Random Descent

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Evaluating an Expression

expression:

p 1

Evaluating an Expression

expression:

$$\mathsf{p}\,1$$

evaluation rules:

$$\begin{aligned}
\mathsf{p}\,x &\longrightarrow \mathsf{if}\ x = 0\ \mathsf{then}\ 1\ \mathsf{else}\ 2 \cdot \mathsf{p}\,(x-1) \\
\mathsf{if\ false\ then}\ x\ \mathsf{else}\ y &\longrightarrow y \\
\mathsf{if\ true\ then}\ x\ \mathsf{else}\ y &\longrightarrow x \\
&\vdots
\end{aligned}$$

Evaluation of the Expression

$$\mathsf{p}\,1$$

$$\downarrow$$

$$\text{if } 1 = 0 \text{ then } 1 \text{ else } 2 \cdot \mathsf{p}\,(1 - 1)$$

$$\downarrow$$

$$\text{if false then } 1 \text{ else } 2 \cdot \mathsf{p}\,(1 - 1)$$

$$\downarrow$$

$$2 \cdot \mathsf{p}\,(1 - 1)$$

$$\downarrow$$

$$2 \cdot \mathsf{p}\,0$$

$$\downarrow$$

$$2 \cdot (\text{if } 0 = 0 \text{ then } 1 \text{ else } 2 \cdot \mathsf{p}\,(0 - 1))$$

$$\downarrow$$

$$2 \cdot (\text{if true then } 1 \text{ else } 2 \cdot \mathsf{p}\,(0 - 1))$$

$$\downarrow$$

$$2 \cdot 1$$

$$\downarrow$$

$$2$$

Another Evaluation of the Expression

$$\mathsf{p}\,1$$
$$\downarrow$$
$$\text{if } 1 = 0 \text{ then } 1 \text{ else } 2 \cdot \mathsf{p}\,(1 - 1)$$
$$\downarrow$$
$$\text{if } 1 = 0 \text{ then } 1 \text{ else } 2 \cdot \mathsf{p}\,0$$
$$\downarrow$$
$$\text{if false then } 1 \text{ else } 2 \cdot \mathsf{p}\,0$$
$$\downarrow$$
$$2 \cdot \mathsf{p}\,0$$
$$\downarrow$$
$$2 \cdot (\text{if } 0 = 0 \text{ then } 1 \text{ else } 2 \cdot \mathsf{p}\,(0 - 1))$$
$$\downarrow$$
$$2 \cdot (\text{if true then } 1 \text{ else } 2 \cdot \mathsf{p}\,(0 - 1))$$
$$\downarrow$$
$$2 \cdot 1$$
$$\downarrow$$
$$2$$

And Yet Another

$$\mathsf{p}\,1$$

$$\downarrow$$

$$\text{if } 1 = 0 \text{ then } 1 \text{ else } 2 \cdot \mathsf{p}\,(1-1)$$

$$\downarrow$$

$$\text{if } 1 = 0 \text{ then } 1 \text{ else } 2 \cdot (\text{if } (1-1) = 0 \text{ then } 1 \text{ else } 2 \cdot \mathsf{p}\,((1-1)-1))$$

$$\downarrow$$

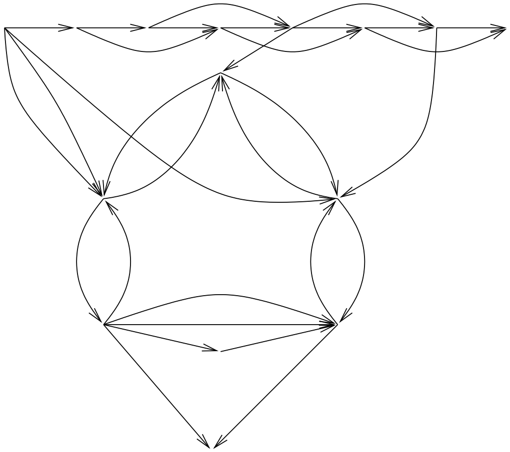
$$\text{if } 1 = 0 \text{ then } 1 \text{ else } 2 \cdot (\text{if } (1-1) = 0 \text{ then } 1 \text{ else } 2 \cdot (\text{if } ((1-1)-1) = 0 \text{ then } 1 \text{ else } 2 \cdot \mathsf{p}\,(((1-1)-1)-1)))$$

$$\downarrow$$

$$\vdots$$

Abstracting away Expressions and Evaluation

Graph: Nodes as Expressions, Edges as Evaluation Steps



Abstract Rewrite System = Graph

We are concerned with two kinds of entities, "objects" and the "moves" performed on them, and each move is associated with two objects, "initial" and "final." We are therefore dealing essentially with *indexed* 1-*complexes* (in which, therefore, a positive sense is assigned in each 1-cell), the vertices being the "objects," and the positive 1-cells the "moves." It will be convenient to make use of this topological terminology.[3] The incidence relations are in no way restricted: there may be many cells with the same vertices, and the initial and final vertices of a cell may coincide. In diagrams the positive 1-cells slope down the paper, and some of the terms used are chosen accordingly.

[3] The notions that arise are closely related to those of the theory of partially ordered sets, but usually not identical. Except in the case of identity the terms of that theory are therefore avoided.

Newman 1942, page 224

ARSs are not relations

Rewrite relation instead of system ??

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Rewrite relation instead of system ??

$I(I(a)) \rightrightarrows I(a)$ (two evaluations) not expressible !!

Strategy?

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No general definition on Wikipedia . . .

Strategy?

No general definition on Wikipedia ...
No uniform definition in rewriting papers ...

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. . . nor in Baader & Nipkow 1998.

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(neither are ARSs, only rewrite relations)

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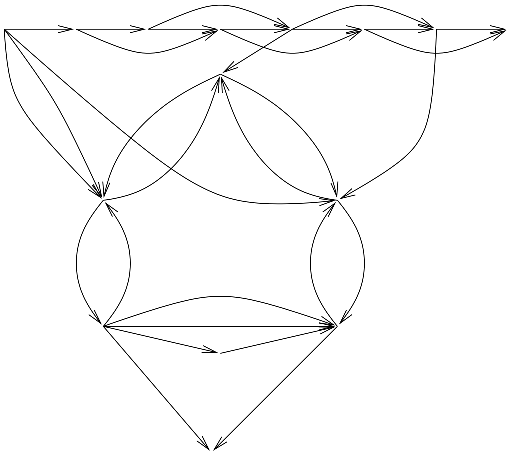
. . . nor in Baader & Nipkow 1998.
(neither are ARSs, only rewrite relations)

Definition (Terese 2003)

Strategy : sub-ARS having same objects, normal forms

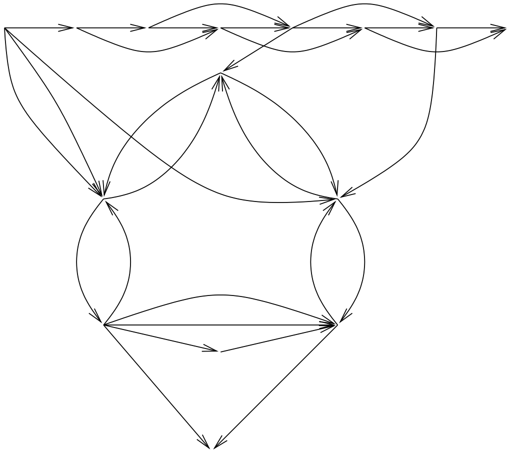
Strategy Examples

ARS



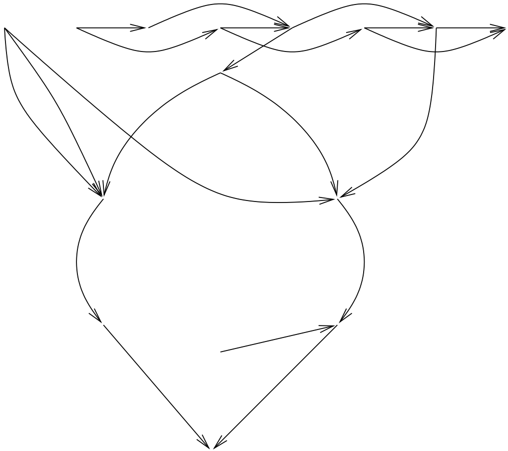
Strategy Examples

ARS strategy for itself!



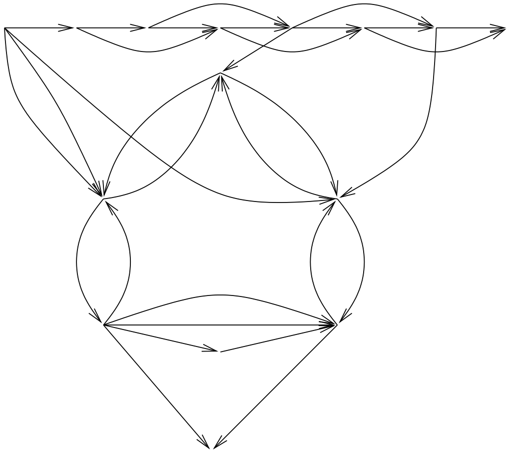
Strategy Examples

An optimal strategy



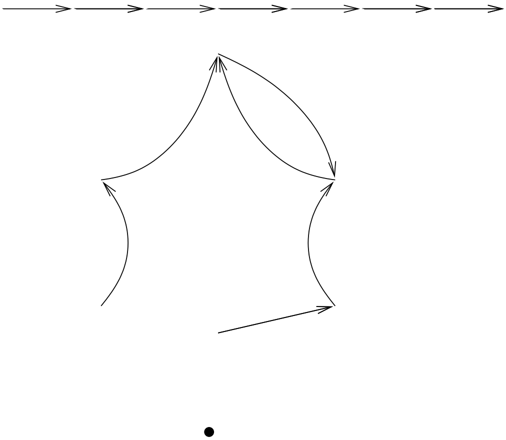
Strategy Examples

Original ARS again

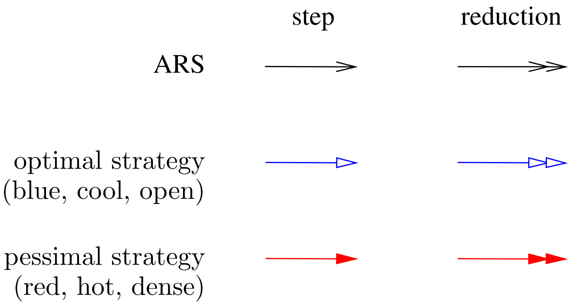


Strategy Examples

A pessimal strategy

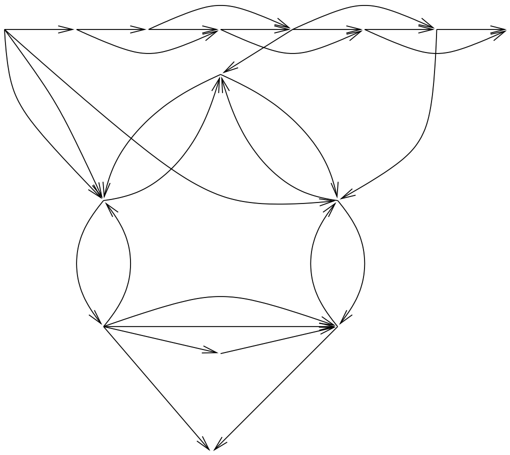


Arrows colour convention



Examples of colour convention

ARS



Examples of colour convention

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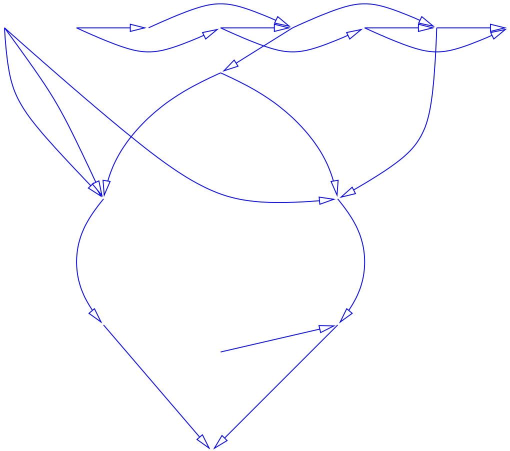

Examples of colour convention

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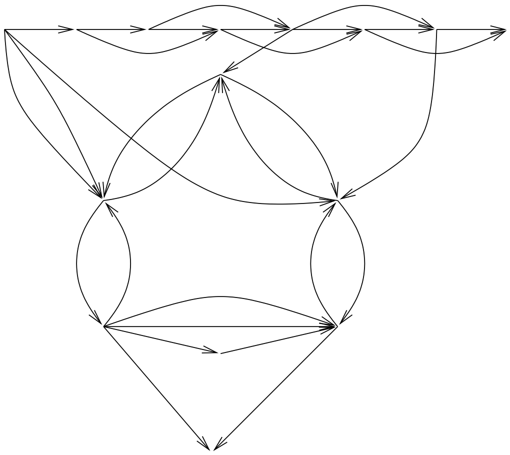

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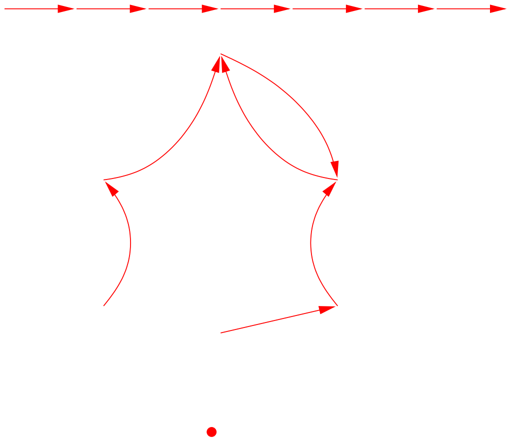
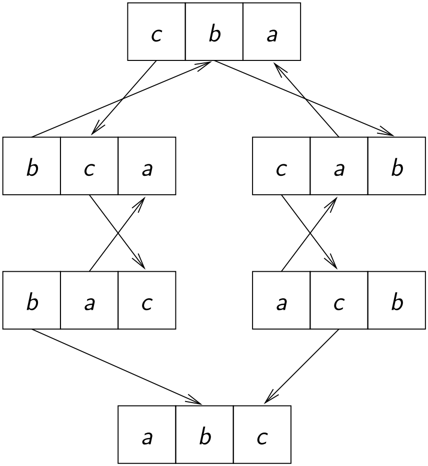
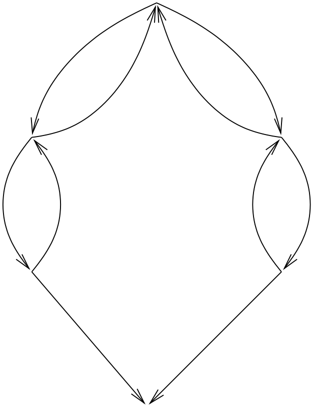


Sorting by Swapping

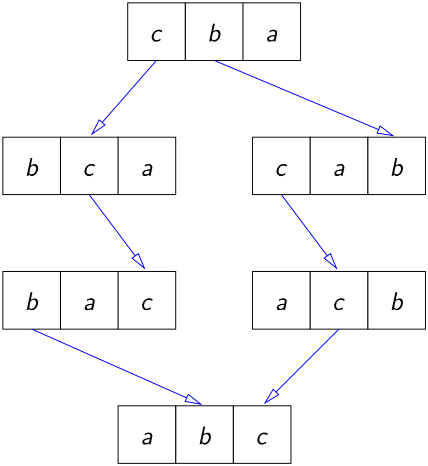


Reduction graph: Arrows start at first element swapped

Sorting by Swapping Abstractly

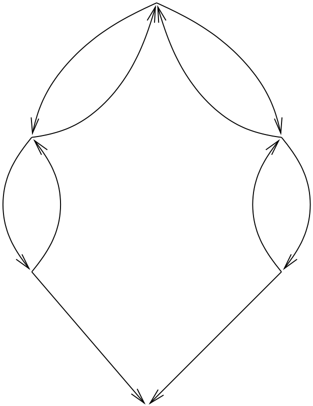


Sorting Strategy: Inversion

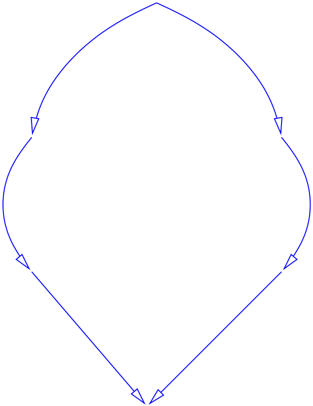


Inversion: only swap elements in wrong order

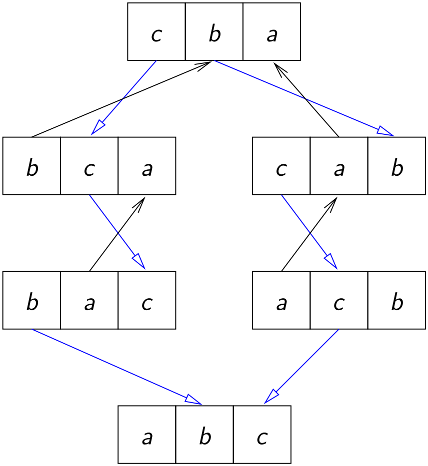
Sorting Strategy: Inversion Abstractly



Sorting Strategy: Inversion Abstractly

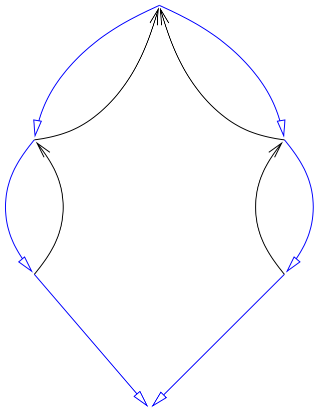


Sorting by Swapping



Reduction graph: inversions vs. anti-inversions

Sorting by Swapping Abstractly



Strategy Analysis

- Normalising: if normal form exists, it is found

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Inversion sort optimal (normalising and minimal)?

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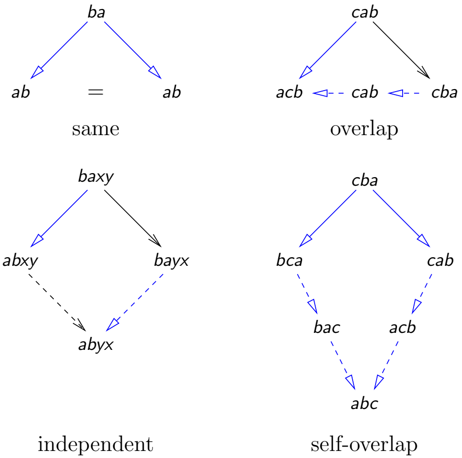
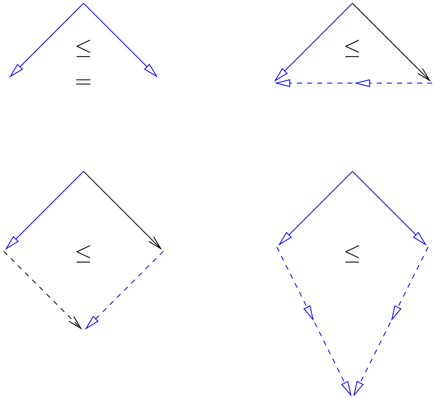
Inversion sort optimal (normalising and minimal)?

By local commutation of $\triangleright$ and $\rightarrow$!

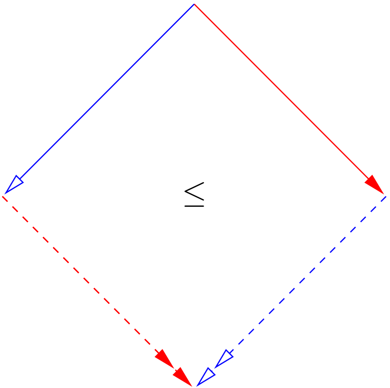
Local Commutation of $\triangleright$ and $\longrightarrow$



Ordered Local Commutation of $\triangleright$ and $\longrightarrow$



Ordered Local Commutation



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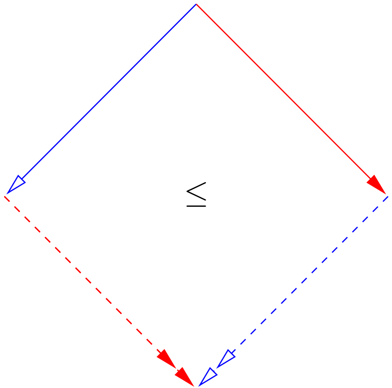
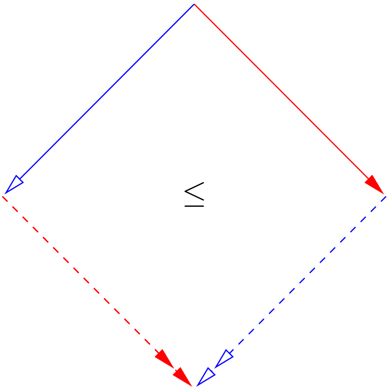


∀ local peak

Ordered Local Commutation



∀ local peak ∃ valley

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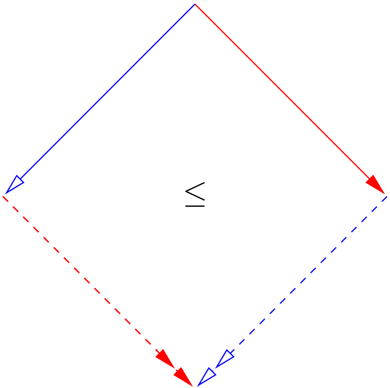
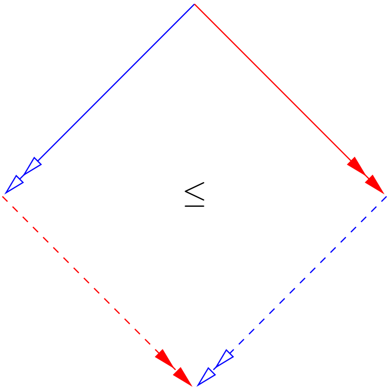


∀ local peak ∃ valley s.t. left path not longer than right path

Ordered Commutation



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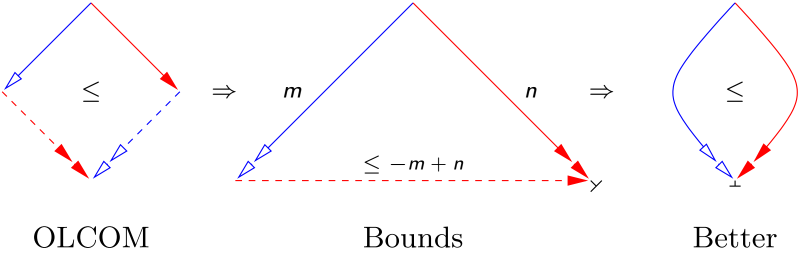
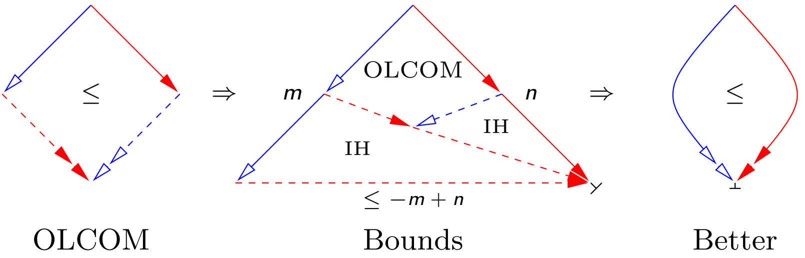
OLCOM $\Rightarrow$ Better



Better: max $\triangleright$ reduction not longer than max $\blacktriangleright$ reduction

OLCOM ⇒ Better (Proof)



Induction on n

Better ⇒ Normalising and Minimal

Theorem

- *better than* ▶ ⇒
- *normalising and minimal for* ▶

Proof.

□

Better $\Rightarrow$ Normalising and Minimal

Theorem

$\triangleright$ *better than* $\blacktriangleright$ $\Rightarrow$

$\triangleright$ *normalising and minimal for* $\blacktriangleright$

Proof.

- Normalising: a reduction to normal form is upper bound

$\square$

Better $\Rightarrow$ Normalising and Minimal

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$\square$

Corollary

Inversion sort normalising and minimal w.r.t. swapping

Proof.

OLCOM($\triangleright$,$\longrightarrow$) $\Rightarrow$

Better($\triangleright$,$\longrightarrow$) $\Rightarrow$

$\triangleright$ normalising and minimal for $\longrightarrow$. $\square$

Better ⇒ Perpetual and Maximal

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$\triangleright$ *better than* $\blacktriangleright$ ⇒

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Applications

- Internal needed strategy normalising, minimal (Khasidashvili)
  variations: Alves et al., Machkasova (WRS 2007)

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Proofs by 'critical pair' analysis and setting up 'simulations'.

Completeness

Theorem (Completeness)

$\triangleright$ *is better than* $\blacktriangleright \Rightarrow$ *OLCOM*($\triangleright$,$\blacktriangleright$),
if $\blacktriangleright$ *or* $\triangleright$ *equal to* $\rightarrow$ *and* $\rightarrow$ *has unique normal forms.*

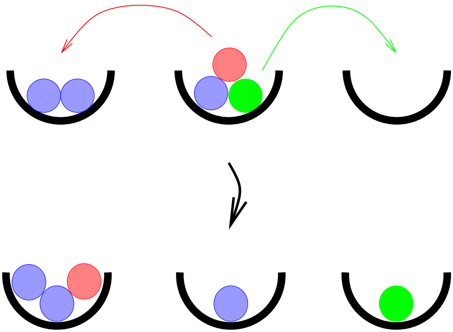
Completeness

Theorem (Completeness)

$\triangleright$ is better than $\blacktriangleright$ $\Rightarrow$ OLCOM($\triangleright$,$\blacktriangleright$), if $\blacktriangleright$ or $\triangleright$ equal to $\longrightarrow$ and $\longrightarrow$ has unique normal forms.

OLCOM is always applicable!

Bowls and Beans



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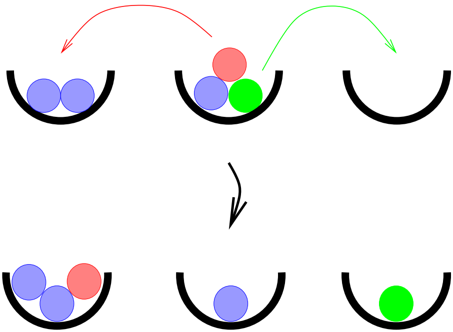
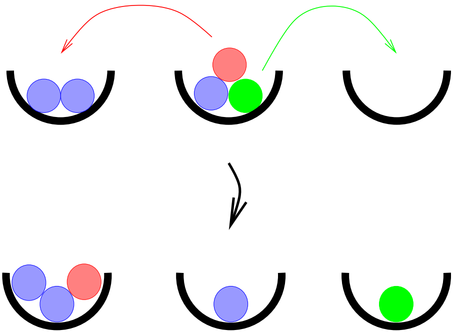


- Two-sided infinite sequence of bowls with beans: $\mathbb{Z} \to \mathbb{N}$.

Bowls and Beans



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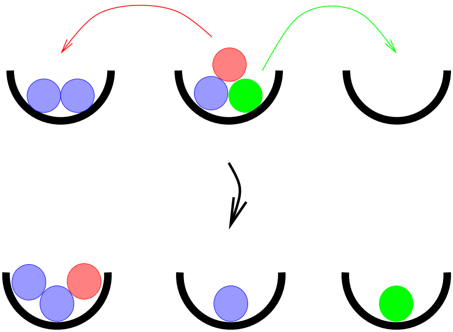
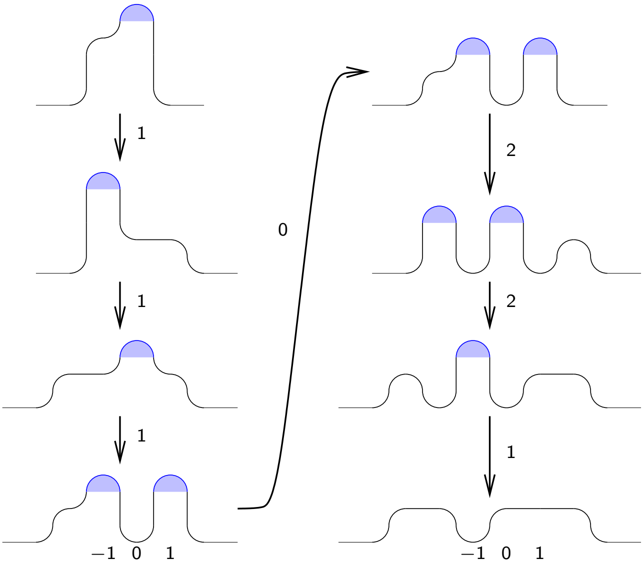


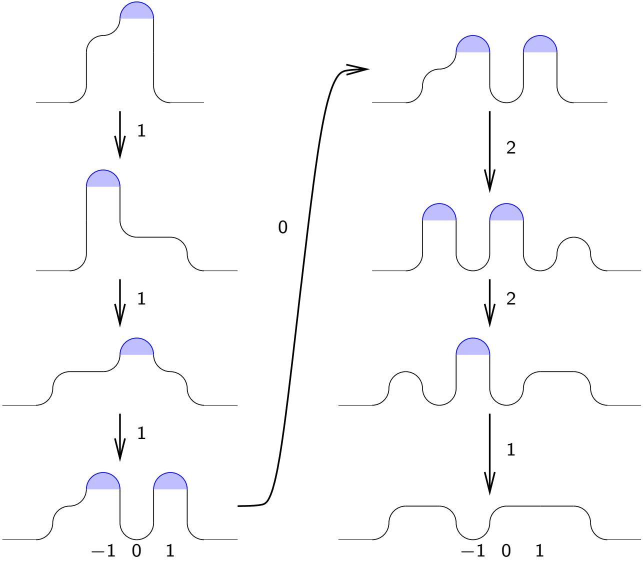
- Two-sided infinite sequence of bowls with beans: $\mathbb{Z} \to \mathbb{N}$.
- Finite number of beans $\Rightarrow$ finite number of steps
- Independent of strategy, same number of steps, final state

Bean run



Terminates . . .

Bean run



Terminates . . . but always so?

Analysis

Random Descent: all maximal reductions have same length

Newman 1942, page 226

5

In these examples it is obvious that if an end-form exists it is reached by random descent. This is necessarily so in all systems with non-interference of moves:

Theorem 2. *Under the conditions of Theorem 1, if there is a descending path of k cells from a to an end e, no descending path from a contains more than k cells.*

If $k = 1$, Σ cannot contain a cell ay with $y \neq e$, since if it does b exists such that $y\mu b$ and $e\mu b$, and e is not an end. In the general case let π be a descending path $\xi_1 + \xi_2 + \cdots + \xi_k$ joining a to e, and let $\eta_1 + \eta_2 + \cdots + \eta_j$ be any descending path from a. Let ξ_1 and η_1 be cells ax and ay. If $x = y$ it follows immediately from an induction that $j \leqq k$. If not, let the cells ζ and ω descend from x and y to the common vertex w. By Theorem 1 there is a descending path σ from w to a vertex $\leqq e$, i.e., since e is an end, to e itself. Since $\xi_2 + \cdots + \xi_k$ has $k - 1$ cells, $\zeta + \sigma$ has, by an inductive hypothesis, at most $k - 1$ cells; therefore $\omega + \sigma$, and finally also $\eta_2 + \cdots + \eta_j$, have at most $k - 1$ cells,—i.e. $j \leqq k$.

Corollary 2.1. *Every descending path from a is part of a descending path of k cells from a to e (i.e. there is "random descent" to e).*

Conditions of Theorem 1: join local peak in 0 or 1 steps

Analysis

Random Descent: all maximal reductions have same length

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Bean run has random descent?

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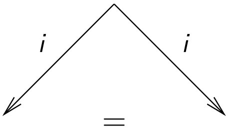
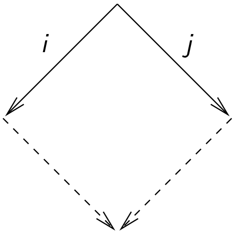
Bean run has random descent?

By local confluence of $\longrightarrow$!

Local Confluence of $\longrightarrow$

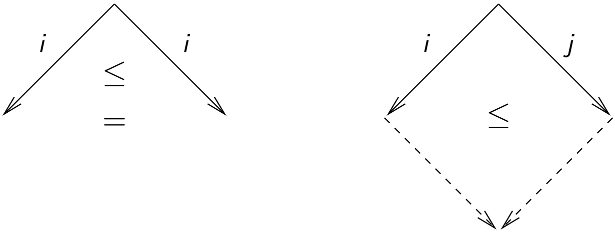


same

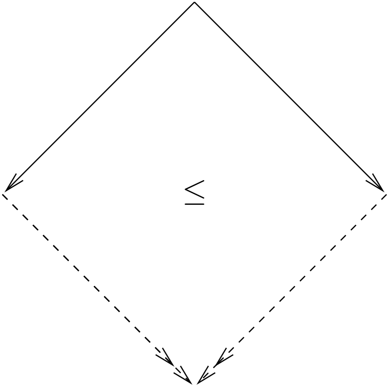


distinct

Ordered Local Confluence of $\longrightarrow$

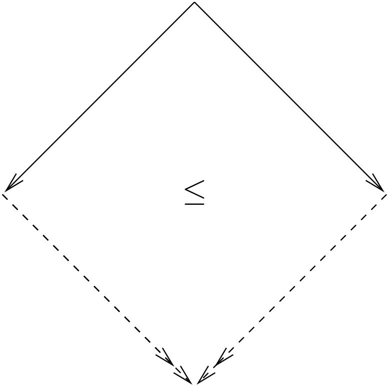


Ordered Local Confluence (OWCR)



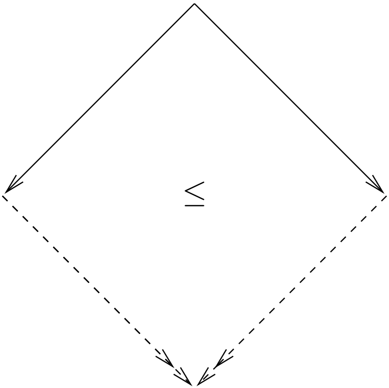
∀ local peak

Ordered Local Confluence (OWCR)



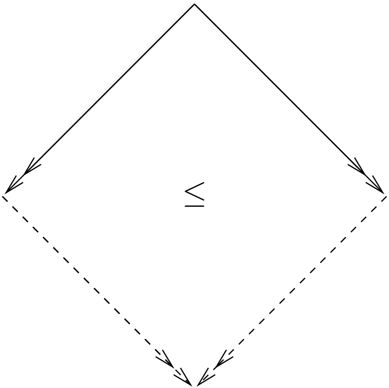
∀ local peak ∃ valley

Ordered Local Confluence (OWCR)



∀ local peak ∃ valley s.t. left path not longer than right path

Ordered Confluence



∀ peak ∃ valley s.t. left path not longer than right path

OWCR $\Rightarrow$ Random Descent

OWCR($\rightarrow$) $\Leftrightarrow$
OLCOM($\rightarrow$,$\rightarrow$) $\Rightarrow$
$\rightarrow$ better than itself $\Rightarrow$
$\rightarrow$ maximal, minimal!

Corollary

Bowls and beans has random descent

Proof.

OWCR $\Rightarrow$
RD $\Rightarrow$
all reductions to final state have same length $\square$

Solving bowls and beans

RD $\Rightarrow$ normalisation (WN) suffices for termination

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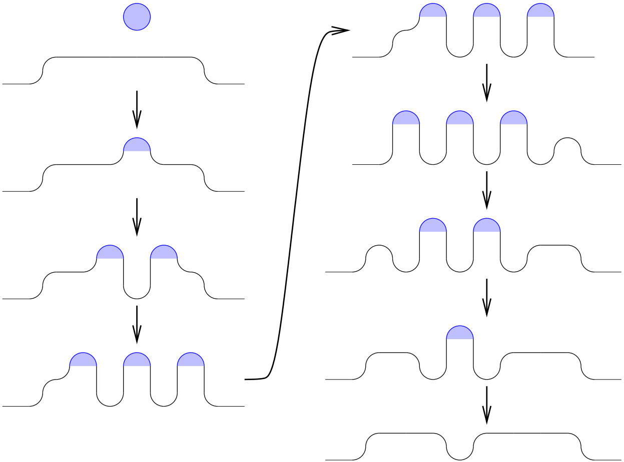

By induction on the number of beans

Existing generalizations of Newman's condition

Let $A = \langle D, \rightarrow \rangle$ be an abstract reduction system.

Definition 3. *$A = \langle D, \rightarrow \rangle$ (or $\rightarrow$) is balanced weakly Church-Rosser (BWCR) iff $\forall x, y, z \in D, [x \rightarrow y \wedge x \rightarrow z \Rightarrow \exists w \in D, \exists k \geq 0, y \rightarrow^k w \wedge z \rightarrow^k w]$ (Figure 1).*

Lemma 1 (BWCR Lemma). *Let $A = \langle D, \rightarrow \rangle$ be BWCR. Let $x = y$ and $y \in NF$. Then,*

(1) *x is complete,*
(2) *all the reductions from x to y have the same length (i.e., the same number of reduction steps).*

BWCR (Toyama 92/05): join local peak in same number of steps

Existing generalizations of Newman

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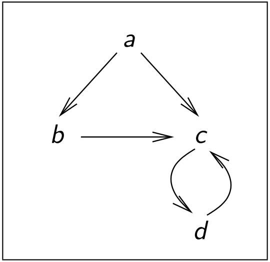
None covers:



Out of sync

Applications of OLCON

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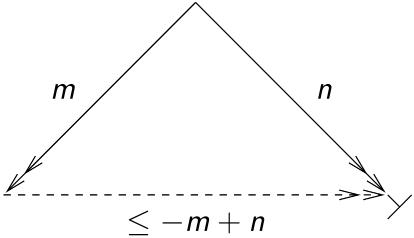
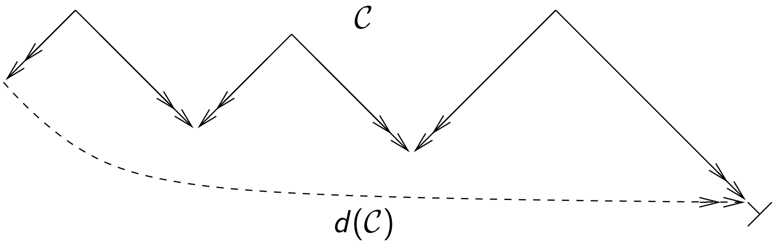
Proofs by ‘critical pair’ analysis and setting up ‘simulations’.

Completeness



OWCR $\Leftrightarrow$ RD (cf. confluence)

Completeness



$d(\mathcal{C})$ = number of forward steps minus number of backward steps

OWCR $\Leftrightarrow$ RD (cf. Church–Rosser)

Conclusions

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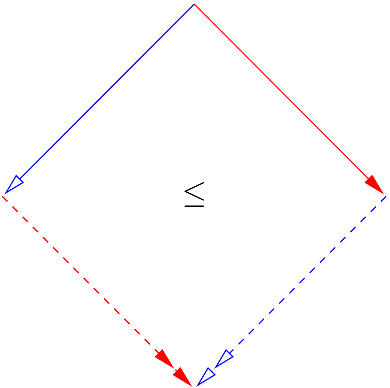
  $\forall\exists$ instead of $\forall\forall$ notion of better

Ordered Local Commutation



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