Abstract Rewriting

ISR 2008, Obergurgl, Austria

Vincent van Oostrom

Theoretical Philosophy
Utrecht University
Netherlands

16:00 – 17:30, Mon/Wednesday July 21, ISR 2008

Introduction

Motivation

various concrete rewrite systems

string rewrite systems

graph rewrite systems

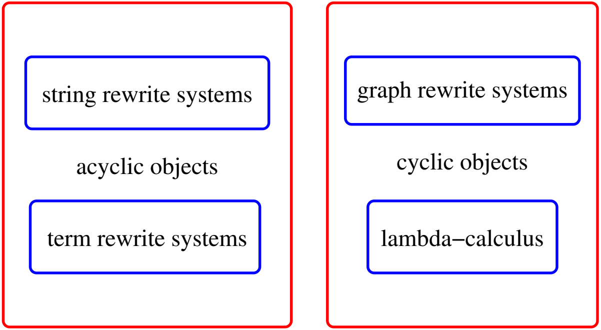
term rewrite systems

lambda–calculus

how to prove properties uniformly?

Motivation

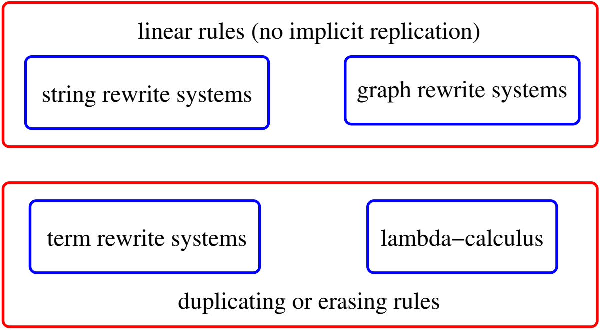
objects structured in different ways



how to prove properties uniformly?

Motivation

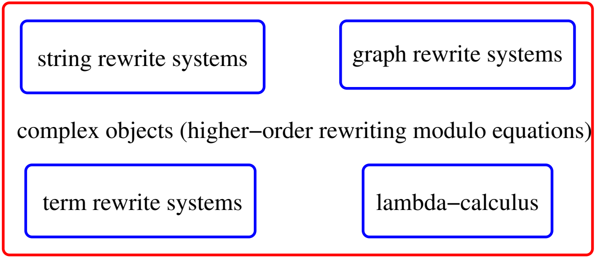
steps manipulate objects in different ways



how to prove properties uniformly?

More complex or more simple?

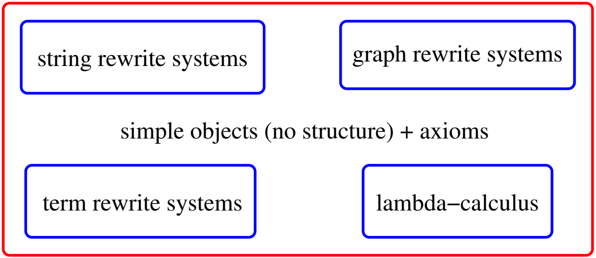
one complex format to rule them all?



never complex enough, complexity needs learning

More complex or more simple?

one simple format to rule them all?



different axioms, simplicity needs imagination/axiom checking

Abstract rewriting

- Newman 1942 (confluence, orthogonality)
- Hindley, Rosen, de Bruijn (orthogonality, commutation)
- Klop, Huet, Geser (abstract reduction as framework)
- Jouannaud/Kirchner, Ohlebusch (rewriting modulo)
- Melliès, Khasidashvili (standardisation, neededness)
- Ghani/Lüth (substitution)
- ...

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Abstract rewriting

On Theories with a Combinatorial Definition of "Equivalence"

M. H. A. Newman

The Annals of Mathematics, 2nd Ser., Vol. 43, No. 2. (Apr., 1942), pp. 223-243.

Stable URL:
http://links.jstor.org/sici?sici=0003-486X%28194204%292%3A43%3A2%3C223%3AOTWACD%3E2.0.CO%3B2-C

The Annals of Mathematics is currently published by Annals of Mathematics.

- abstract study of confluence (examples from mathematics)
- abstract study of orthogonality (application to λ-calculus)

Abstract rewriting

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- confluence $\Rightarrow$ uniqueness of normal forms
- confluence $\Rightarrow$ consistency (Church–Rosser)
- confuence $\Rightarrow$ decidable convertibility, if $\rightarrow$ is terminating

Standard notions

Newman	modern	notations I use
cell	step	$\to$
path	conversion	$\leftrightarrow^*$
descending path	reduction/rewriting seq.	$\twoheadrightarrow$
lower bound	common reduct	$\downarrow$
upper bound	common ancestor	$\uparrow$
property A	Church–Rosser property	$\leftrightarrow^* \subseteq \twoheadrightarrow \cdot \twoheadleftarrow$
property B	confluence property	$\twoheadleftarrow \cdot \twoheadrightarrow \subseteq \twoheadrightarrow \cdot \twoheadleftarrow$ $(\uparrow \subseteq \downarrow)$
property C	semi-confluence	$\leftarrow \cdot \twoheadrightarrow \subseteq \twoheadrightarrow \cdot \twoheadleftarrow$
property D	local confluence	$\leftarrow \cdot \to \subseteq \twoheadrightarrow \cdot \twoheadleftarrow$
derivate	residual	$/$
conversion calc.	λ-calculus	

Plan

- Monday
- formalism: abstract rewrite relations (whether, Terese Ch. 1)
- A set of objects
- $\rightarrow \subseteq A \times A$ rewrite relation on A
- confluence property, lower bounds
- proof method: decreasing diagrams (Terese Ch. 14)
- proof method: Z property

Plan

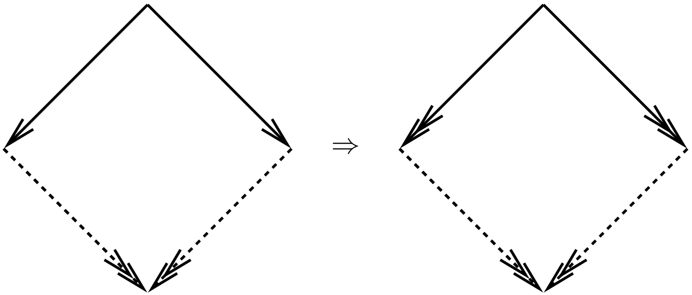
- Wednesday
- formalism: abstract rewrite systems (how, Terese Ch. 8)
- A set of objects
- $\to$ set of rewrite steps with source/target maps
- orthogonality, greatest lower bounds
- axiomatisation: residual systems (Terese Ch. 8.7)
- proof method: confluification into multi-steps

Confluence by decreasing diagrams

Newman's Lemma

Theorem (Newman 1942)

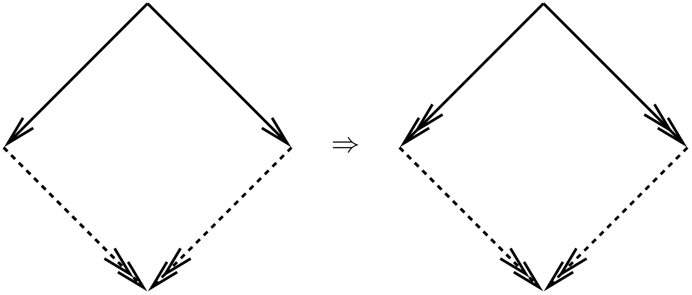
THEOREM 3. *In an indexed complex in which all descending paths are finite, (D) implies (A).*



Newman's Lemma

Theorem (Newman 1942)

local confluence implies confluence, if $\longrightarrow$ *terminating*



Newman's Lemma

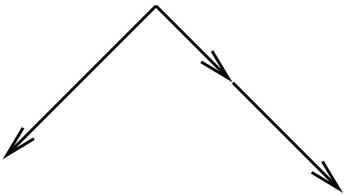
various proofs in literature:

- Newman: see my homepage for modern rendering
- Huet: short proof by Noetherian induction on $\to^+$
- Klop: proving absence of ambiguous points
- ...
- here: by analysing tiling of Kleene counterexample

Newman's Lemma

Proof.

non-terminating Kleene system: $b \leftarrow a \leftrightarrow a' \rightarrow c$



if repeated tiling terminates then confluent. . . $\square$

Newman's Lemma

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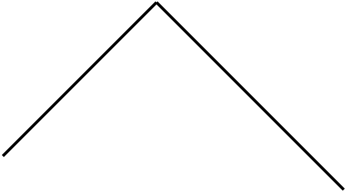

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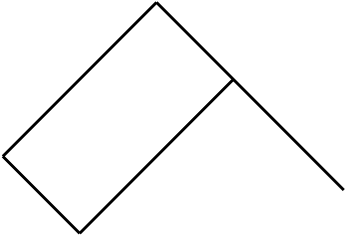

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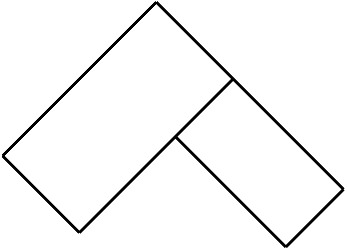

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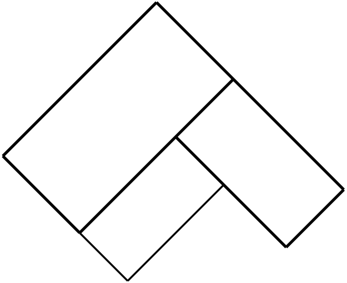

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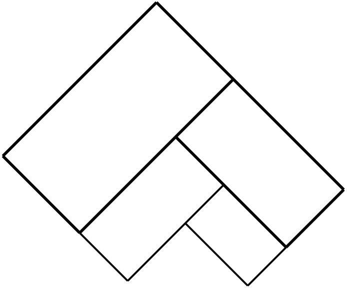

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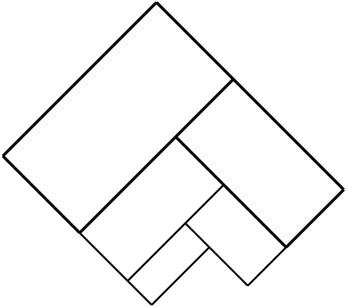

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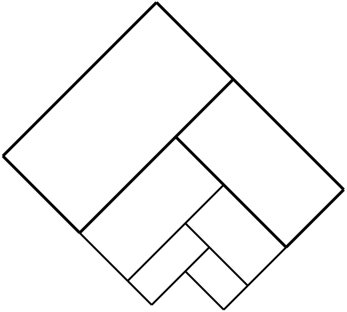

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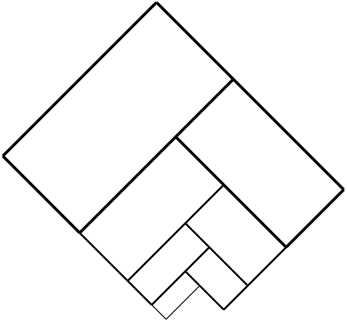

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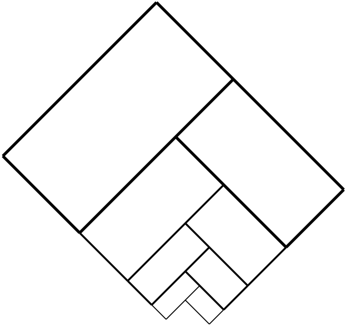

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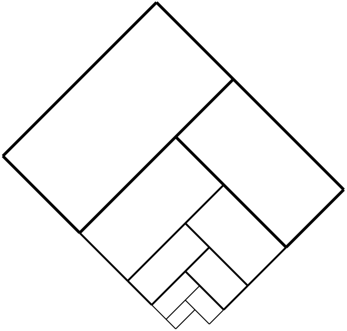

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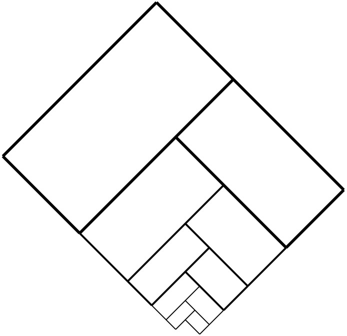

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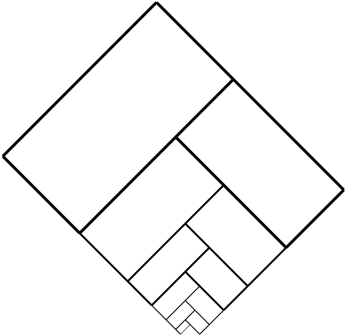

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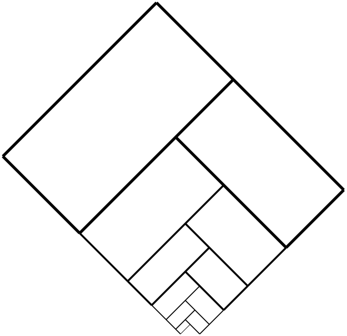

. . . but if it does not, then infinite reduction? $\square$

Newman's Lemma

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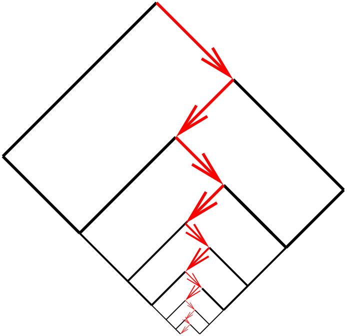


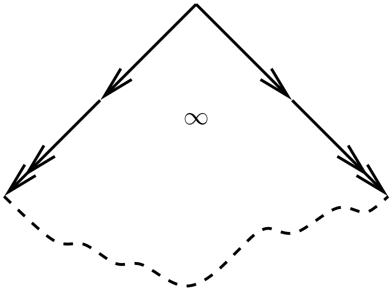
...but if it does not, then infinite reduction! □

Newman's Lemma

Proof.

Escher diagram: finite peak, infinite filled diagram

try to construct an infinite reduction in Escher diagram



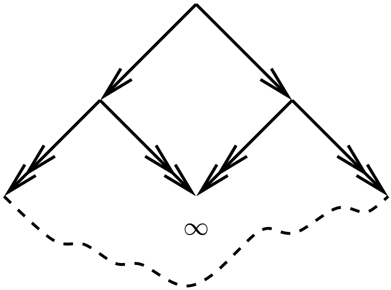
□

Newman's Lemma

Proof.

Escher diagram: finite peak, infinite filled diagram

if infinite, there's a filled local peak



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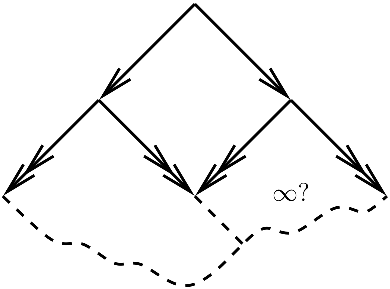
Newman's Lemma

Proof.

Escher diagram: finite peak, infinite filled diagram

is right sub-diagram Escher?



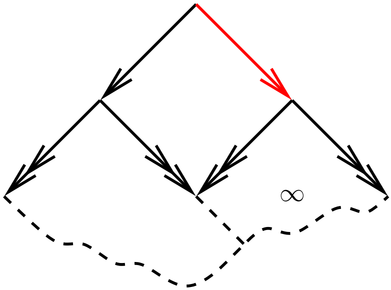
□

Newman's Lemma

Proof.

Escher diagram: finite peak, infinite filled diagram

if right sub-diagram Escher, go right



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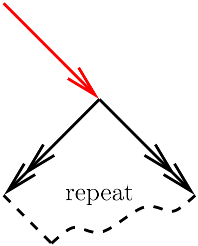

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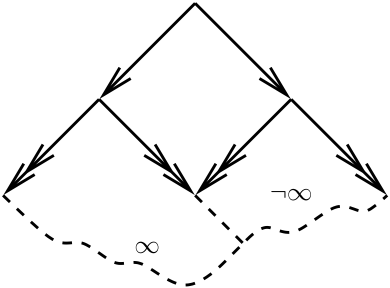
Newman's Lemma

Proof.

Escher diagram: finite peak, infinite filled diagram

if right sub-diagram not Escher, left sub-diagram is



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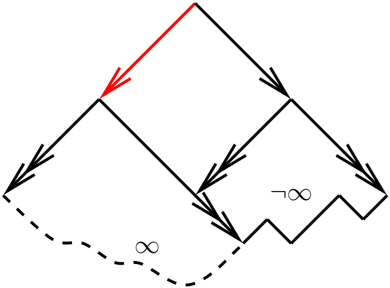
Newman's Lemma

Proof.

Escher diagram: finite peak, infinite filled diagram

if right sub-diagram not Escher, go left



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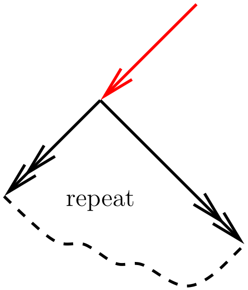

Diamond property

Theorem (Newman 1942)

THEOREM 1. *Let* Σ *be such that if* $a\mu x$ *and* $a\mu y$, *and* $x \neq y$, *there exists* b *such that* $x\mu b$ *and* $y\mu b$. *Then property* (A) *holds:*

Proof.

Diamond property

Theorem (Newman 1942)

diamond property ($\leftarrow \cdot \rightarrow \subseteq \rightarrow \cdot \leftarrow$) *implies confluence*

Proof.

$\square$

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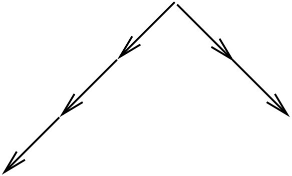

repeat: fill in diamonds □

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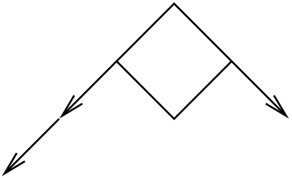

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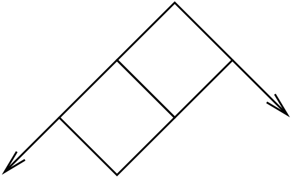

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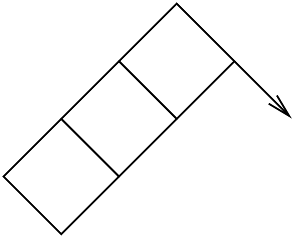

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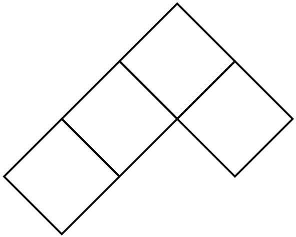

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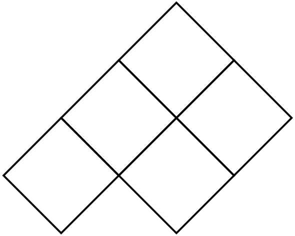

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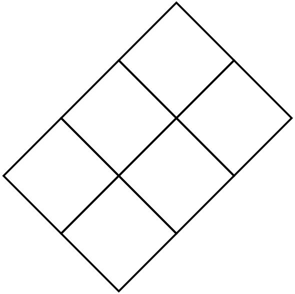


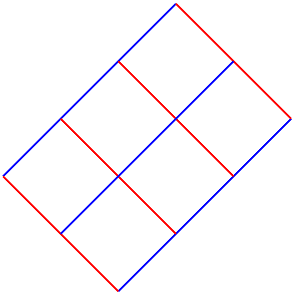
must stop: area to fill becomes smaller $\square$

From confluence to commutation

Theorem

$\triangleleft \cdot \blacktriangleright \subseteq \blacktriangleright \cdot \triangleleft$ *implies* $\triangleleft\!\triangleleft \cdot \blacktriangleright\!\blacktriangleright \subseteq \blacktriangleright\!\blacktriangleright \cdot \triangleleft\!\triangleleft$

Proof.



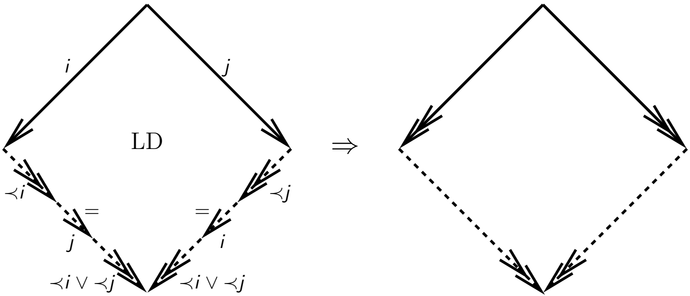
$\square$

One method to rule them all (Newman,diamond)?

Decreasing Diagrams

Theorem (de Bruijn 1978,vO 1994)

locally decreasing implies confluence

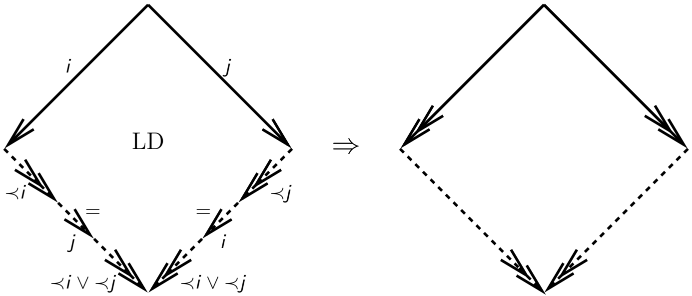


$\rightarrow = \bigcup_{i \in I} \rightarrow_i$, $\prec$ *well-founded order on* I

Decreasing Diagrams

Theorem (de Bruijn 1978,vO 1994)

locally decreasing implies confluence



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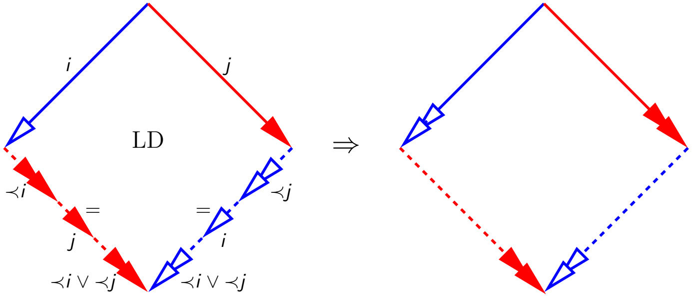
Decreasing Diagrams

Theorem (vO 1994)

locally decreasing implies commutation

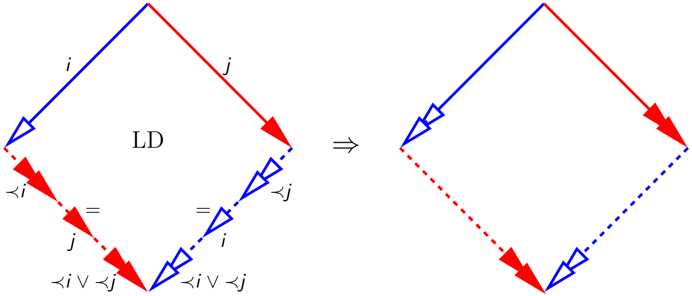


$\triangleright = \bigcup_{i \in I} \triangleright_i$, $\blacktriangleright = \bigcup_{j \in J} \blacktriangleright_j$, $\prec$ *well-founded order on* $I \cup J$

Decreasing Diagrams

Theorem (vO 1994)

locally decreasing implies commutation



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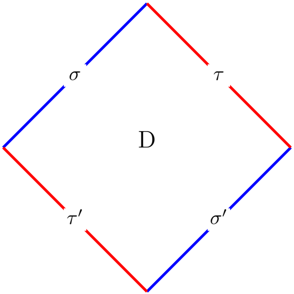
Decreasing Diagrams

Proof.

by decreasingness



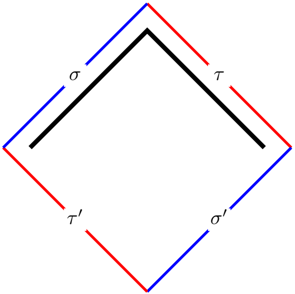
peak σ,τ as large as lhs $\sigma\tau'$ and rhs $\tau\sigma'$ after filtering

□

Decreasing Diagrams

Proof.

by decreasingness



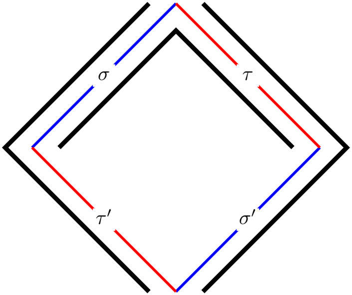
measure peak by multiset sum $|\sigma| \uplus |\tau|$

$|\_|$ filters smaller labels to right, $|32343| = [3, 3, 4]$ $\square$

Decreasing Diagrams

Proof.

by decreasingness

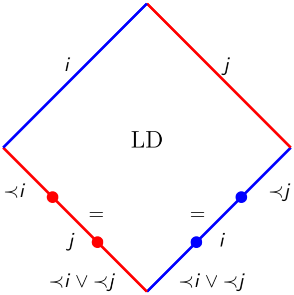


decreasing if $|\sigma| \uplus |\tau|$ as large as both $|\sigma\tau'|$ and $|\tau\sigma'|$ in multiset-extension of $\prec$ $\square$

Decreasing Diagrams

Proof.

(1) locally decreasing $\Rightarrow$ decreasing



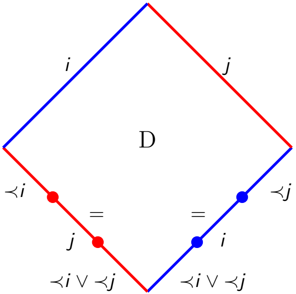
peak $|i| \uplus |j|$

lhs $|i(\prec i)^*(j + \varepsilon)(\prec i + \prec j)^*|$ □

Decreasing Diagrams

Proof.

(1) locally decreasing $\Rightarrow$ decreasing



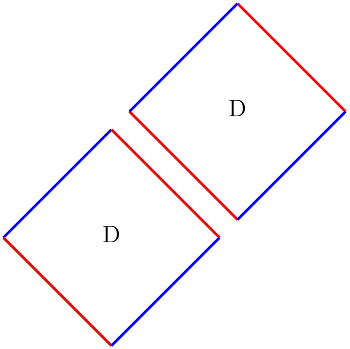
$|i| \uplus |j|$ is $[i] \uplus [j] = [i, j]$

$|i(\prec i)^*(j + \varepsilon)(\prec i + \prec j)^*|$ is $[i]$, $[i, j]$ or $[i, j_1, \ldots, j_n]$ $\square$

Decreasing Diagrams

Proof.

(2) decreasingness preserved under pasting

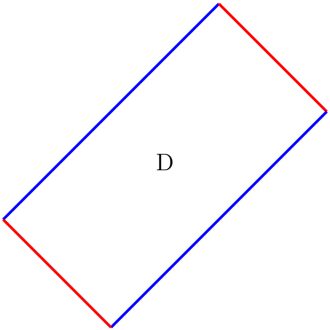


□

Decreasing Diagrams

Proof.

(2) decreasingness preserved under pasting on left

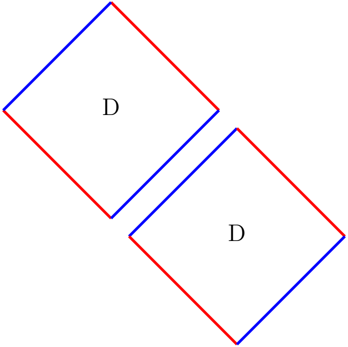


□

Decreasing Diagrams

Proof.

(2) decreasingness preserved under pasting

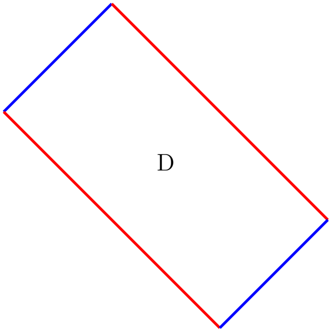


□

Decreasing Diagrams

Proof.

(2) decreasingness preserved under pasting on right

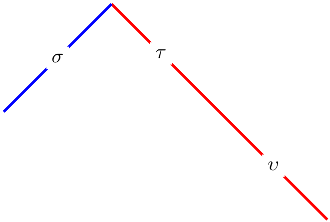


□

Decreasing Diagrams

Proof.

(3) filling with decreasing diagram decreases measure

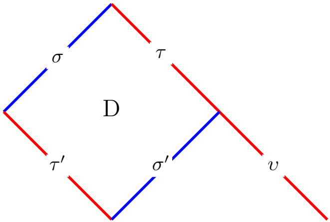


□

Decreasing Diagrams

Proof.

(3) filling with decreasing diagram decreases measure

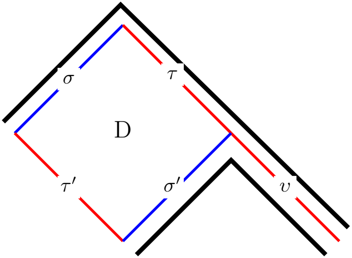


□

Decreasing Diagrams

Proof.

(3) filling with decreasing diagram decreases measure

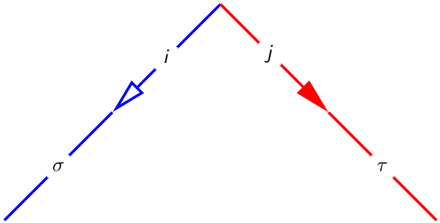


$|\sigma| \uplus |\tau\upsilon|$ greater than $|\sigma'| \uplus |\upsilon|$ □

Decreasing Diagrams

Proof.

LD $\Rightarrow$ D by (1) assumption, (2) pasting, and (3) filling

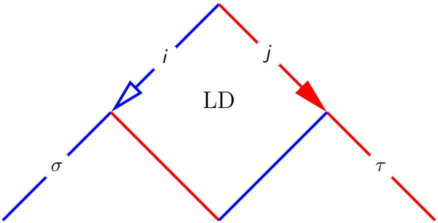


□

Decreasing Diagrams

Proof.

LD $\Rightarrow$ D by (1) assumption, (2) pasting, and (3) filling

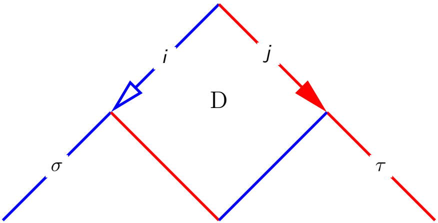


□

Decreasing Diagrams

Proof.

LD ⇒ D by (1) assumption, (2) pasting, and (3) filling

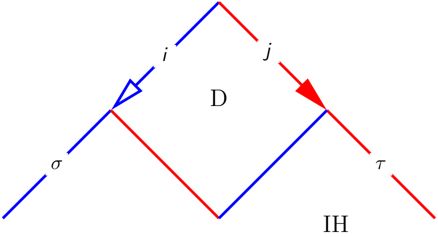


□

Decreasing Diagrams

Proof.

LD $\Rightarrow$ D by (1) assumption, (2) pasting, and (3) filling

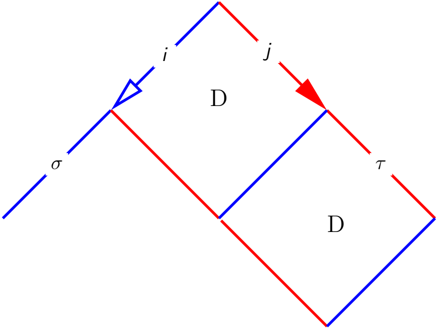


□

Decreasing Diagrams

Proof.

LD ⇒ D by (1) assumption, (2) pasting, and (3) filling



□

Decreasing Diagrams

Proof.

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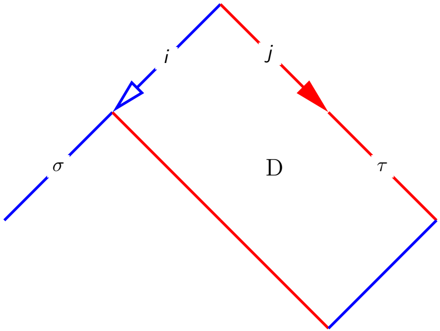
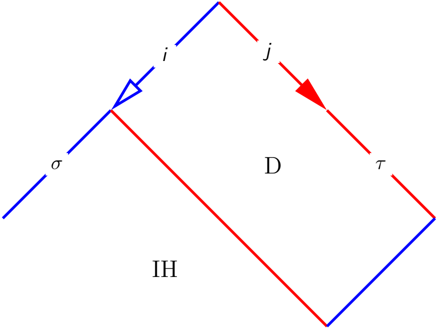


□

Decreasing Diagrams

Proof.

LD $\Rightarrow$ D by (1) assumption, (2) pasting, and (3) filling

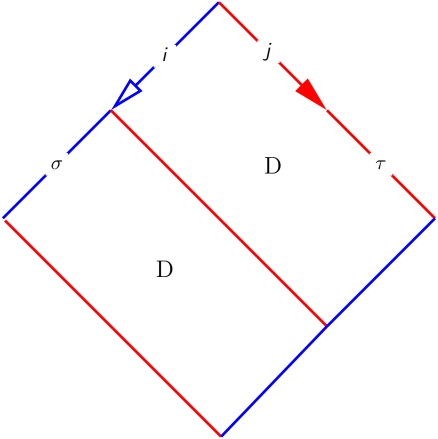


□

Decreasing Diagrams

Proof.

LD $\Rightarrow$ D by (1) assumption, (2) pasting, and (3) filling

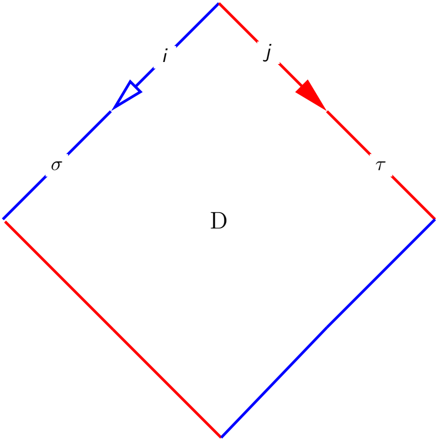


□

Decreasing Diagrams

Proof.

LD $\Rightarrow$ D by (1) assumption, (2) pasting, and (3) filling



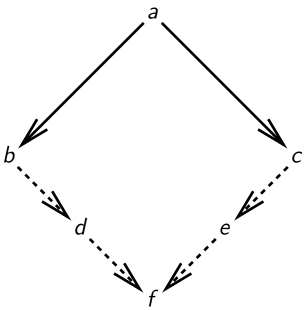
□

Newman's Lemma by decreasingness

Newman's Lemma by decreasingness

Proof.

local confluence $\Rightarrow$ confluence, if $\rightarrow$ terminating



label steps by their source, order labels by $\rightarrow^{+}$ □

Newman's Lemma by decreasingness

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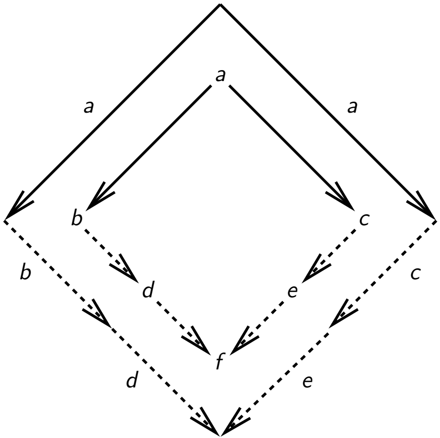

label steps by their source, order labels by $\rightarrow^+$ $\square$

Newman's Lemma by decreasingness

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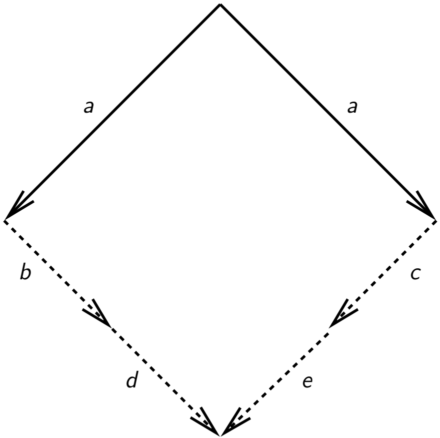
local confluence $\Rightarrow$ confluence, if $\rightarrow$ terminating



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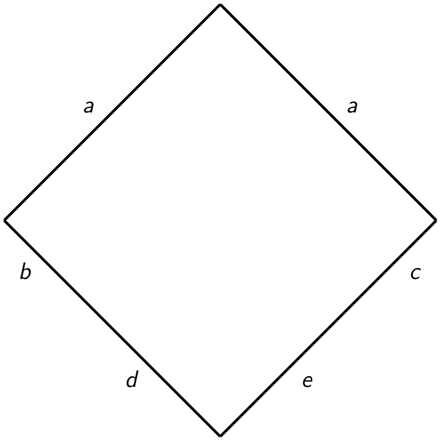

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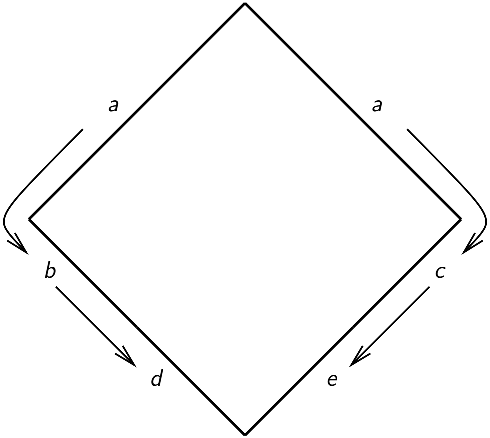

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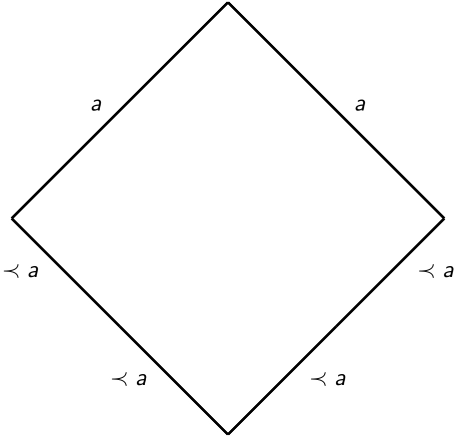


label steps by their source, order labels by $\rightarrow^+$ □

Diamond by decreasingness

Theorem

diamond property $(\leftarrow \cdot \rightarrow \subseteq \rightarrow \cdot \leftarrow)$ *implies confluence*

Diamond by decreasingness

Theorem

diamond property ($\leftarrow \cdot \rightarrow \subseteq \rightarrow \cdot \leftarrow$) *implies confluence*

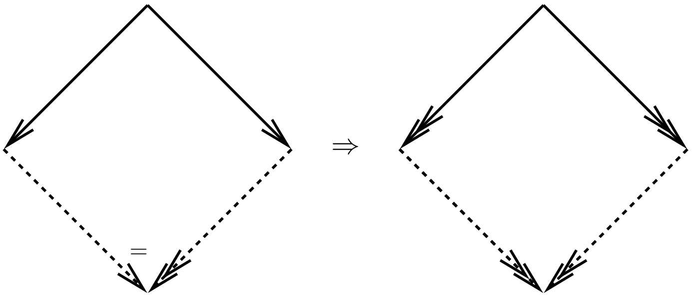
Proof.

take empty labelling □

Lemma of Hindley/uet

Theorem (Huet 1980)

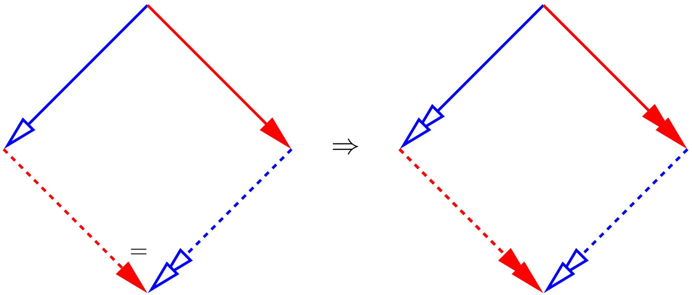
strong confluence implies confluence



Lemma of Hindley/uet

Theorem (Hindley 1964)

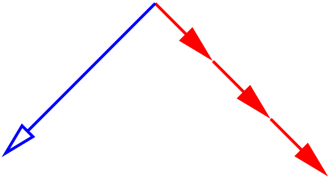
strong commutation implies commutation



Lemma of Hindley/uet

Proof.

intuition: tiling terminates since only ▷ steps are split

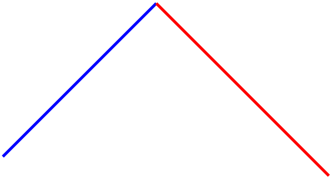


repeat: fill in local peak with local diagram □

Lemma of Hindley/uet

Proof.

intuition: tiling terminates since only $\triangleright$ steps are split



repeat: fill in local peak with local diagram $\square$

Lemma of Hindley/uet

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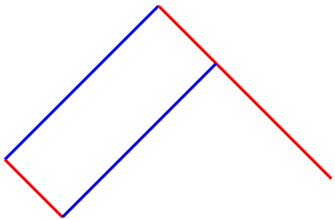

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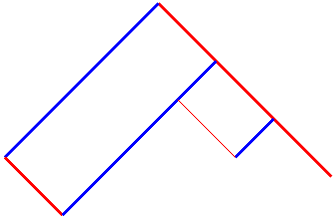

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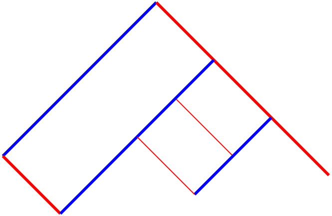

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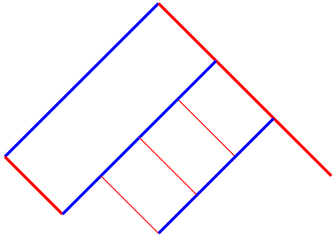

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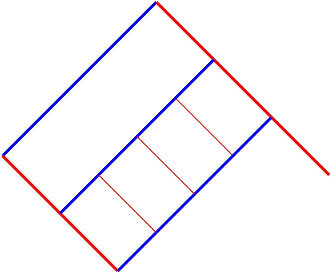

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Lemma of Hindley/uet

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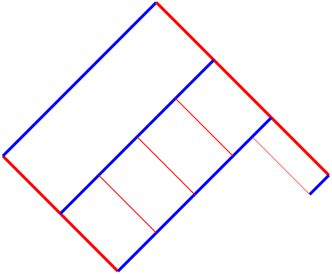

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Lemma of Hindley/uet

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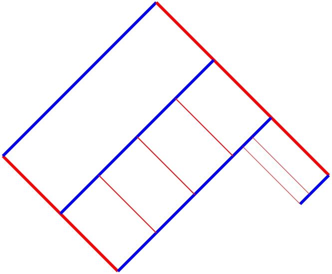

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Lemma of Hindley/uet

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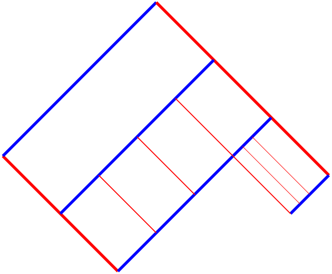

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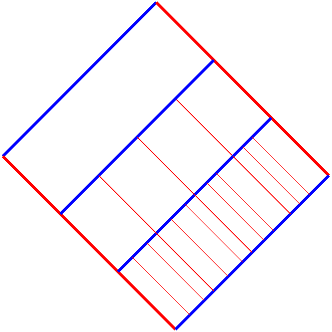

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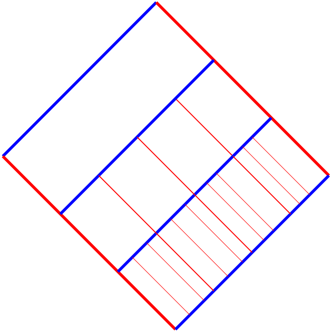

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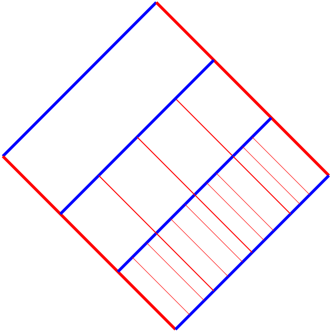


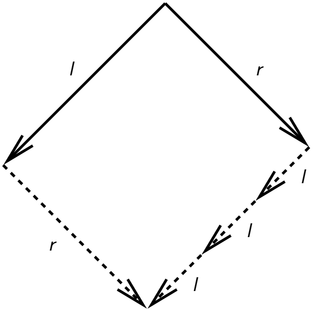
must stop: each ▶ stripe is eventualy filled □

Lemma of Hindley/uet by decreasingness

Lemma of Hindley/uet by decreasingness

Proof.

strong confluence ⇒ confluence

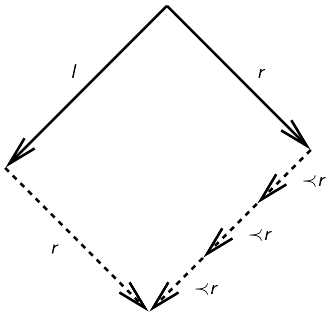


label steps by their direction (l or r), order r above l □

Lemma of Hindley/uet by decreasingness

Proof.

strong confluence $\Rightarrow$ confluence

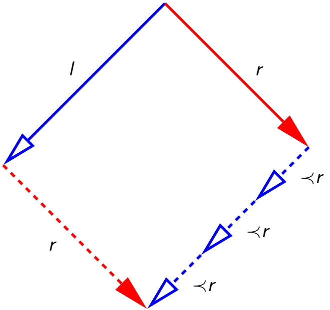


label steps by their direction (l or r), order r above l □

Lemma of Hindley/uet by decreasingness

Proof.

strong commutation $\Rightarrow$ commutation

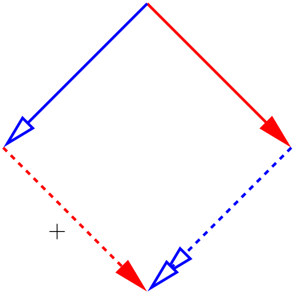


label steps by their direction ($\triangleright$ by l, $\blacktriangleright$ by r), order r above l $\square$

Geser/Di Cosmo/Piperno Lemma by decreasingness

Theorem (Geser)

commutation holds, if ▶ *terminating and*

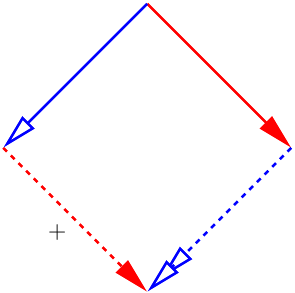


Geser/Di Cosmo/Piperno Lemma by decreasingness

equivalent (Bachmair & Dershowitz) to:

Theorem (Geser)

commutation holds, if $\blacktriangleright / \vartriangleleft \, (= \vartriangleleft\!\vartriangleleft \cdot \blacktriangleright \cdot \vartriangleleft\!\vartriangleleft)$ *terminating and*



Geser/Di Cosmo/Piperno Lemma by decreasingness

equivalent (Bachmair & Dershowitz) to:

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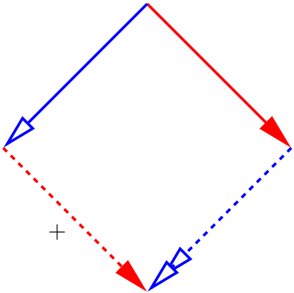
commutation holds, if $\blacktriangleright/\triangleleft\,(= \triangleleft\!\!\triangleleft \cdot \blacktriangleright \cdot \triangleleft\!\!\triangleleft)$ *terminating and*



Proof.

label steps by their target, order by $\blacktriangleright/\triangleleft$ $\square$

But isn't this just another ad hoc method?

(In)completeness of decreasing diagrams

Theorem

if a countable rewrite relation is confluent, then it can be proven so by decreasing diagrams.

Proof.

□

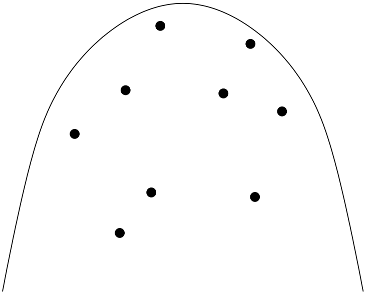
(In)completeness of decreasing diagrams

Theorem

if a countable rewrite relation is confluent, then it can be proven so by decreasing diagrams.

Proof.

suffices to consider $\leftrightarrow^*$-equivalence class



□

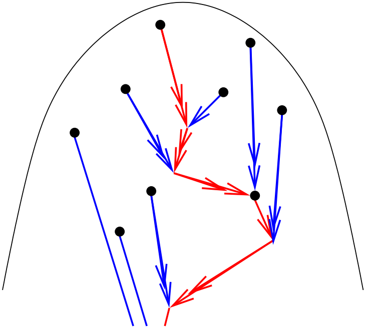
(In)completeness of decreasing diagrams

Theorem

if a countable rewrite relation is confluent, then it can be proven so by decreasing diagrams.

Proof.

construct a cofinal reduction (use countability)



□

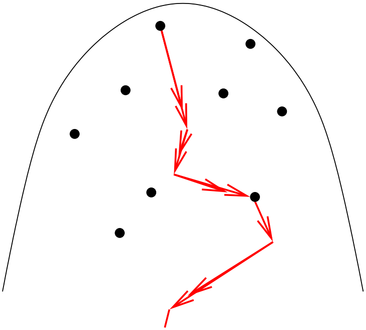
(In)completeness of decreasing diagrams

Theorem

if a countable rewrite relation is confluent, then it can be proven so by decreasing diagrams.

Proof.

cofinal reduction: such that all objects reduce to it



□

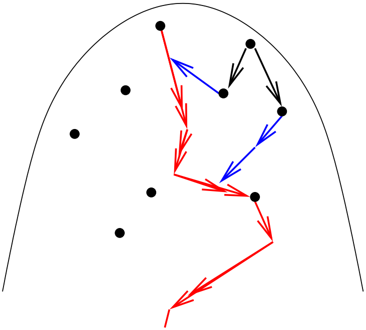
(In)completeness of decreasing diagrams

Theorem

if a countable rewrite relation is confluent, then it can be proven so by decreasing diagrams.

Proof.

complete local peaks by reducing to cofinal reduction



□

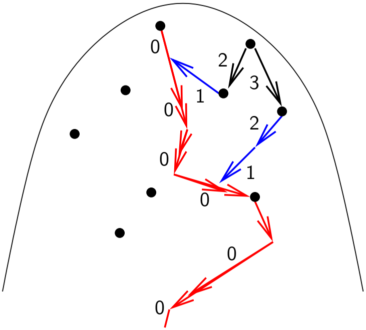
(In)completeness of decreasing diagrams

Theorem

if a countable rewrite relation is confluent, then it can be proven so by decreasing diagrams.

Proof.

label steps with distance to cofinal reduction



□

(In)completeness of decreasing diagrams

decreasing diagrams is incomplete for commutation
(Endrullis,Grabmayer, ISR 2008)

Example

$$d \blacktriangleleft b \lhd a_1 \stackrel{\blacktriangleright}{\lhd} a_2 \blacktriangleright c \rhd d$$

Proof by contradiction.

consider triples of shape $b \lhd_i a_1 \stackrel{\blacktriangleright}{\lhd}_j a_2 \blacktriangleright_k c$ with labels $[i, j, k]$. suppose w.l.o.g. $a_1 \blacktriangleright_j a_2$. then $b \lhd_i a_1 \blacktriangleright_j a_2$ can only be closed by $b \lhd_{i'} a_1 \lhd_{j'} a_2$. distinguish cases on the origin of the label j':

- if $j' < j$, then consider the triple with labels $[i, j', k]$.
- suppose $j' = i$. if $i' < i$ consider the triple with labels $[i', j, k]$, else $i' < j$ and consider the triple with labels $[i', i, k]$.

□

Application to TRSs

heuristic: label step by rule-name in a term rewriting system

Application to TRSs

heuristic: label step by rule-name in a term rewriting system

Theorem

linear TRS is confluent, if critical peaks are locally decreasing.

Application to TRSs

heuristic: label step by rule-name in a term rewriting system

Theorem

linear TRS is confluent, if critical peaks are locally decreasing.

Example

1. $\mathsf{nats} \to 0 : \mathsf{inc}(\mathsf{nats})$
2. $\mathsf{inc}(x : y) \to \mathsf{s}(x) : \mathsf{inc}(y)$
3. $\mathsf{hd}(x : y) \to x$
4. $\mathsf{tl}(x : y) \to y$
5. $\mathsf{inc}(\mathsf{tl}(\mathsf{nats})) \to \mathsf{tl}(\mathsf{inc}(\mathsf{nats}))$

one critical peak

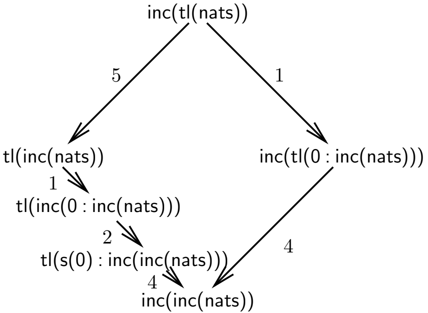
Application to TRSs

heuristic: label step by rule-name in a term rewriting system

Theorem

linear TRS is confluent, if critical peaks are locally decreasing.

Example

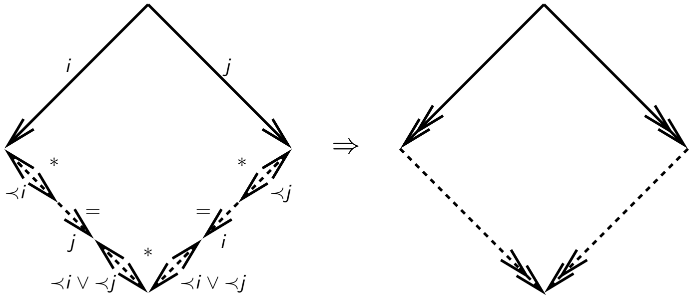


order?

(Trough) decreasingness $\rightsquigarrow$ seascape decreasingness

Theorem

locally decreasing seascape $\Rightarrow$ confluence

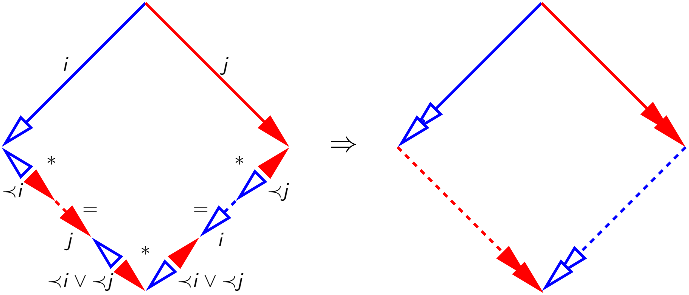


$\to = \bigcup_{i \in I} \to_i$, $\prec$ *well-founded order on I*

(Trough) decreasingness ⇝ seascape decreasingness

Theorem

locally decreasing seascape $\Rightarrow$ *commutation*

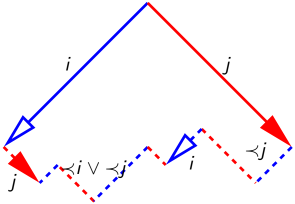


$\triangleright = \bigcup_{i \in I} \triangleright_i$, $\blacktriangleright = \bigcup_{j \in J} \blacktriangleright_j$, $\prec$ *well-founded order on* $I \cup J$

(Trough) decreasingness $\rightsquigarrow$ seascape decreasingness

Proof.

same measure of peaks, but local peak may not be base case

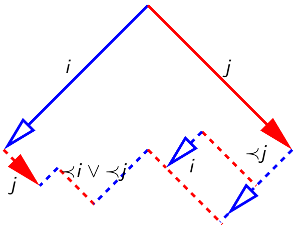


□

(Trough) decreasingness ⇝ seascape decreasingness

Proof.

but its peaks can be filled in by induction

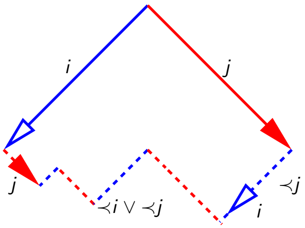


□

(Trough) decreasingness $\rightsquigarrow$ seascape decreasingness

Proof.

but its peaks can be filled in by induction.

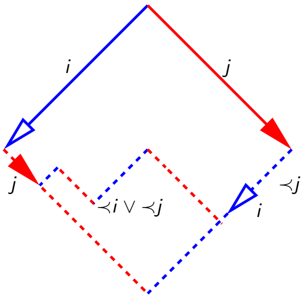


□

(Trough) decreasingness $\rightsquigarrow$ seascape decreasingness

Proof.

but its peaks can be filled in by induction..

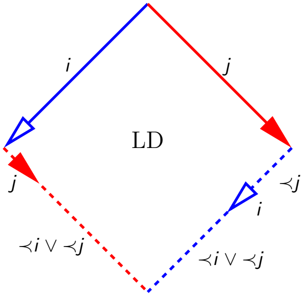


□

(Trough) decreasingness $\rightsquigarrow$ seascape decreasingness

Proof.

giving in the end a (trough) locally decreasing diagram



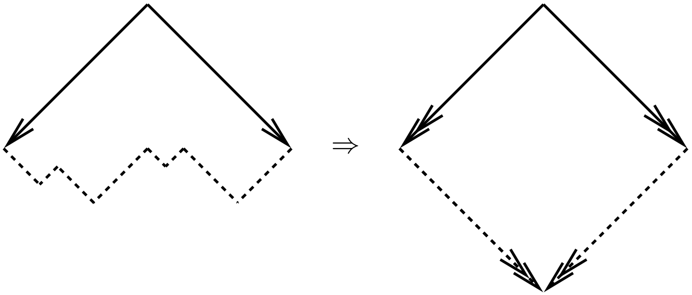
□

Winkler & Buchberger's Lemma

Winkler & Buchberger's Lemma

Theorem (Winkler & Buchberger 1983)

local confluence below $\Rightarrow$ *confluence, if* $\rightarrow$ *terminating*



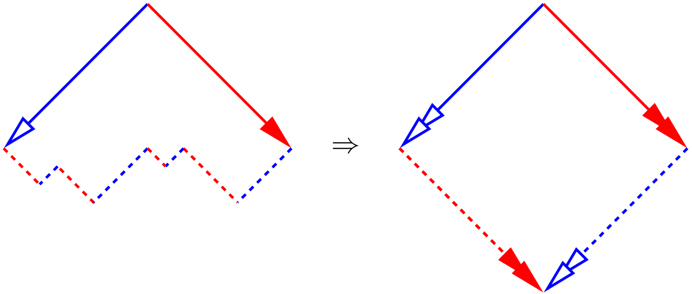
Definition

below: all objects in seascape $\rightarrow^+$-reachable from top

Winkler & Buchberger's Lemma

Theorem

local commutation below $\Rightarrow$ *commutation, if* $\triangleright \cup \blacktriangleright$ *terminating*



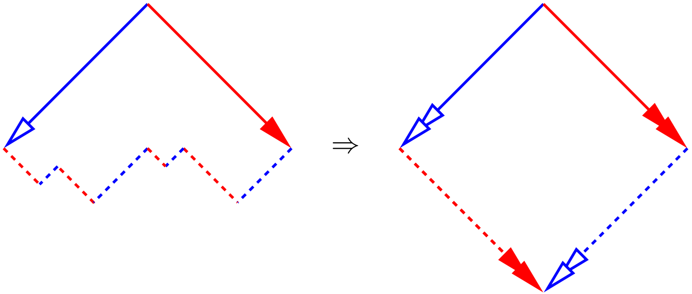
Definition

below: all objects in seascape $(\triangleright \cup \blacktriangleright)^+$-reachable from top

Winkler & Buchberger's Lemma

Theorem

local commutation below $\Rightarrow$ *commutation, if* $\triangleright^+ \cdot \blacktriangleright^+$ *terminating*



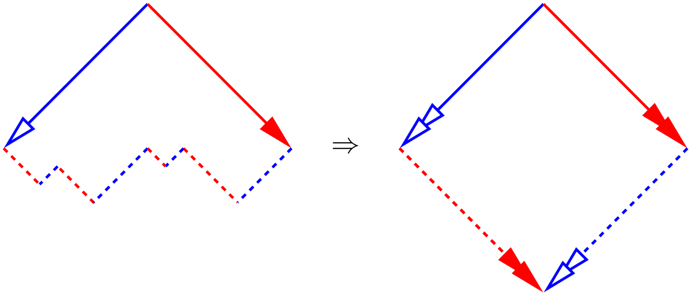
Definition

below: if $a \triangleright b$ in seascape, a $(\triangleright \cup \blacktriangleright)^+$-reachable from top with $\blacktriangleright$

Winkler & Buchberger's Lemma

Theorem

local commutation *below* $\Rightarrow$ *commutation, if* $\triangleright^+ \cdot \blacktriangleright^+$ *terminating*



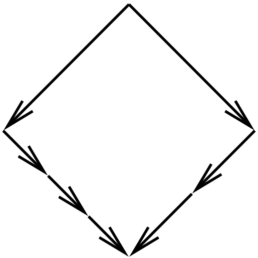
Definition

below: if $a \blacktriangleright b$ in seascape, a $(\triangleright \cup \blacktriangleright)^+$-reachable from top with $\triangleright$

Exercises on decreasing diagrams

Exercise

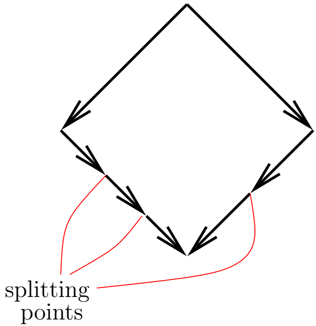
(splitting headache)
Any Escher diagram for a locally confluent rewrite relation $\rightarrow$ has an infinite path through infinitely many splitting points.



Exercises on decreasing diagrams

Exercise

(splitting headache)
Any Escher diagram for a locally confluent rewrite relation $\rightarrow$ has an infinite path through infinitely many splitting *points.*



Exercises on decreasing diagrams

Exercise

(commuting splitting headache)
Let $\triangleright$, $\blacktriangleright$ be locally commuting rewrite relations. Show that any (commutation) Escher diagram has

- *an infinite path,*
- *which is zigzagging, and*
- *goes through infinitely many splitting points*

Exercises on decreasing diagrams

Exercise

(Pous Lemma for process algebra)
Show that if $\triangleright, \blacktriangleright$ *commute locally, and* $\triangleright^{+} \cdot \blacktriangleright^{+}$ *is terminating, then* $\triangleright$, $\blacktriangleright$ *commute by*

- *an infinite diagram argument using the previous exercise;*
- *decreasing diagrams.*

Exercises on decreasing diagrams

Exercise

(Geser)

Fully prove Geser's Lemma, as found in the slides above, by means of the decreasing diagrams technique.

Exercises on decreasing diagrams

Exercise

(Rosen)

Show that if $\textcolor{blue}{\triangleright}, \textcolor{red}{\blacktriangleright}$ *both have the diamond property, and* $\textcolor{blue}{\triangleleft} \cdot \textcolor{red}{\blacktriangleright} \subseteq \textcolor{red}{\blacktriangleright} \cdot \textcolor{red}{\blacktriangleleft} \cdot \textcolor{blue}{\triangleleft}$ *($\textcolor{blue}{\triangleright}$ requests $\textcolor{red}{\blacktriangleright}$), then* $\textcolor{blue}{\triangleright} \cup \textcolor{red}{\blacktriangleright}$ *is confluent.*

Could you think of how to weaken the requests-condition without losing confluence?

Exercises on decreasing diagrams

Exercise

*(decreasing critical peaks)**

For a left-linear term rewrite system, is it true that if all critical peaks can be completed into decreasing diagrams, when indexing steps by the rule applied and well-foundedly ordering these, then the term rewrite system itself is confluent?

Exercises on decreasing diagrams

Exercise

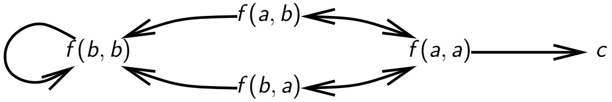
*(Newman's error)**

Newman's axioms for proving confluence as developed in Sections 8–12 of his paper, fail in their

13. Application to the conversion calculus *. Find what goes wrong!*

Answers to exercises

(decreasing critical peaks)
No, $f(b,b)$ and c convertible but not joinable:



for left-linear (but not right-linear) TRS:

$$
\begin{array}{lrcl}
1: & a & \to & b \\
2: & f(b,x) & \to & f(x,x) \\
3: & f(x,b) & \to & f(x,x) \\
4: & f(a,a) & \to & c
\end{array}
$$

(three) critical pairs decreasing ordering rules as $1 > 2, 3, 4$
Remark 'half' of Lévy's TRS (see e.g. Section 2.8, Terese)