Modularity of Confluence

Constructed

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modularity of confluence

constructing confluence

proof idea

- ranking the terms
- balancing the terms
- constructing confluence by decreasing diagrams

Modularity of confluence

proving property of rewrite system via its components

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proving property of rewrite system via its components

Definition

property P is modular if $P(\mathcal{T}_1 \uplus \mathcal{T}_2) \iff P(\mathcal{T}_1)\ \&\ P(\mathcal{T}_2)$

Modularity of confluence

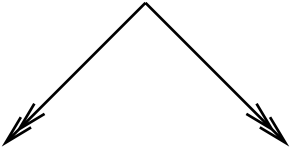
proving confluence of rewrite system via its components

Definition

confluence is modular if $\mathrm{Con}(\mathcal{T}_1 \uplus \mathcal{T}_2) \iff \mathrm{Con}(\mathcal{T}_1)\ \&\ \mathrm{Con}(\mathcal{T}_2)$

Modularity of confluence

for each pair of reductions with common source



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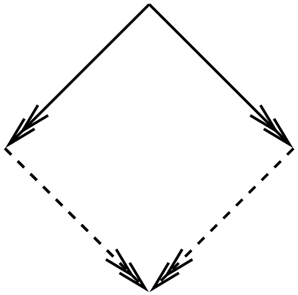

exists pair of reductions with common reduct

Modularity of confluence

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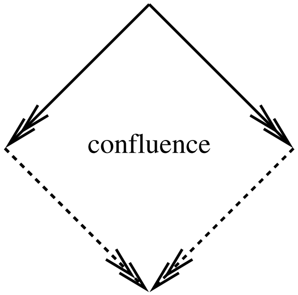

exists pair of reductions with common reduct

Consequences:

- consistency (if distinct normal forms)
- uniqueness of results (normal forms)
- decidable convertibility (if $\rightarrow$ terminating)
- existence of lower bounds (w.r.t. $\twoheadrightarrow$ order)

Confluence is not modular

Example (Klop 1980)

$$(\lambda x.M(x))N \quad \to \quad M(N)$$

$$\uplus$$

$$e(x,x) \quad \to \quad \top$$

disjoint confluent components

Confluence is not modular

Example (Klop 1980)

$$\begin{array}{rcl} (\lambda x.M(x))N & \to & M(N) \\ e(x,x) & \to & \top \end{array}$$

disjoint union of components not confluent

Confluence is not modular

Example (Klop 1980)

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disjoint union of components not confluent

$$\begin{array}{ccc} A & \twoheadrightarrow & CA \\ \downarrow & & \downarrow \\ CA & & C(CA) \\ \downarrow & & \downarrow \\ e(A,CA) & & Ce(A,CA) \\ \downarrow & & \downarrow \\ e(CA,CA) & & Ce(CA,CA) \\ \downarrow & & \downarrow \\ \top & \downarrow ?? & C\top \end{array}$$

where $C =^{def} Y(\lambda ca.e(a,ca))$ $\quad A =^{def} YC$

Confluence is modular

Theorem (Toyama 1987)

confluence is modular for TRSs

Confluence is modular

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confluence is modular for TRSs

Example

$$
\begin{array}{rcl}
@(I(), x) & \to & x \\
@(@(K(), x), y) & \to & x \\
@(@(@(S(), x), y), z) & \to & @(@(x, z), @(y, z)) \\
*(x, x) & \to & x \\
a() & \to & b()
\end{array}
$$

confluent?

Confluence is modular

Theorem (Toyama 1987)

confluence is modular for TRSs

Example

$$\begin{array}{rcl} Ix & \rightarrow & x \\ Kxy & \rightarrow & x \\ Sxyz & \rightarrow & xz(yz) \\ x * x & \rightarrow & x \\ a & \rightarrow & b \end{array}$$

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$\mathcal{CL} \uplus \mathcal{E}$ confluent?

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$\mathcal{CL}$ orthogonal $\Rightarrow$ confluent (Rosen)

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$\mathcal{E}$ terminating and no critical pairs $\Rightarrow$ confluent (Huet)

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disjoint confluent components $\Rightarrow$ union confluent (Toyama)

Can common reduct be constructed ?

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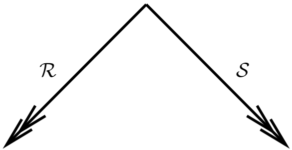
exist constructions f, g such that
for each pair $\mathcal{R}$, $\mathcal{S}$ of reductions with common source



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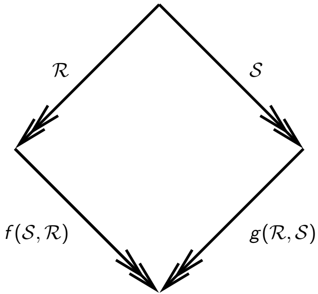

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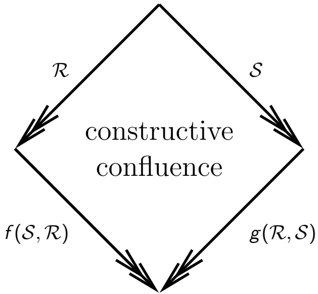

$f(\mathcal{S},\mathcal{R})$, $g(\mathcal{R},\mathcal{S})$ is pair of reductions with common reduct
Extra consequence:

- computation of lower bounds (w.r.t. $\twoheadrightarrow$ order)

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$$Ix \to x \qquad Kxy \to x \qquad Sxyz \to xz(yz)$$

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construction for $\mathcal{CL} \uplus \mathcal{E}$?

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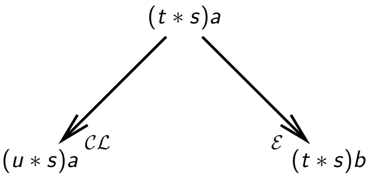
construction for $\mathcal{CL} \uplus \mathcal{E}$? extract from Toyama?

Can common reduct be constructed modularly ?

Example

for $\mathcal{CL}$-terms t,s,u with $t \rightarrow_{\mathcal{CL}} u$



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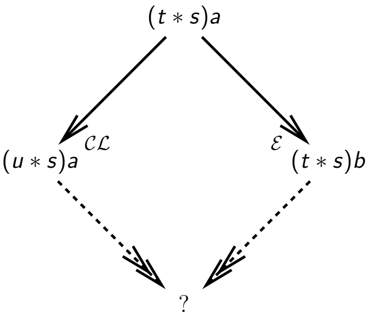

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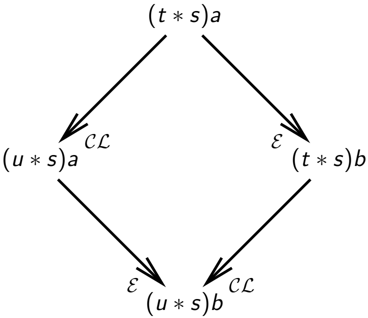

expected. . .

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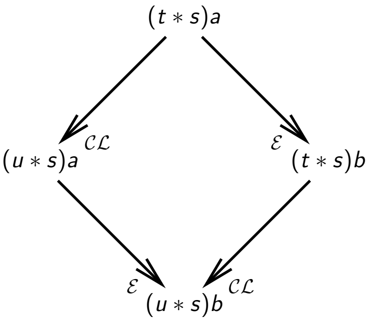

expected... not given by proofs (Toyama, Klop et al., Jouannaud)

Can common reduct be constructed modularly ?

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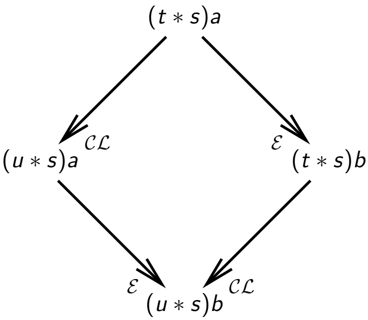

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Can common reduct be constructed modularly ?

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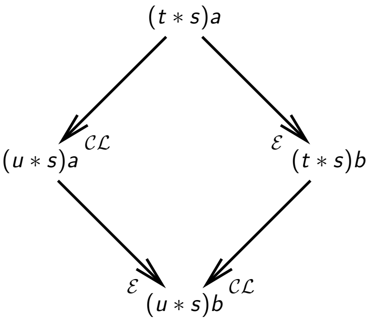

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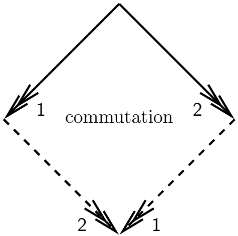
rely on test: can $t * s$ collapse?
but undecidable whether t, s have common reduct

Proof by commutation?

Lemma (Hindley–Rosen)

$\rightarrow_1 \cup \rightarrow_2$ *confluent,*
if $\rightarrow_i$ *are, and commute:*



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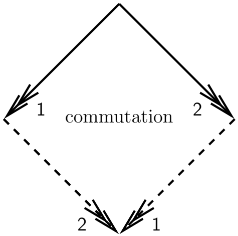
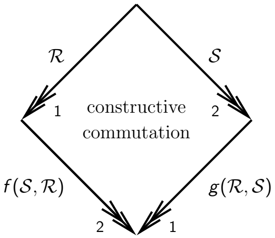


hence would suffice to show commutation ...

Proof by commutation?

Lemma (constructive Hindley–Rosen)

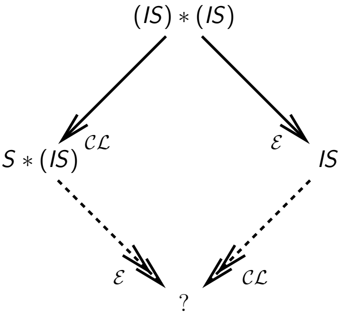
$\rightarrow_1 \cup \rightarrow_2$ constructively confluent,
if $\rightarrow_i$ are, and commute constructively:



hence would suffice to show constructive commutation . . .

Proof by commutation fails

Example



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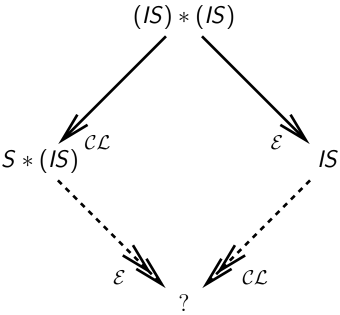

impossible because of non-left-linearity of rule $x * x \to x$

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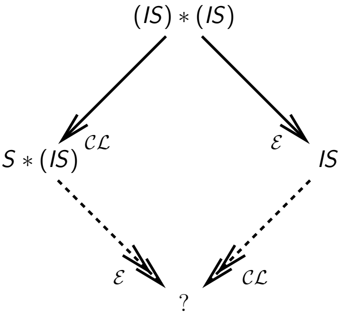
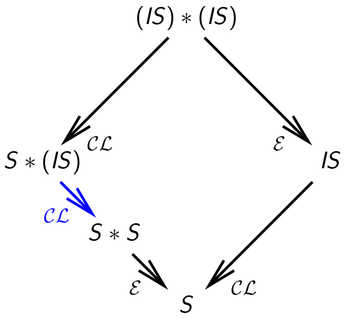


impossible because of non-left-linearity of rule $x * x \rightarrow x$
$S * (IS)$ needs to be balanced first

Proof by commutation fails

Example



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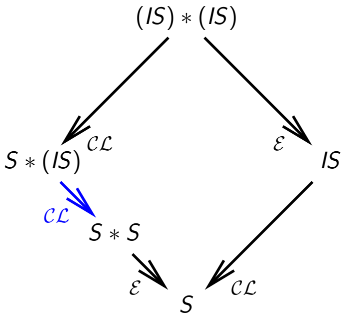
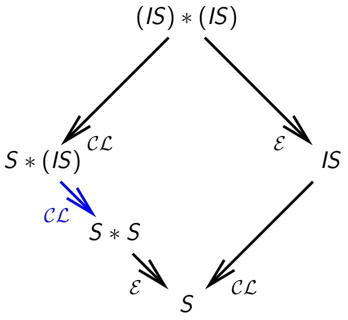


commute 'up to' balancing $\rightarrow_{\mathcal{CL}}$-step which is smaller:

Proof by commutation fails

Example



commute 'up to' balancing $\rightarrow_{\mathcal{CL}}$-step which is smaller:

- $\mathcal{CL}$-term rewritten has lower rank than whole term

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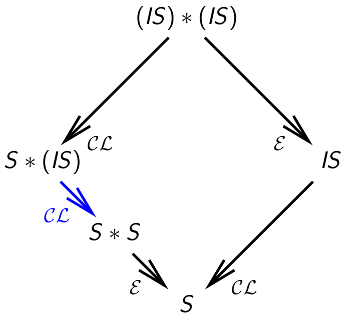

commute 'up to' balancing $\rightarrow_{\mathcal{CL}}$-step which is smaller:

- $\mathcal{CL}$-term rewritten has lower rank than whole term
- step $S * (IS) \rightarrow_{\mathcal{CL}} S * S$ decreases imbalance of whole term

Proof idea

proof by induction on rank
per rank: proof by induction on imbalance

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proof by induction on rank
per rank: proof by induction on imbalance

Example (Running)

$\mathcal{T}_1$ over alphabet $\{a, f\}$ (small)

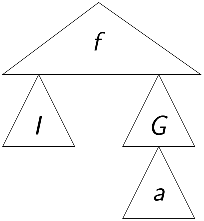
$$f(x,x) \rightarrow x$$

$\mathcal{T}_2$ over alphabet $\{I, J, K, G, H\}$ (caps)

$$\begin{aligned} G(x) &\rightarrow I \\ I &\rightarrow K \\ G(x) &\rightarrow H(x) \\ H(x) &\rightarrow J \\ J &\rightarrow K \end{aligned}$$

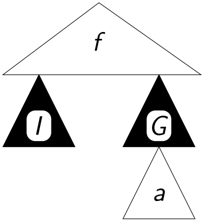
Proof by induction on rank: ranking the terms

$f(I, G(a))$ first stratified into layers:



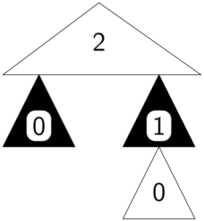
Proof by induction on rank: ranking the terms

rank of $f(I, G(a))$ is #alternations of layers:



Proof by induction on rank: ranking the terms

$f(I, G(a))$ has rank 2:



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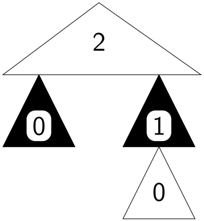

Fact

rank does not increase along rewriting in TRSs

Proof by induction on rank

Theorem

for every rank, reductions from terms up to that rank are constructively confluent, if components are

Proof.

□

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- base case 0: peak entirely within one TRS
  use constructive confluence on components...
  (standard)

□

Proof by induction on rank

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for every rank, reductions from terms up to that rank are constructively confluent, if components are

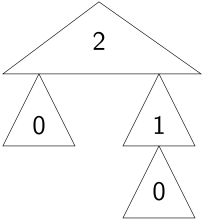
Proof.

- base case 0: peak entirely within one TRS
  use constructive confluence on components...
  (standard)
- step case $r + 1$: by induction on imbalance ...
  (novel)

□

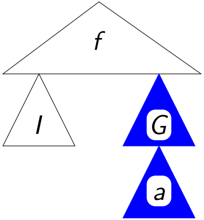
Proof by induction on imbalance: term decomposition

$f(I, G(a))$ for rank 2: first find tall aliens (aliens of rank 1):



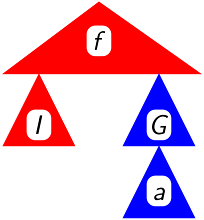
Proof by induction on imbalance: term decomposition

$f(I, G(a))$ for rank 2: next base is context of tall aliens:



Proof by induction on imbalance: term decomposition

$f(I, [G(a)])$ for rank 2: base–tall alien decomposition



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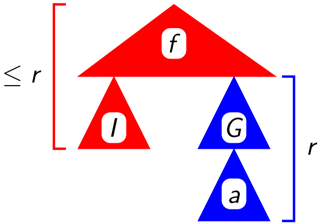

Fact

term decomposes uniquely into base and vector of tall aliens both with rank up to r, so both constructively confluent by IH

Definition

imbalance of term is #tall aliens (as set)

Proof by induction on imbalance: step decomposition

classify steps according to location of redex-pattern:

- base-step $\blacktriangleright$: redex-pattern in base
- tall alien-step $\triangleright$: redex-pattern in tall alien

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$f(I, G(a)) \to f(I, I) \to f(I, K)$

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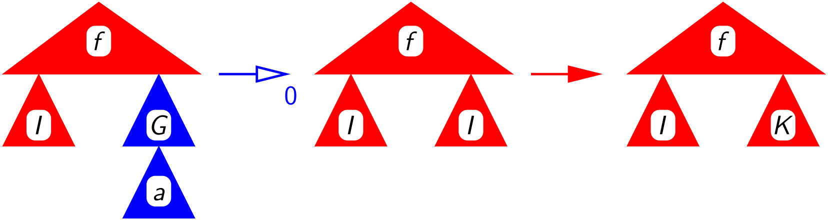

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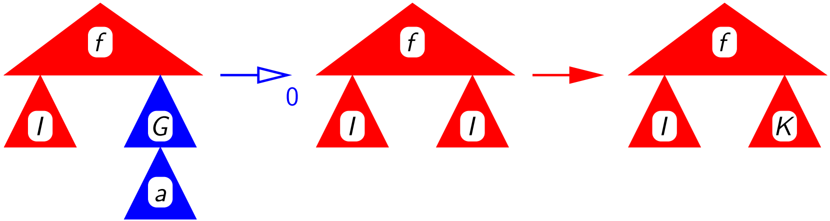

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every redex-pattern either base or tall alien (by disjointness)

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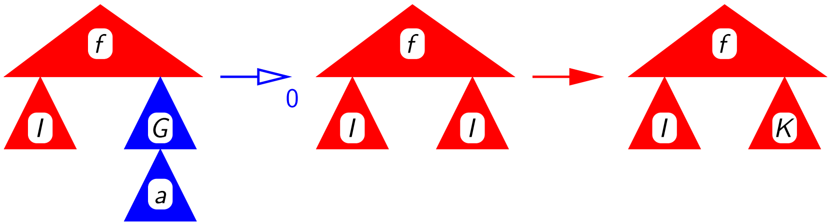

Fact

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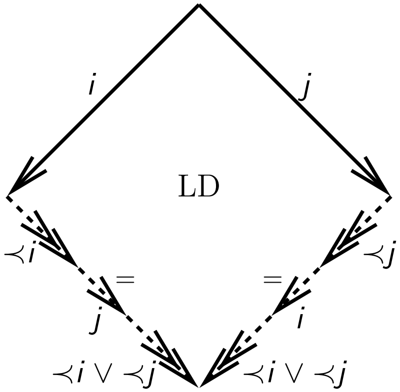
- base-reduction ▶▶: ▶-steps, ends when collapsed to tall alien
- tall alien-reduction ▷▷: ▷-steps, labelled with imbalance target

Constructive confluence by decreasing diagrams

Theorem (de Bruijn 1978,vO 1994)

$\rightarrow$ *confluent,*
if $\rightarrow = \bigcup_{i \in I} \rightarrow_i$, $\prec$ *well-founded order on* I, *such that:*



Constructive confluence by decreasing diagrams

Theorem (DD special case needed here)

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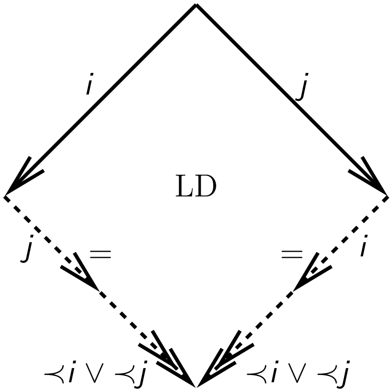
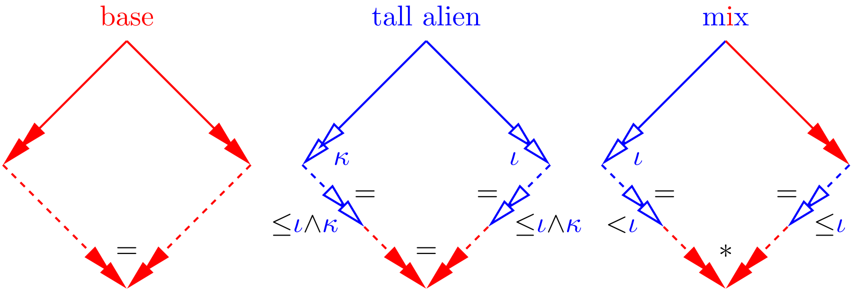


Constructive confluence by decreasing diagrams: by cases on base–tall alien decomposition



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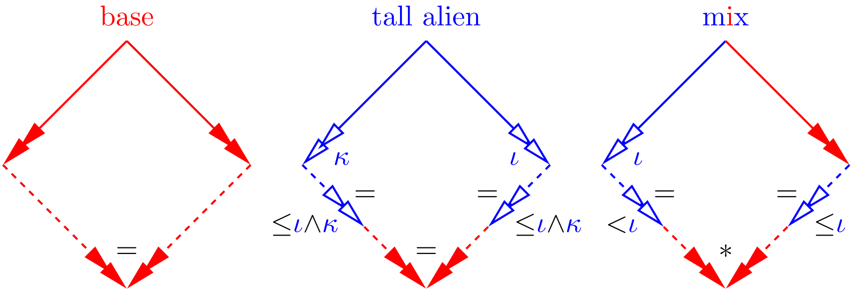


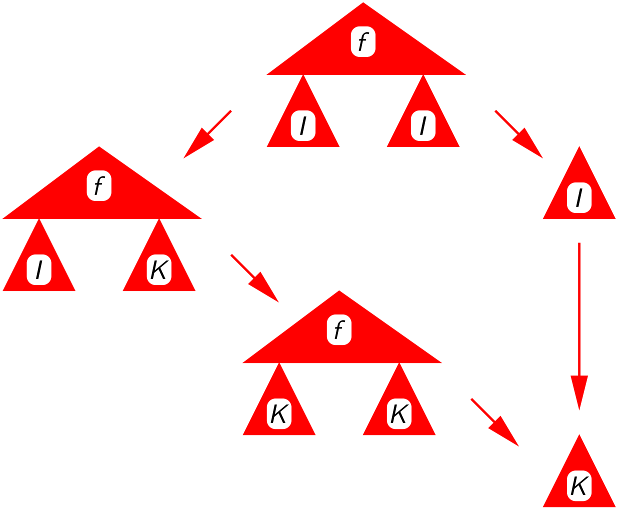
Theorem

each case is decreasing

Proof.

Set $\twoheadrightarrow < \rhd\!\!\rhd_\iota < \rhd\!\!\rhd_\kappa$, for $\iota < \kappa$. $\square$

Decreasing diagram: base case



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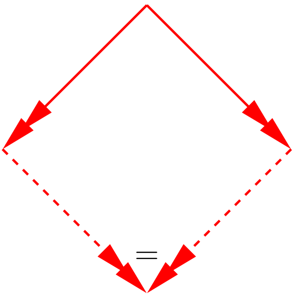

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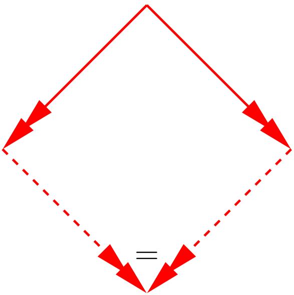

- base reduction confluent by induction hypothesis (bases of rank up to r)

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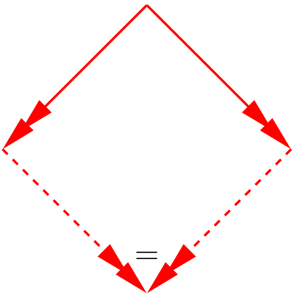

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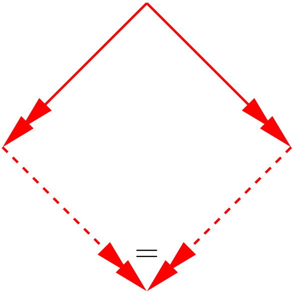
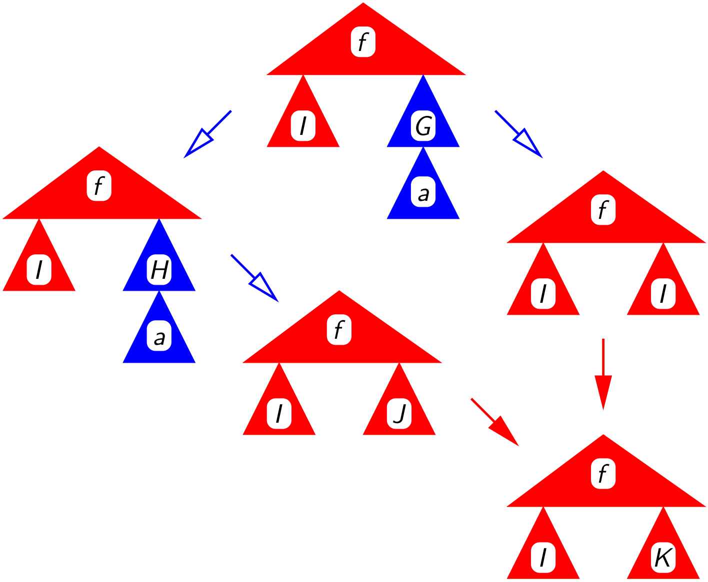


- base reduction confluent by induction hypothesis (bases of rank up to r)
- cannot create new tall aliens (only replicate existing ones)
- may collapse to tall alien (then results in base term)

Decreasing diagram: tall alien case



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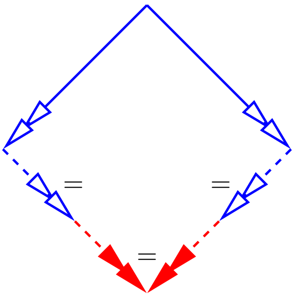

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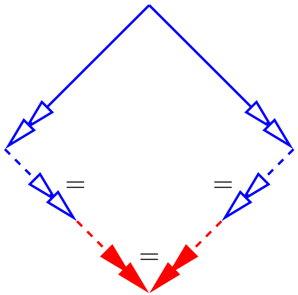

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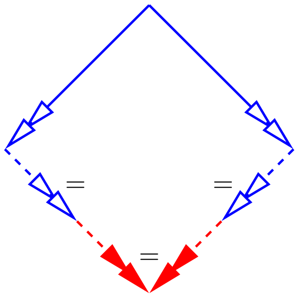

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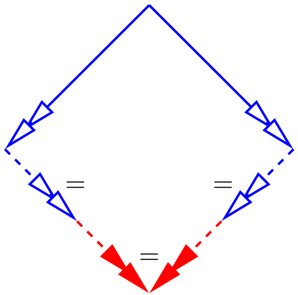

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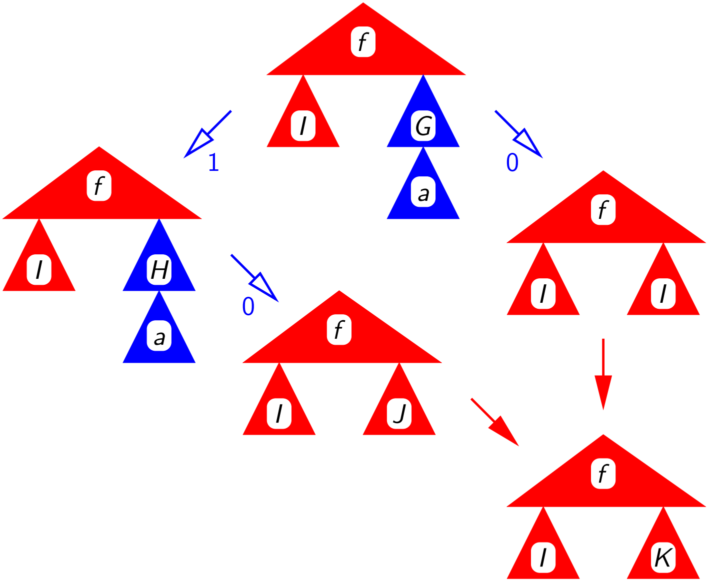

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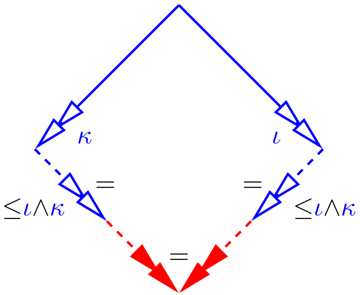

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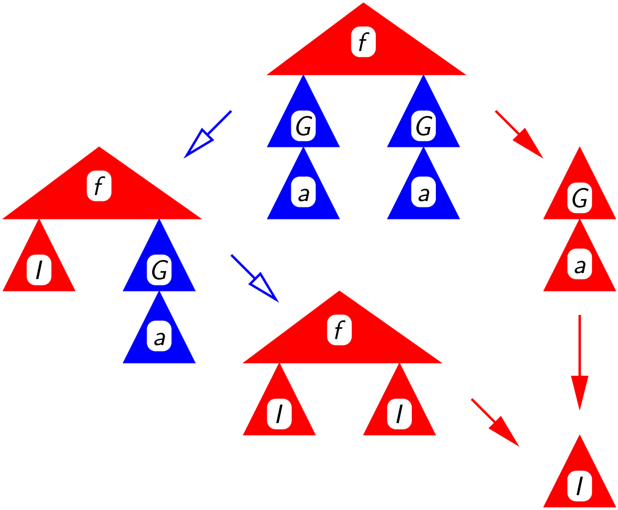
Decreasing diagram: mix case



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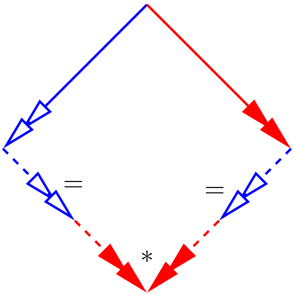

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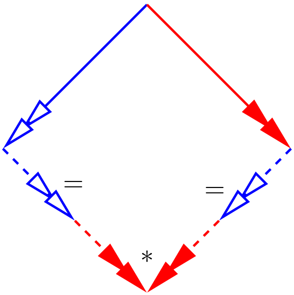

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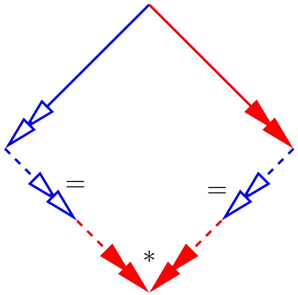

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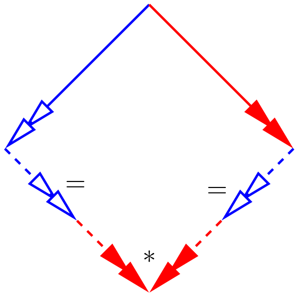

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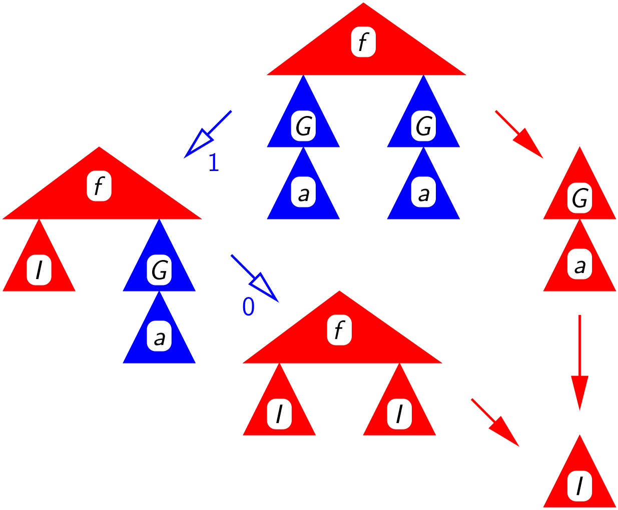

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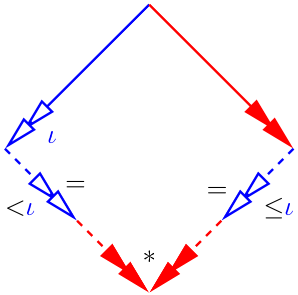

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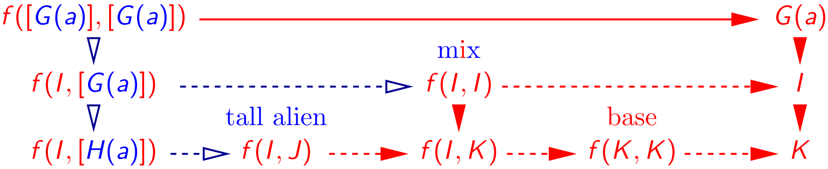
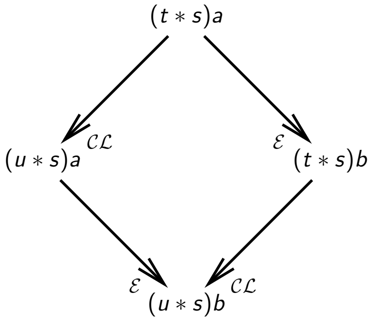
Example displaying all three cases



confluence constructed by tiling!

Motivating example is trivial

for $\mathcal{CL}$-terms t,s,u with $t \to_{\mathcal{CL}} u$



expected

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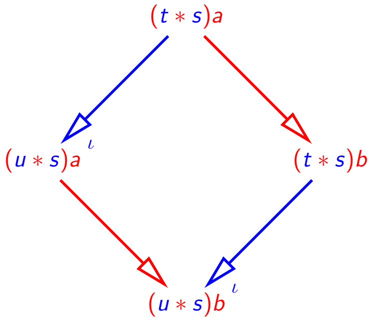

yes, in case $t \neq s$ or $t = s = u$

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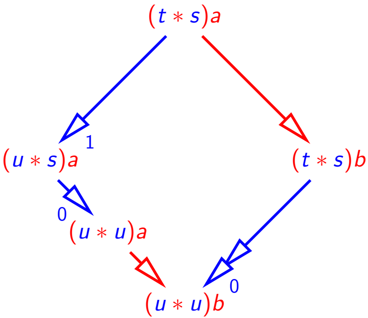

in case $t = s \neq u$, balancing is performed

Extensions

Theorem

constructive confluence is modular when sharing constructors, if opaque: no constructor lifting, no collapse

Proof.

reduction to modularity by combining non-shared-constructors with all shared constructors below them. □

Extensions

extra-variable TRSs: confluence not preserved under decomposition

Example

$$
\begin{array}{rclrcl}
 & \mathcal{T}_1 & & & \mathcal{T}_2 & \\
f(x,y) & \to & f(z,z) & m(y,x,x) & \to & y \\
f(b,c) & \to & a & m(x,x,y) & \to & y \\
b & \to & d & & & \\
c & \to & d & & &
\end{array}
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$\mathcal{T}_1 \uplus \mathcal{T}_2$ confluent, $\mathcal{T}_1$ not: $a \leftarrow f(b,c) \to f(z,z)$

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