Home Page

Title Page

Contents

◀◀ ▶▶

◀ ▶

Go Back

Full Screen

Close

Quit

Sub-Birkhoff

Vincent van Oostrom
Universiteit Utrecht
Department of Philosophy
Vincent.vanOostrom@phil.uu.nl

Contents

1 Motivation **4**

2 Equational specification **5**
2.1 Validity . . . 7
2.2 Derivability . . . 8
2.3 Convertibility . . . 9
2.4 Proofs for equational specifications . . . 10
2.5 Birkhoff soundness . . . 11
2.6 Birkhoff completeness . . . 12
2.7 Logicality soundness . . . 14
2.8 Logicality completeness . . . 15

3 Sub-equational specifications **16**
3.1 Sub-equational specifications . . . 17
3.2 Sub-equational specification examples . . . 18
3.3 Sub-validity . . . 19
3.4 Sub-validity examples . . . 20
3.5 Sub-derivability . . . 21
3.6 Sub-convertibility . . . 22
3.7 Proofs for sub-equational specifications . . . 23
3.8 Sub-Birkhoff soundness . . . 24
3.9 Sub-Birkhoff completeness . . . 25

Home Page

Title Page

Contents

◀◀ ▶▶

◀ ▶

Page 3 of 28

Go Back

Full Screen

Close

Quit

3.10 Logicality soundness 26
3.11 Logicality completeness 27

4 Conclusion **28**

1. Motivation

valid $\Longleftrightarrow$ derivable $\Longleftrightarrow$ convertible

1. Motivation

valid $\iff$ derivable $\iff$ convertible

For equational specification $\mathcal{E}$

$\mathcal{E} \models s = t \iff \mathcal{E} \vdash s = t \iff s \leftrightarrow^*_{\mathcal{E}} t$

1. Motivation

valid $\iff$ derivable $\iff$ convertible

For equational specification $\mathcal{E}$

$$\mathcal{E} \models s = t \iff \mathcal{E} \vdash s = t \iff s \leftrightarrow^*_{\mathcal{E}} t$$

For rewriting logic specification $\mathcal{R}$

$$\mathcal{R} \models s \geq t \iff \mathcal{R} \vdash s \geq t \iff s \to^*_{\mathcal{R}} t$$

1. Motivation

valid $\iff$ derivable $\iff$ convertible

For equational specification $\mathcal{E}$

$\mathcal{E} \models s = t \iff \mathcal{E} \vdash s = t \iff s \leftrightarrow^*_{\mathcal{E}} t$

For rewriting logic specification $\mathcal{R}$

$\mathcal{R} \models s \geq t \iff \mathcal{R} \vdash s \geq t \iff s \rightarrow^*_{\mathcal{R}} t$

For term rewriting system $\mathcal{T}$

$\mathcal{T}$ admits a compatible well-founded monotone algebra

$\iff \rightarrow^+_{\mathcal{T}}$ is terminating

1. Motivation

valid $\iff$ derivable $\iff$ convertible

For equational specification $\mathcal{E}$

$\mathcal{E} \models s = t \iff \mathcal{E} \vdash s = t \iff s \leftrightarrow^*_{\mathcal{E}} t$

For rewriting logic specification $\mathcal{R}$

$\mathcal{R} \models s \geq t \iff \mathcal{R} \vdash s \geq t \iff s \rightarrow^*_{\mathcal{R}} t$

For term rewriting system $\mathcal{T}$

$\mathcal{T}$ admits a compatible well-founded monotone algebra

$\iff \rightarrow^+_{\mathcal{T}}$ is terminating

follows from:

$\mathcal{T} \models s > t \iff \mathcal{T} \vdash s > t \iff s \rightarrow^+_{\mathcal{T}} t$

1. Motivation

valid $\iff$ derivable $\iff$ convertible

For equational specification $\mathcal{E}$

$\mathcal{E} \models s = t \iff \mathcal{E} \vdash s = t \iff s \leftrightarrow^*_{\mathcal{E}} t$

For rewriting logic specification $\mathcal{R}$

$\mathcal{R} \models s \geq t \iff \mathcal{R} \vdash s \geq t \iff s \rightarrow^*_{\mathcal{R}} t$

For term rewriting system $\mathcal{T}$

$\mathcal{T}$ admits a compatible well-founded monotone algebra

$\iff \rightarrow^+_{\mathcal{T}}$ is terminating

follows from:

$\mathcal{T} \models s > t \iff \mathcal{T} \vdash s > t \iff s \rightarrow^+_{\mathcal{T}} t$

Problem 1 *Same result?*

2. Equational specification

Equational specification $\mathcal{EM}ul$

2. Equational specification

Equational specification $\mathcal{EM}ul$

signature Σ

`O`, `S`, `A`, `M`.

2. Equational specification

Equational specification $\mathcal{EM}ul$

signature Σ

$\mathtt{O}$, $\mathtt{S}$, $\mathtt{A}$, $\mathtt{M}$.

equations over Σ

$$\begin{aligned} \mathtt{A}(x, \mathtt{O}) &\approx x \\ \mathtt{A}(x, \mathtt{S}(y)) &\approx \mathtt{S}(\mathtt{A}(x, y)) \\ \mathtt{M}(x, \mathtt{O}) &\approx \mathtt{O} \\ \mathtt{M}(x, \mathtt{S}(y)) &\approx \mathtt{A}(x, \mathtt{M}(x, y)) \end{aligned}$$

2. Equational specification

Equational specification $\mathcal{EM}ul$

signature Σ

$\mathtt{0}$, $\mathtt{S}$, $\mathtt{A}$, $\mathtt{M}$.

equations over Σ

$$\begin{aligned} \mathtt{A}(x,\mathtt{0}) &\approx x \\ \mathtt{A}(x,\mathtt{S}(y)) &\approx \mathtt{S}(\mathtt{A}(x,y)) \\ \mathtt{M}(x,\mathtt{0}) &\approx \mathtt{0} \\ \mathtt{M}(x,\mathtt{S}(y)) &\approx \mathtt{A}(x,\mathtt{M}(x,y)) \end{aligned}$$

equation considered w.r.t. $\mathcal{EM}ul$

$$\mathtt{M}(\mathtt{S}(x),\mathtt{S}(\mathtt{0})) \approx \mathtt{S}(x) \quad (1)$$

2.1. Validity

algebra $\mathcal{A}$ interprets signature (carrier, operations)
$\mathcal{N}at$ interprets Σ as set of natural numbers
$\mathcal{N}at$ interprets `0`, `S`, `A` and `M` as
zero, successor, addition and multiplication

2.1. Validity

algebra $\mathcal{A}$ interprets signature (carrier, operations)
$\mathcal{N}at$ interprets Σ as set of natural numbers
$\mathcal{N}at$ interprets $\mathtt{0}$, $\mathtt{S}$, $\mathtt{A}$ and $\mathtt{M}$ as
zero, successor, addition and multiplication

equation $s \approx t$ holds in $\mathcal{A}$
$\mathcal{N}at \models \mathtt{A}(\mathtt{0}, \mathtt{S}(\mathtt{0})) \approx \mathtt{A}(\mathtt{S}(\mathtt{0}), \mathtt{0})$ (since $1 = 1$)
$\mathcal{N}at \not\models \mathtt{A}(\mathtt{0}, \mathtt{0}) \approx \mathtt{S}(\mathtt{0})$ (since $0 \neq 1$)
open equation holds, if so for all assignments α

2.1. Validity

algebra $\mathcal{A}$ interprets signature (carrier, operations)
$\mathcal{N}at$ interprets Σ as set of natural numbers
$\mathcal{N}at$ interprets $\mathtt{0}$, $\mathtt{S}$, $\mathtt{A}$ and $\mathtt{M}$ as
zero, successor, addition and multiplication

equation $s \approx t$ holds in $\mathcal{A}$
$\mathcal{N}at \models \mathtt{A}(\mathtt{0}, \mathtt{S}(\mathtt{0})) \approx \mathtt{A}(\mathtt{S}(\mathtt{0}), \mathtt{0})$ (since $1 = 1$)
$\mathcal{N}at \not\models \mathtt{A}(\mathtt{0}, \mathtt{0}) \approx \mathtt{S}(\mathtt{0})$ (since $0 \neq 1$)
open equation holds, if so for all assignments α

$\mathcal{A}$ models $\mathcal{E}$, if all equations hold
$\mathcal{N}at \models \mathcal{E}\mathcal{M}ul$
$s \approx t$ valid in $\mathcal{E}$, if holds in any model
$\mathcal{E} \models \mathtt{M}(\mathtt{S}(x), \mathtt{S}(\mathtt{0})) \approx \mathtt{S}(x)$

2.2. Derivability

$s \approx t$ derivable (in equational logic) from $\mathcal{E}$

$$\frac{(s \approx t \in E)}{s \approx t} \qquad \frac{s \approx t}{\sigma(s) \approx \sigma(t)} \qquad \frac{s_1 \approx t_1 \quad \dots \quad s_n \approx t_n}{f(s_1, \dots, s_n) \approx f(t_1, \dots, t_n)}$$

$$\frac{}{s \approx s} \qquad \frac{s \approx t}{t \approx s} \qquad \frac{s \approx t \quad t \approx u}{s \approx u}$$

σ substitution, f function symbol

$$\dfrac{\dfrac{\dfrac{}{\mathtt{M}(x,\mathtt{S}(y)) \approx \mathtt{A}(x,\mathtt{M}(x,y))}}{\mathtt{M}(\mathtt{S}(x),\mathtt{S}(0)) \approx \mathtt{A}(\mathtt{S}(x),\mathtt{M}(\mathtt{S}(x),0))} \quad \dfrac{\dfrac{\dfrac{}{\mathtt{S}(x) \approx \mathtt{S}(x)} \quad \dfrac{\dfrac{}{\mathtt{M}(x,0) \approx 0}}{\mathtt{M}(\mathtt{S}(x),0) \approx 0}}{\mathtt{A}(\mathtt{S}(x),\mathtt{M}(\mathtt{S}(x),0)) \approx \mathtt{A}(\mathtt{S}(x),0)} \quad \dfrac{\dfrac{}{\mathtt{A}(x,0) \approx x}}{\mathtt{A}(\mathtt{S}(x),0) \approx \mathtt{S}(x)}}{\mathtt{A}(\mathtt{S}(x),\mathtt{M}(\mathtt{S}(x),0)) \approx \mathtt{S}(x)}}{\mathtt{M}(\mathtt{S}(x),\mathtt{S}(0)) \approx \mathtt{S}(x)}$$

2.3. Convertibility

reduction step from s to t

$s \to_{\mathcal{E}} t$, if $s = C[\sigma(l)]$ and $t = C[\sigma(r)]$

C context, σ substitution, $l \approx r \in \mathcal{E}$

$\mathtt{S}(\mathtt{A}(0,0)) \to_{\mathcal{E}} \mathtt{S}(0)$

$C := \mathtt{S}([\,])$, $\sigma(x) := 0$, $\mathtt{A}(x,0) \approx x \in \mathcal{E}\mathcal{M}ul$

2.3. Convertibility

reduction step from s to t

$s \to_{\mathcal{E}} t$, if $s = C[\sigma(l)]$ and $t = C[\sigma(r)]$

C context, σ substitution, $l \approx r \in \mathcal{E}$

$\mathtt{S}(\mathtt{A}(0, 0)) \to_{\mathcal{E}} \mathtt{S}(0)$

$C := \mathtt{S}([\,])$, $\sigma(x) := 0$, $\mathtt{A}(x, 0) \approx x \in \mathcal{E}\mathcal{M}ul$

s convertible to t

connected by backward and forward reduction steps

$\underline{\mathtt{M}(\mathtt{S}(x), \mathtt{S}(0))} \to \mathtt{A}(\mathtt{S}(x), \underline{\mathtt{M}(\mathtt{S}(x), 0)}) \to \underline{\mathtt{A}(\mathtt{S}(x), 0)} \to \mathtt{S}(x)$

2.4. Proofs for equational specifications

valid $\Leftrightarrow_{\text{Birkhoff}}$ derivable $\Leftrightarrow_{\text{Logicality}}$ convertible

2.4. Proofs for equational specifications

valid $\Leftrightarrow_{\text{Birkhoff}}$ derivable $\Leftrightarrow_{\text{Logicality}}$ convertible

- Soundness of Birkhoff by induction on derivations
- Completeness of Birkhoff by term model
- Soundness of logicality by simulation
- Completeness of logicality by derivation standardisation

2.5. Birkhoff soundness

Thm 2 *valid* $\Leftarrow$ *derivable*

Proof
by induction on derivations

2.5. Birkhoff soundness

Thm 2 *valid* $\Leftarrow$ *derivable*

Proof
by induction on derivations

All inference rules trivially preserve validity ...

2.5. Birkhoff soundness

Thm 2 *valid* $\Leftarrow$ *derivable*

Proof
by induction on derivations

All inference rules trivially preserve validity . . .

$$\frac{s \approx t}{\sigma(s) \approx \sigma(t)}$$

needs semantic substitution lemma

$$[\![\mathcal{A} \cup \alpha]\!](\sigma(u)) \;=\; [\![\mathcal{A} \cup \alpha_\sigma]\!](u) \qquad (2)$$

2.6. Birkhoff completeness

Thm 3 *valid* $\Rightarrow$ *derivable*

Proof

Derivable equality ‘is’ a model

2.6. Birkhoff completeness

Thm 3 *valid* $\Rightarrow$ *derivable*

Proof

Derivable equality 'is' a model

Term algebra $\mathcal{T}(\Sigma)$ (interpret terms as themselves)

not yet a model e.g. $\mathtt{A}(\mathtt{0},\mathtt{0}) \neq \mathtt{0}$

2.6. Birkhoff completeness

Thm 3 *valid* $\Rightarrow$ *derivable*

Proof

Derivable equality 'is' a model

Term algebra $\mathcal{T}(\Sigma)$ (interpret terms as themselves)

not yet a model e.g. $\mathtt{A}(0,0) \neq 0$

Quotient algebra $\mathcal{T}(\Sigma)/_{\approx}$ (terms modulo derivability)

2.6. Birkhoff completeness

Thm 3 *valid* $\Rightarrow$ *derivable*

Proof

Derivable equality 'is' a model

Term algebra $\mathcal{T}(\Sigma)$ (interpret terms as themselves)

not yet a model e.g. $\mathtt{A}(0,0) \neq 0$

Quotient algebra $\mathcal{T}(\Sigma)/_{\approx}$ (terms modulo derivability)

$\mathcal{T}(\Sigma)/_{\approx}$ is algebra (derivable equality is congruence)

2.6. Birkhoff completeness

Thm 3 *valid* $\Rightarrow$ *derivable*

Proof
Derivable equality 'is' a model
Term algebra $\mathcal{T}(\Sigma)$ (interpret terms as themselves)
not yet a model e.g. $\mathtt{A}(0,0) \neq 0$
Quotient algebra $\mathcal{T}(\Sigma)/_{\approx}$ (terms modulo derivability)
$\mathcal{T}(\Sigma)/_{\approx}$ is algebra (derivable equality is congruence)
$\mathcal{T}(\Sigma)/_{\approx}$ is a model
all derivable equalities hold by induction on derivation...

2.6. Birkhoff completeness

Thm 3 *valid ⇒ derivable*

Proof
Derivable equality 'is' a model
Term algebra $\mathcal{T}(\Sigma)$ (interpret terms as themselves)
not yet a model e.g. $\mathtt{A}(\mathtt{0},\mathtt{0}) \neq \mathtt{0}$
Quotient algebra $\mathcal{T}(\Sigma)/_{\approx}$ (terms modulo derivability)
$\mathcal{T}(\Sigma)/_{\approx}$ is algebra (derivable equality is congruence)
$\mathcal{T}(\Sigma)/_{\approx}$ is a model
all derivable equalities hold by induction on derivation...

$$s \approx t \in \mathcal{E}$$

needs syntactic substitution lemma

$$[\![\mathcal{T}(\Sigma)/_{\approx} \cup \beta]\!](u) \;=\; [[\![\mathcal{T}(\Sigma) \cup \alpha]\!](u)]_{\approx} \qquad (3)$$

2.7. Logicality soundness

Thm 4 *derivable* $\Leftarrow$ *convertible*

Proof

2.7. Logicality soundness

Thm 4 *derivable* $\Leftarrow$ *convertible*

Proof

reduction step $C[\sigma(l)] \rightarrow_{\mathcal{E}} C[\sigma(r)]$ 'is' a derivation

2.7. Logicality soundness

Thm 4 *derivable* $\Leftarrow$ *convertible*

Proof

reduction step $C[\sigma(l)] \rightarrow_{\mathcal{E}} C[\sigma(r)]$ 'is' a derivation

$l \rightarrow r$ simulated by $\dfrac{(s \approx t \in E)}{s \approx t}$

2.7. Logicality soundness

Thm 4 *derivable* $\Leftarrow$ *convertible*

Proof

reduction step $C[\sigma(l)] \rightarrow_{\mathcal{E}} C[\sigma(r)]$ ‘is’ a derivation

$l \rightarrow r$ simulated by $\dfrac{(s \approx t \in E)}{s \approx t}$

$\sigma(l) \rightarrow \sigma(r)$ simulated by $\dfrac{s \approx t}{\sigma(s) \approx \sigma(t)}$

2.7. Logicality soundness

Thm 4 *derivable* $\Leftarrow$ *convertible*

Proof

reduction step $C[\sigma(l)] \rightarrow_{\mathcal{E}} C[\sigma(r)]$ 'is' a derivation

$l \rightarrow r$ simulated by $\dfrac{(s \approx t \in E)}{s \approx t}$

$\sigma(l) \rightarrow \sigma(r)$ simulated by $\dfrac{s \approx t}{\sigma(s) \approx \sigma(t)}$

$C[\sigma(l)] \rightarrow_{\mathcal{E}} C[\sigma(r)]$ simulated by

$\dfrac{s_1 \approx t_1 \quad \ldots \quad s_n \approx t_n}{f(s_1, \ldots, s_n) \approx f(t_1, \ldots, t_n)}$ and $\dfrac{}{s \approx s}$

2.7. Logicality soundness

Thm 4 *derivable* $\Leftarrow$ *convertible*

Proof

reduction step $C[\sigma(l)] \rightarrow_{\mathcal{E}} C[\sigma(r)]$ 'is' a derivation

$l \rightarrow r$ simulated by $\dfrac{(s \approx t \in E)}{s \approx t}$

$\sigma(l) \rightarrow \sigma(r)$ simulated by $\dfrac{s \approx t}{\sigma(s) \approx \sigma(t)}$

$C[\sigma(l)] \rightarrow_{\mathcal{E}} C[\sigma(r)]$ simulated by

$\dfrac{s_1 \approx t_1 \quad \ldots \quad s_n \approx t_n}{f(s_1, \ldots, s_n) \approx f(t_1, \ldots, t_n)}$ and $\dfrac{}{s \approx s}$

conversion (back/forward steps) 'is' a derivation

2.7. Logicality soundness

Thm 4 *derivable* $\Leftarrow$ *convertible*

Proof

reduction step $C[\sigma(l)] \rightarrow_{\mathcal{E}} C[\sigma(r)]$ 'is' a derivation

$l \rightarrow r$ simulated by $\dfrac{(s \approx t \in E)}{s \approx t}$

$\sigma(l) \rightarrow \sigma(r)$ simulated by $\dfrac{s \approx t}{\sigma(s) \approx \sigma(t)}$

$C[\sigma(l)] \rightarrow_{\mathcal{E}} C[\sigma(r)]$ simulated by

$\dfrac{s_1 \approx t_1 \quad \ldots \quad s_n \approx t_n}{f(s_1, \ldots, s_n) \approx f(t_1, \ldots, t_n)}$ and $\dfrac{}{s \approx s}$

conversion (back/forward steps) 'is' a derivation

backward simulated by $\dfrac{s \approx t}{t \approx s}$

2.7. Logicality soundness

Thm 4 *derivable* $\Leftarrow$ *convertible*

Proof

reduction step $C[\sigma(l)] \rightarrow_{\mathcal{E}} C[\sigma(r)]$ 'is' a derivation

$l \rightarrow r$ simulated by $\dfrac{(s \approx t \in E)}{s \approx t}$

$\sigma(l) \rightarrow \sigma(r)$ simulated by $\dfrac{s \approx t}{\sigma(s) \approx \sigma(t)}$

$C[\sigma(l)] \rightarrow_{\mathcal{E}} C[\sigma(r)]$ simulated by

$\dfrac{s_1 \approx t_1 \quad \dots \quad s_n \approx t_n}{f(s_1, \dots, s_n) \approx f(t_1, \dots, t_n)}$ and $\dfrac{}{s \approx s}$

conversion (back/forward steps) 'is' a derivation

backward simulated by $\dfrac{s \approx t}{t \approx s}$

steps simulated by

$\dfrac{}{s \approx s}$ and $\dfrac{s \approx t \qquad t \approx u}{s \approx u}$

2.8. Logicality completeness

Thm 5 *derivable* $\Rightarrow$ *convertible*

Proof

Derivation standardises to conversion

2.8. Logicality completeness

Thm 5 *derivable* $\Rightarrow$ *convertible*

Proof

Derivation standardises to conversion

standardisation: commute derivations in wrong order

$$\dfrac{\dfrac{s \approx t \quad t \approx u}{s \approx u}}{u \approx s} \rightsquigarrow \dfrac{\dfrac{s \approx t}{t \approx s} \qquad \dfrac{t \approx u}{u \approx t}}{u \approx s}$$

2.8. Logicality completeness

Thm 5 *derivable ⇒ convertible*

Proof

Derivation standardises to conversion

standardisation: commute derivations in wrong order

$$\dfrac{\dfrac{s \approx t \quad t \approx u}{s \approx u}}{u \approx s} \rightsquigarrow \dfrac{\dfrac{s \approx t}{t \approx s} \qquad \dfrac{t \approx u}{u \approx t}}{u \approx s}$$

process terminates, by recursive path order

$r_1(r_2(\vec{x_1}), \ldots, r_2(\vec{x_n})) \rightsquigarrow$

$r_2(r_1(x_{11}, \ldots, x_{1n}), \ldots, r_1(x_{m1}, \ldots, x_{mn}))$

3. Sub-equational specifications

Arise by removing some of derivation rules

3. Sub-equational specifications

Arise by removing some of derivation rules

equational specification: remove nothing
rewriting logic specification: remove symmetry
strict specification: remove reflexivity as well
term rewriting specification: also transitivity
etc.

3. Sub-equational specifications

Arise by removing some of derivation rules

equational specification: remove nothing
rewriting logic specification: remove symmetry
strict specification: remove reflexivity as well
term rewriting specification: also transitivity
etc.

Problem 6 *Proofs are not parametric*

3. Sub-equational specifications

Arise by removing some of derivation rules

equational specification: remove nothing
rewriting logic specification: remove symmetry
strict specification: remove reflexivity as well
term rewriting specification: also transitivity
etc.

Problem 6 *Proofs are not parametric*

Solution 7 *Make proofs parametric*
Remove dependencies between derivation rules

3.1. Sub-equational specifications

Sub-equational specification $\mathcal{M}ul$

3.1. Sub-equational specifications

Sub-equational specification $\mathcal{M}ul$

signature Σ

$\mathtt{O}$, $\mathtt{S}$, $\mathtt{A}$, $\mathtt{M}$.

3.1. Sub-equational specifications

Sub-equational specification $\mathcal{M}ul$

signature Σ

$\mathtt{O}$, $\mathtt{S}$, $\mathtt{A}$, $\mathtt{M}$.

statements over Σ

$(\mathtt{A}(x, \mathtt{O}), x)$

$(\mathtt{A}(x, \mathtt{S}(y)), \mathtt{S}(\mathtt{A}(x, y)))$

$(\mathtt{M}(x, \mathtt{O}), \mathtt{O})$

$(\mathtt{M}(x, \mathtt{S}(y)), \mathtt{A}(x, \mathtt{M}(x, y)))$

subset of inference modes

{(embedding), (compatibility),
(reflexivity), (symmetry), (transitivity)}

3.2. Sub-equational specification examples

equational spec {(embedding), (compatibility), (reflexivity), (symmetry), (transitivity)}

rewriting logic spec {(embedding), (compatibility), (reflexivity), (transitivity)}

strict spec {(embedding), (compatibility), (transitivity)}

3.3. Sub-validity

algebra $\mathcal{A}$ interprets signature (carrier, operations)
$\mathcal{N}at$ interprets Σ as set of natural numbers
$\mathcal{N}at$ interprets `0`, `S`, `A` and `M` as
zero, successor, addition and multiplication

3.3. Sub-validity

algebra $\mathcal{A}$ interprets signature (carrier, operations)
$\mathcal{N}at$ interprets Σ as set of natural numbers
$\mathcal{N}at$ interprets `O`, `S`, `A` and `M` as
zero, successor, addition and multiplication

relation R models statements which hold
statement (s, t) holds in $(\mathcal{A}, R)$ if $s^{\mathcal{A}}\ R\ t^{\mathcal{A}}$
open statement holds, if so for all assignments α

3.3. Sub-validity

algebra $\mathcal{A}$ interprets signature (carrier, operations)
$\mathcal{N}at$ interprets Σ as set of natural numbers
$\mathcal{N}at$ interprets `0`, `S`, `A` and `M` as
zero, successor, addition and multiplication

relation R models statements which hold
statement (s,t) holds in $(\mathcal{A},R)$ if $s^{\mathcal{A}}\ R\ t^{\mathcal{A}}$
open statement holds, if so for all assignments α

relational model of $\mathcal{M}ul$ if

$$\frac{(s,t)\in\mathcal{M}ul}{s\ R\ t}\ (\text{emb}) \qquad \frac{a_1,\ldots,a_n =[R]\ b_1,\ldots,b_n}{f^{\mathcal{A}}(a_1,\ldots,a_n)\ R\ f^{\mathcal{A}}(b_1,\ldots,b_n)}\ (\text{comp})$$

$$\frac{}{a\ R\ a}\ (\text{ref}) \quad \frac{a\ R\ b}{b\ R\ a}\ (\text{sym}) \quad \frac{a\ R\ b \quad b\ R\ c}{a\ R\ c}\ (\text{trans})$$

3.4. Sub-validity examples

$(\mathcal{N}at, =)$ is a relational model of equational logic spec $\mathcal{M}ul$

$(\mathcal{T}(\Sigma), \leftrightarrow^*)$ is a relational model of equational logic spec $\mathcal{M}ul$

$(\mathcal{T}(\Sigma), \rightarrow^*)$ is a relational model of rewriting logic spec $\mathcal{M}ul$

3.4. Sub-validity examples

$(\mathcal{N}at, =)$ is a relational model of equational logic spec $\mathcal{M}ul$

$(\mathcal{T}(\Sigma), \leftrightarrow^*)$ is a relational model of equational logic spec $\mathcal{M}ul$

$(\mathcal{T}(\Sigma), \rightarrow^*)$ is a relational model of rewriting logic spec $\mathcal{M}ul$
but not of $\mathcal{M}ul$ as equational logic spec

$(\mathcal{T}(\Sigma), =)$ is not a relational model of equational logic spec $\mathcal{M}ul$

3.4. Sub-validity examples

$(\mathcal{N}at, =)$ is a relational model of equational logic spec $\mathcal{M}ul$

$(\mathcal{T}(\Sigma), \leftrightarrow^*)$ is a relational model of equational logic spec $\mathcal{M}ul$

$(\mathcal{T}(\Sigma), \rightarrow^*)$ is a relational model of rewriting logic spec $\mathcal{M}ul$ but not of $\mathcal{M}ul$ as equational logic spec

$(\mathcal{T}(\Sigma), =)$ is not a relational model of equational logic spec $\mathcal{M}ul$

model: relational model no non-trivial congruences

3.5. Sub-derivability

(s,t) derivable from $\mathcal{S}$

$$\frac{(s,t)\in S}{\sigma(s)\ \underline{\mathcal{S}}\ \sigma(t)}\ (\mathrm{emb}) \qquad \frac{s_1,\ldots,s_n =[\underline{\mathcal{S}}]\ t_1,\ldots,t_n}{f(s_1,\ldots,s_n)\ \underline{\mathcal{S}}\ f(t_1,\ldots,t_n)}\ (\mathrm{comp})$$

$$\frac{}{s\ \underline{\mathcal{S}}\ s}\ (\mathrm{ref}) \qquad \frac{s\ \underline{\mathcal{S}}\ t}{t\ \underline{\mathcal{S}}\ s}\ (\mathrm{sym}) \qquad \frac{s\ \underline{\mathcal{S}}\ t \quad t\ \underline{\mathcal{S}}\ u}{s\ \underline{\mathcal{S}}\ u}\ (\mathrm{trans})$$

Rule only if allowed inference mode

Note: no congruence, no substitution

Home Page

Title Page

Contents

◀◀ ▶▶

◀ ▶

Go Back

Full Screen

Close

Quit

3.5. Sub-derivability

(s,t) derivable from $\mathcal{S}$

$$\frac{(s,t)\in S}{\sigma(s)\ \underline{\mathcal{S}}\ \sigma(t)}\ (\mathrm{emb}) \qquad \frac{s_1,\ldots,s_n =[\underline{\mathcal{S}}]\ t_1,\ldots,t_n}{f(s_1,\ldots,s_n)\ \underline{\mathcal{S}}\ f(t_1,\ldots,t_n)}\ (\mathrm{comp})$$

$$\frac{}{s\ \underline{\mathcal{S}}\ s}\ (\mathrm{ref}) \qquad \frac{s\ \underline{\mathcal{S}}\ t}{t\ \underline{\mathcal{S}}\ s}\ (\mathrm{sym}) \qquad \frac{s\ \underline{\mathcal{S}}\ t \quad t\ \underline{\mathcal{S}}\ u}{s\ \underline{\mathcal{S}}\ u}\ (\mathrm{trans})$$

Rule only if allowed inference mode

Note: no congruence, no substitution

$$\frac{\dfrac{}{(\mathtt{M}(\mathtt{S}x,\mathtt{S0}),\mathtt{A}(\mathtt{S}x,\mathtt{M}(\mathtt{S}x,\mathtt{0})))}\ (\sigma) \qquad \dfrac{\dfrac{\dfrac{}{(\mathtt{M}(\mathtt{S}x,\mathtt{0}),\mathtt{0})}\ (\sigma)}{(\mathtt{A}(\mathtt{S}x,\mathtt{M}(\mathtt{S}x,\mathtt{0})),\mathtt{A}(\mathtt{S}x,\mathtt{0}))}\ (\mathrm{comp},\ \mathtt{A}) \qquad \dfrac{}{(\mathtt{A}(\mathtt{S}x,\mathtt{0}),\mathtt{S}x)}\ (\sigma)}{(\mathtt{A}(\mathtt{S}x,\mathtt{M}(\mathtt{S}x,\mathtt{0})),\mathtt{S}x)}\ (\mathrm{trans})}{(\mathtt{M}(\mathtt{S}x,\mathtt{S0}),\mathtt{S}x)}\ (\mathrm{trans})$$

derivation for equational/rewriting logic/strict spec $\mathcal{M}ul$

3.6. Sub-convertibility

The sub-convertibility relation obtained as closure under inference modes in order

$$(\mathrm{emb}), (\mathrm{comp}), (\mathrm{ref}), (\mathrm{sym}), (\mathrm{trans})$$

3.6. Sub-convertibility

The sub-convertibility relation obtained as closure under inference modes in order

$$(\mathrm{emb}), (\mathrm{comp}), (\mathrm{ref}), (\mathrm{sym}), (\mathrm{trans})$$

for equational specification
sub-convertibility is convertibility $\leftrightarrow^*_{\mathcal{M}ul}$

for rewriting logic specification
sub-convertibility is rewritability/reachability $\longrightarrow^*_{\mathcal{M}ul}$

for strict specification
sub-convertibility is strict reachability $\longrightarrow^+_{\mathcal{M}ul}$

3.7. Proofs for sub-equational specifications

$$\text{sub-valid} \Leftrightarrow_{\text{Sub-Birkhoff}} \text{sub-derivable} \Leftrightarrow_{\text{Sub-logicality}} \text{sub-convertible}$$

3.7. Proofs for sub-equational specifications

sub-valid $\Leftrightarrow_{\text{Sub-Birkhoff}}$ sub-derivable $\Leftrightarrow_{\text{Sub-logicality}}$ sub-convertible

- Soundness of sub-Birkhoff by induction on derivations
- Completeness of sub-Birkhoff by relational term model followed by quotient construction
- Soundness of sub-logicality by simulation
- Completeness of sub-logicality by derivation standardisation

3.8. Sub-Birkhoff soundness

Thm 8 *sub-valid* $\Leftarrow$ *sub-derivable*

Proof
by induction on derivations

3.8. Sub-Birkhoff soundness

Thm 8 *sub-valid* $\Leftarrow$ *sub-derivable*

Proof
by induction on derivations

All inference rules trivially preserve validity . . .

3.8. Sub-Birkhoff soundness

Thm 8 *sub-valid* $\Leftarrow$ *sub-derivable*

Proof
by induction on derivations

All inference rules trivially preserve validity ...

$$\frac{(s,t) \in S}{\sigma(s)\ \underline{S}\ \sigma(t)}\ (\mathrm{emb})$$

needs semantic substitution lemma

3.9. Sub-Birkhoff completeness

Thm 9 *sub-valid* $\Rightarrow$ *sub-derivable*

Proof

Sub-derivability 'is' a relational model

Quotiented sub-derivability 'is' a model

3.9. Sub-Birkhoff completeness

Thm 9 *sub-valid* $\Rightarrow$ *sub-derivable*

Proof

Sub-derivability ‘is’ a relational model

Quotiented sub-derivability ‘is’ a model

Relational term model:

Term algebra $\mathcal{T}(\Sigma)$ paired up with derivable equality

Note: this is a relational model

3.9. Sub-Birkhoff completeness

Thm 9 *sub-valid* $\Rightarrow$ *sub-derivable*

Proof
Sub-derivability 'is' a relational model
Quotiented sub-derivability 'is' a model
Relational term model:
Term algebra $\mathcal{T}(\Sigma)$ paired up with derivable equality
Note: this is a relational model

Quotienting out maximal congruence yields model
all derivable equalities hold by induction on derivation

$$\frac{(s,t) \in S}{\sigma(s)\ \underline{S}\ \sigma(t)}$$

needs syntactic substitution lemma

3.10. Logicality soundness

Thm 10 *sub-derivable* $\Leftarrow$ *sub-convertible*

Proof

sub-conversion is a sub-derivation

3.11. Logicality completeness

Thm 11 *derivable* $\Rightarrow$ *convertible*

Proof

Derivation standardises to conversion

3.11. Logicality completeness

Thm 11 *derivable* $\Rightarrow$ *convertible*

Proof
Derivation standardises to conversion
standardisation: commute derivations in wrong order

	(emb)	(comp)	(ref)	(sym)	(trans)
(emb)	x	mon	mon	mon	mon
(comp)	x	x	(ref)	(sym)	(trans)
(ref)	x	x	x	mon	mon
(sym)	x	x	x	x	(trans)
(trans)	x	x	x	x	x

Vertically: property to be preserved under closing
Horizontally: with respect to indicated inference rule
Note: chosen order is important

4. Conclusion

Parametrised sound- and completeness results

4. Conclusion

Parametrised sound- and completeness results

Attend RTA 2004

Regisration is open now