y-cluster results in the same term. By applying this argument for the result of replacing the subterms outside of the Y-cluster by fresh variables, we conclude that none of these subterms can be affected (moved, copied, deleted) by contracting a redex from the Y-cluster. ∎

Lemma 6.7. *Let R be a weakly orthogonal TRS, $t \in Ter^\infty(\Sigma)$ a term. Let U be a set of non-overlapping redexes in t. Furthermore, let $V \subseteq U$ be such that every redex in V is contained in a Y-cluster of t. Then the multi-step with respect to $U \backslash V$ results in the same term as the multi-step with respect to U.*

Proof. We reduce in the complete development first all Y-redexes: by Lemma 6.6 this leaves the term as well as all redexes outside of Y-clusters untouched. As a consequence, the result of the complete development (multi-step) depends only on the redexes outside of Y-clusters. ■

Definition 6.8. Let $\sigma : s \mathrel{-\!\!\circ\!\!\to}_U t_1$ and $\delta : s \mathrel{-\!\!\circ\!\!\to}_V t_2$ be multi-steps. An *orthogonalization witness* for the pair $\langle \sigma, \delta \rangle$ of multi-steps is a pair $\langle f_U, f_V \rangle$ of injective partial functions $f_U : U \rightharpoonup U \cup V$ and $f_V : V \rightharpoonup U \cup V$ such that it holds: (i) $\mathit{ran}(f_U)$ and $\mathit{ran}(f_V)$ are orthogonal sets of redexes in t ; (ii) for all $u \in \mathit{dom}(f_U)$, $f_U(u) \leftrightsquigarrow u$, as well as, for all $v \in \mathit{dom}(f_V)$, $f_V(v) \leftrightsquigarrow v$; and (iii) $(U \setminus \mathit{dom}(f_U)) \cup (V \setminus \mathit{dom}(f_V)) \subseteq \{v : v \text{ is Y-redex in } t\}$.

Informally, an orthogonalization witness of multi-steps w.r.t. redex sets U and V defines (as stated in the proposition below) an orthogonalization consisting of multi-steps w.r.t. redex sets U' and V' that arise from U and V by exchanging redexes with equivalent, overlapping ones, and by possibly dropping some Y-redexes which have no effect.

Proposition 6.9. *Let $\sigma : s \mathrel{-\!\!\circ\!\!\to}_U t_1$ and $\delta : s \mathrel{-\!\!\circ\!\!\to}_V t_2$ be multi-steps, and let $\langle f_U, f_V \rangle$ be an orthogonalization witness for $\langle \sigma, \delta \rangle$. Then $U' = \mathit{ran}(f_U)$ and $V' = \mathit{ran}(f_V)$ are orthogonal sets of redexes in s, and there exist multi-steps $\sigma' : s \mathrel{-\!\!\circ\!\!\to}_{U'} t_1$ and $\delta' : s \mathrel{-\!\!\circ\!\!\to}_{V'} t_2$, and hence an orthogonalization $\langle \sigma', \delta' \rangle$ of $\langle \sigma, \delta \rangle$.*

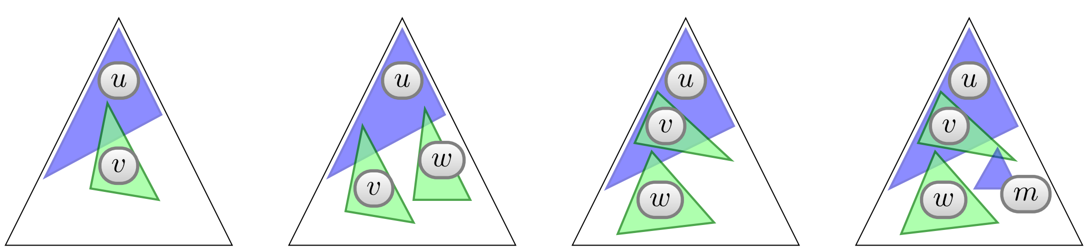
We now define a top–down orthogonalization algorithm. Roughly speaking, we start at the top of the term and replace overlapping redexes with the outermost one. However, care has to be taken in situations as depicted in Figure 9.

Theorem 6.10. *Let R be a weakly orthogonal TRS, $t \in \mathit{Ter}^\infty(\Sigma)$ a (possibly infinite) term. Every pair $\langle \sigma, \delta \rangle$ of multi-steps $\sigma : t \mathrel{-\!\!\circ\!\!\to} t'$ and $\delta : t \mathrel{-\!\!\circ\!\!\to} t''$ has an orthogonalization.*

Proof. Let $\sigma : s \mathrel{-\!\!\circ\!\!\to}_U t_1$ and $\delta : s \mathrel{-\!\!\circ\!\!\to}_V t_2$ be multi-steps with respect to sets U and V of (non-overlapping) redexes. In view of Proposition 6.9 it suffices to construct an orthogonalization witness $\langle f_U, f_V \rangle$ for $\langle \sigma, \delta \rangle$. Briefly, we will show that it is always possible to solve outermost conflicts without creating fresh ones. After solving a conflict, the orthogonalization continues with the next conflict that is now at a top-most position.

If U and V are orthogonal, then we are finished (then f_U and f_V are both the identity). In this proof, by overlap we mean non-identical redexes whose patterns overlap. If there exist overlaps, let $u \in U \cup V$ be a topmost redex (that is, having minimal depth) among the redexes which have an overlap. Without loss of generality (by symmetry) we assume that $u \in U$ and let $v \in V$ be a topmost redex among the redexes in V overlapping u.

We distinguish the following cases:



case (i) case (ii) case (iii) case (iv)

Figure 10: *Case distinction for the orthogonalization algorithm.*

(i) If v is the only redex in V that overlaps with u, case (i) of Figure 10, then we can safely replace v by u. More precisely, we define $f_U(u) = u$ and $f_V(v) = u$ and continue the orthogonalization with $\langle U \setminus \{u\}, V \setminus \{v\}\rangle$, that is, the remaining redexes. Note that, since $(U \setminus \{u\}) \cup (V \setminus \{v\})$ contains no redexes overlapping u, the orthogonalization of the remainder cannot create overlaps with u.

Otherwise we pick a redex $w \in V$, $w \neq v$ and w overlaps u.

(ii) Assume that v and w are at disjoint positions, case (ii) of Figure 10. Then u, v and w are Y-redexes and can be dropped from U and V by Lemma 6.7. That is, we choose $f_U(u)$, $f_V(v)$ and $f_V(w)$ to be undefined, and continue the orthogonalization with the remainder $\langle U \setminus \{u\}, V \setminus \{v, w\}\rangle$.

Otherwise, v and w are not disjoint, and then w must be nested inside v.

(iii) If u is the only redex from U overlapping v, case (iii) of Figure 10, then we can replace u by v. That is, we define $f_U(u) = v$ and $f_V(v) = v$. We continue the orthogonalization with $\langle U \setminus \{u\}, V \setminus \{v\}\rangle$; that is *including* w since w may have further overlaps that need to be resolved.

(iv) In the remaining case there must be a redex $m \in U$, $m \neq u$ and m overlaps with the redex v, see case (iv) of Figure 10. We pick such an m. Since U and V are developments u cannot overlap with m, and v cannot overlap with w. We have that w is nested in v, both overlapping u, but m is below the pattern of u, overlapping v. Hence w and m must be at disjoint positions (v cannot tunnel through w to touch m); this has also been shown in [5]. Then u, m, v and w are contained in a Y-cluster, and hence they can be removed by Lemma 6.7. We choose $f_U(u)$, $f_U(m)$, $f_V(v)$, $f_V(w)$ to be undefined, and continue the orthogonalization with the remainder $\langle U \setminus \{u, m\}, V \setminus \{v, w\}\rangle$.

For all redexes $u \in U$ and $v \in V$ for which we have not specified $f_U(u)$ or $f_V(v)$, respectively, we define $f_U(u) = u$ or $f_V(v) = v$ (this concerns those u and v that either had no overlaps, or the conflicts have been solved by rearranging another redex positions). ∎

We obtain the diamond property as a corollary.

Corollary 6.11. *For every weakly orthogonal TRS without collapsing rules, (infinite) multi-steps have the diamond property.*

Proof. Let σ, δ be two coinitial complete developments $t_1 \overset{\sigma}{\leftarrow} s \overset{\delta}{\rightarrow} t_2$. Then by Theorem 6.10 there exists an orthogonalization $\langle \sigma', \delta'\rangle$ of σ, δ. The orthogonal projections σ'/δ' and δ'/σ' are complete developments (multi-steps) again, which are strongly convergent since the rules are not collapsing. Hence $t_1 \overset{\delta'/\sigma'}{\rightarrow} s' \overset{\sigma'/\delta'}{\leftarrow} t_2$. ∎

Note that in Corollary 6.11 the non-collapsingness is a necessary condition. To see this, reconsider Example 5.12 and observe that the non-confluent derivations are developments.

In a similar vein, we can prove the triangle property for infinitary weakly orthogonal multi-steps without collapsing rules:

Theorem 6.12. *For every weakly orthogonal TRS without collapsing rules, (infinite) multi-steps have the triangle property.* ∎

7. Conclusions

We have shown the failure of UN^∞ for weakly orthogonal TRSs in the presence of two collapsing rules. For weakly orthogonal TRSs without collapsing rules we established that

CR^∞ (and hence UN^∞) holds, and that this result is optimal in the sense that allowing only one collapsing rule is able to invalidate CR^∞.

However, the failure of UN^∞ for two collapsing rules raises the following question:

Question 7.1. Does UN^∞ hold for weakly orthogonal TRSs with *one* collapsing rule?

Furthermore, we have shown that infinitary developments in weakly orthogonal TRSs without collapsing rules have the diamond property. In general this property fails already in the presence of only one collapsing rule.

The following table summarizes the results of this paper (coloured green) next to known results (black).

		finitary				*infinitary*			
		PML	CR	UN	NF	PML^∞	CR^∞	UN^∞	NF^∞
first-order	OTRS	yes	yes	yes	yes	yes	no	yes	yes
	WOTRS	yes	yes	yes	yes	**yes**	no	**no**	**no**
	nc-WOTRS	yes	yes	yes	yes	**yes**	**yes**	**yes**	**yes**
	1c-WOTRS	yes	yes	yes	yes	**yes**	no	**?**	**?**
higher-order	$\lambda\beta$	yes	yes	yes	yes	no	no	yes	yes
	fe-OCRS	yes	yes	yes	yes	no	no	yes	yes[4]
	$\lambda\beta\eta$	yes[2]	yes	yes	yes	no	no	**no**	**no**
	WOCRS[1]	yes	yes	yes	yes	no	no	**no**	**no**

The nc-WOTRSs are weakly orthogonal TRSs without collapsing rules; 1c-WOTRSs likewise with one collapsing rule. The fe-OCRSs are fully extended orthogonal CRSs, see [4].

References

[1] H. Barendregt and J.W. Klop. Applications of Infinitary Lambda Calculus. *Inf. Comput.*, 207(5):559–582, 2009.

[2] I. Bethke, J.W. Klop, and de Vrijer, R.C. Descendants and Origins in Term Rewriting. *Inf. Comput.*, 159(1–2):59–124, 2000.

[3] R. Kennaway, J.W. Klop, M.R. Sleep, and F.-J. de Vries. Transfinite Reductions in Orthogonal Term Rewriting Systems. *Inf. Comput.*, 119(1):18–38, 1995.

[4] J. Ketema and J. Grue Simonsen. Infinitary Combinatory Reduction Systems: Confluence. *LMCS*, 5(4):1–29, 2009.

[5] J. Ketema, J.W. Klop, and V. van Oostrom. Vicious Circles in Rewriting Systems. CKI Preprint 52, Universiteit Utrecht, 2004. Available at `http://www.phil.uu.nl/preprints/aips/`.

[6] J.W. Klop and R.C. de Vrijer. Infinitary Normalization. In *We Will Show Them: Essays in Honour of Dov Gabbay*, volume 2, pages 169–192. College Publications, 2005.

[7] P. Severi and F.-J. de Vries. An Extensional Böhm Model. In S. Tison, editor, *RTA 2002*, volume 2378 of *LNCS*, pages 159–173, 2002.

[8] P. Severi and F.-J. de Vries. Continuity and Discontinuity in Lambda Calculus. In *TLCA 2005*, volume 3461 of *LNCS*, pages 369–385, 2005.

[9] Terese. *Term Rewriting Systems*, volume 55 of *Cambridge Tracts in Theoretical Computer Science*. Cambridge University Press, 2003.

[10] Vincent van Oostrom. Finite family developments. In Hubert Comon, editor, *Proceedings of RTA 1997*, volume 1232 of *Lecture Notes in Computer Science*, pages 308–322. Springer, 1997.

[2]Beware: in [2] a counterexample is given to PML for $\lambda\beta\eta$, but that pertains to the stronger (classical) version of PML where the 'parallel move' has to consist of contractions of 'residuals' of the originally contracted redex.