Abstract Rewriting

ISR 2008, Obergurgl, Austria

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Theoretical Philosophy
Utrecht University
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16:00 – 17:30, Mon/Wednesday July 23, ISR 2008

Reintroduction

Abstract rewriting

- Newman 1942 (confluence, orthogonality)
- Hindley, Rosen, de Bruijn (orthogonality, commutation)
- Klop, Huet, Geser (abstract reduction as framework)
- Jouannaud/Kirchner, Ohlebusch (rewriting modulo)
- Melliès, Khasidashvili (standardisation, neededness)
- Ghani/Lüth (substitution)
- ...

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Standard notions

Newman	modern	notations I use
cell	step	$\to$
path	conversion	$\leftrightarrow^*$
descending path	reduction/rewriting seq.	$\twoheadrightarrow$
lower bound	common reduct	$\downarrow$
upper bound	common ancestor	$\uparrow$
property A	Church–Rosser property	$\leftrightarrow^* \subseteq \twoheadleftarrow \cdot \twoheadrightarrow$
property B	confluence property	$\twoheadleftarrow \cdot \twoheadrightarrow \subseteq \twoheadrightarrow \cdot \twoheadleftarrow$ $(\uparrow \subseteq \downarrow)$
property C	semi-confluence	$\leftarrow \cdot \twoheadrightarrow \subseteq \twoheadrightarrow \cdot \twoheadleftarrow$
property D	local confluence	$\leftarrow \cdot \to \subseteq \twoheadrightarrow \cdot \twoheadleftarrow$
derivate	residual	$/$
conversion calc.	λ-calculus	

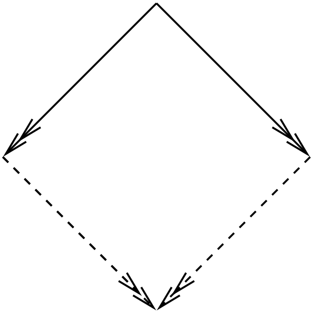
Plan

- Monday
- formalism: abstract rewrite relations (whether, Terese Ch. 1)
- A set of objects
- $\rightarrow \subseteq A \times A$ rewrite relation on A
- confluence property, lower bounds
- proof method: decreasing diagrams (Terese Ch. 14)
- proof method: Z property

Plan

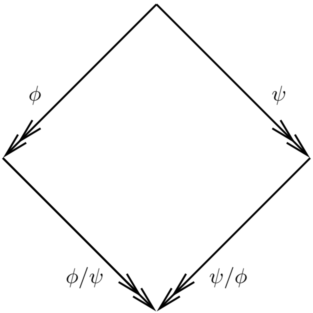
- Wednesday
- formalism: abstract rewrite systems (how, Terese Ch. 8)
- A set of objects
- $\to$ set of rewrite steps with source/target maps
- orthogonality, greatest lower bounds
- axiomatisation: residual systems (Terese Ch. 8.7)
- proof method: confluification into multi-steps

Confluence vs. orthogonality



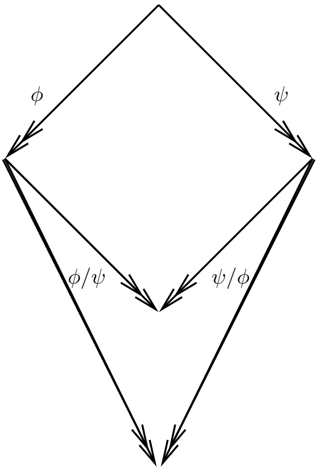
confluence, lower bound

Confluence vs. orthogonality



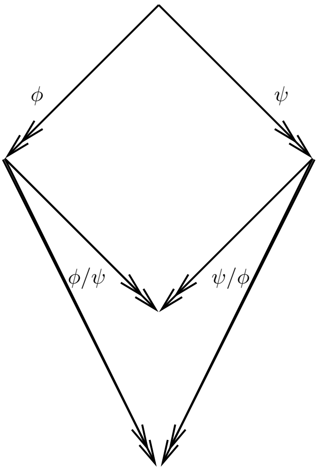
confluence, lower bound via witnessing residual function /

Confluence vs. orthogonality



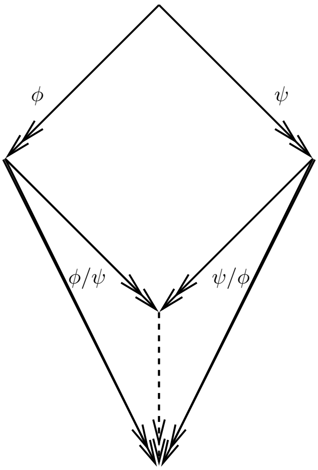
orthogonality, other lower bounds ...

Confluence vs. orthogonality



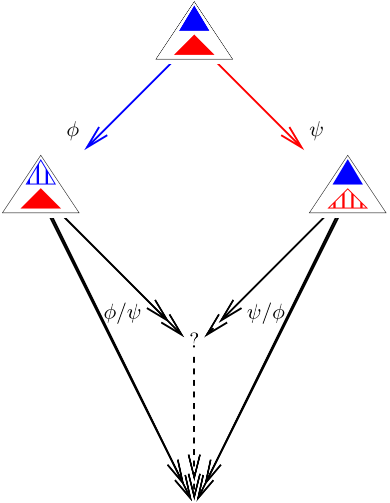
orthogonality, best among lower bounds?

Confluence vs. orthogonality



orthogonality, greatest lower bound

Confluence vs. orthogonality



orthogonality, greatest lower bound = doing work of both?

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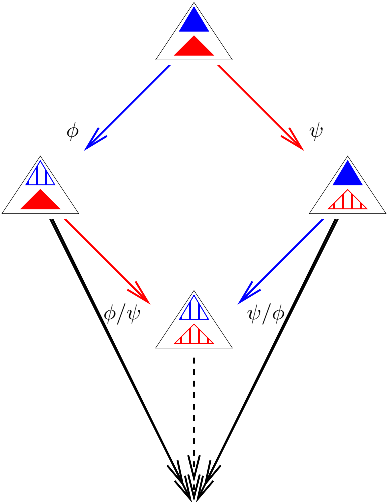

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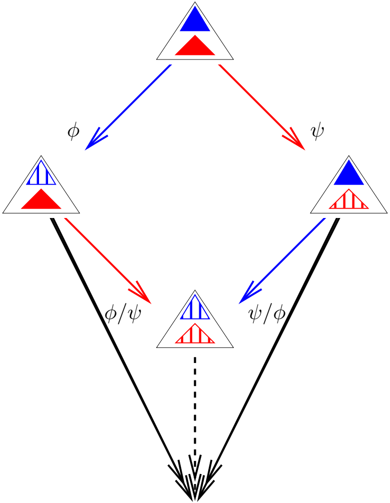
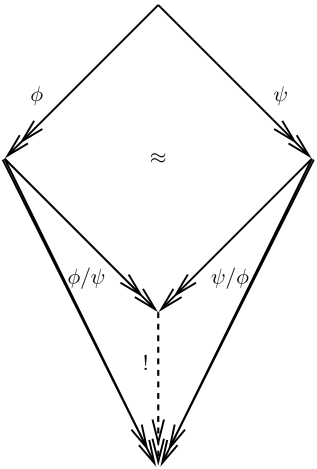


orthogonality, greatest lower bound $\neq$ doing work of both in $I(IK)$

Confluence vs. orthogonality



orthogonality, greatest lower bound w.r.t. notion of same work $\approx$

How to axiomatise orthogonality?

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for rewriting (steps not transitive)

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for rewriting (steps not transitive)
Newman 1942:

> The purpose of this paper is to make a start on a general theory of "sets of moves" by obtaining some conditions under which the answers to both the above questions are favorable. The results are essentially about "partially-ordered" systems, i.e. sets in which there is a transitive relation $>$, and sufficient conditions are given for every two elements to have a lower bound (i.e. for the set to be "directed") if it is known that every two "sufficiently near" elements have a lower bound. What further conditions are required for the existence of a *greatest* lower bound is not relevant to the present purpose, and is reserved for a later discussion.

Abstract rewrite system

Definition

ARS $\rightarrow$ is $\langle A, \Phi, \mathsf{src}, \mathsf{tgt}\rangle$

- A set of objects a, b, c, ...
- Φ set of steps ϕ, ψ, χ, ...
- src, tgt : $\Phi \rightarrow A$
  source and target functions

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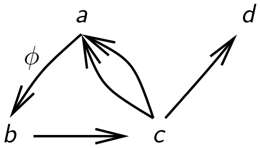
ARS is directed graph, e.g.



Newman's axioms for residuals

(J$_1$) *If* $\xi J \eta$, $\xi \mid \eta$ *has precisely one member.*

(J$_2$) *If* $\eta_1 \in \xi_1 \mid \zeta$ *and* $\eta_2 \in \xi_2 \mid \zeta$, *and if* $\xi_1 J \xi_2$ *or* $\xi_1 = \xi_2$, *then* $\eta_1 J \eta_2$ *or* $\eta_1 = \eta_2$.

J represents non-nesting of redexes

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Example (Schroer)

λ-calculus does not satisfy Newman's axioms
$\omega(\lambda y.\omega y) \to (\lambda y.\omega y)\lambda y.\omega y \to \underline{\omega(\lambda y.\omega y)} \to (\lambda y.\omega y)\lambda y.\omega y$
with $\omega = \lambda x.xx$

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- by (J_2) residuals of ωy are (mutually) J-related.

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- the ωy-redex is duplicated violating (J_1).

From term rewrite system to ARS

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combinatory logic (CL) rules:

$$\begin{array}{rcl} Ix & \rightarrow & x \\ Kxy & \rightarrow & x \\ Sxyz & \rightarrow & xz(yz) \end{array}$$

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named combinatory logic (CL) rules:

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\iota(x) :& \quad Ix &\to& \quad x \\
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multi-step ARS $\multimap\!\!\!\to$:

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step ARS $\to$: restriction of $\multimap$ steps to exactly one rule name

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$\iota(IK): I(IK) \multimap IK$ $\quad$ $I(\iota(K)): I(IK) \multimap IK$

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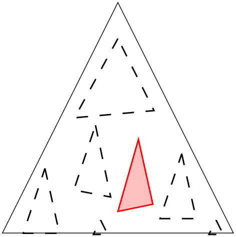
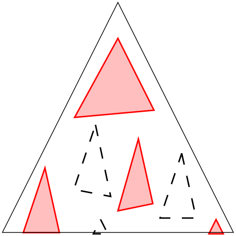
$\iota(IK) : I(IK) \to IK$ $\qquad$ $I(\iota(K)) : I(IK) \to IK$

Steps vs. multi-steps vs. full-developments



step $\rightarrow$: contract one redex-pattern

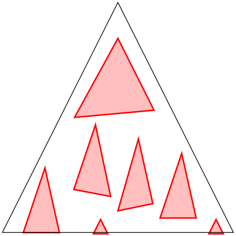
Steps vs. multi-steps vs. full-developments



multi-step $\multimap\!\!\rightarrow$ (development): contract some redex-patterns

$$\rightarrow \;\subseteq\; \multimap\!\!\rightarrow \;\subseteq\; \twoheadrightarrow$$

Steps vs. multi-steps vs. full-developments



full-development $\multimap\!\!\bullet\!\!\to$: contract all redex-patterns

$$\multimap\!\!\bullet\!\!\to \;\subseteq\; \multimap\!\!\circ\!\!\to \;\subseteq\; \twoheadrightarrow$$

Residuals

Intuition

residual of step ϕ after step ψ:
what remains (to be done) of step ϕ after doing ψ.

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and conversely?
same (but now residual is blue!)

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residual of $SIK(IK) \multimap\!\!\rightarrow SIKK$ after
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$I(IK)(K(IK)) \multimap\!\!\to IK(KK)$!

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$SIKK \multimap IK(KK)$!

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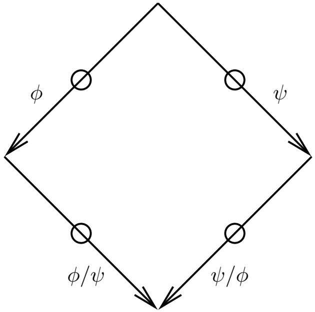

ϕ/ψ and ψ/ϕ: multi-steps ending in same object

Residual system

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residual system is ARS $\rightarrow$ extended with

- 1 the empty step for each object (doing nothing)

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$$\begin{array}{rcll} \phi/\phi & \approx & 1 & \\ \phi/1 & \approx & \phi & \\ 1/\phi & \approx & 1 & \\ (\phi/\psi)/(\chi/\psi) & \approx & (\phi/\chi)/(\psi/\chi) & \text{(cube)} \end{array}$$

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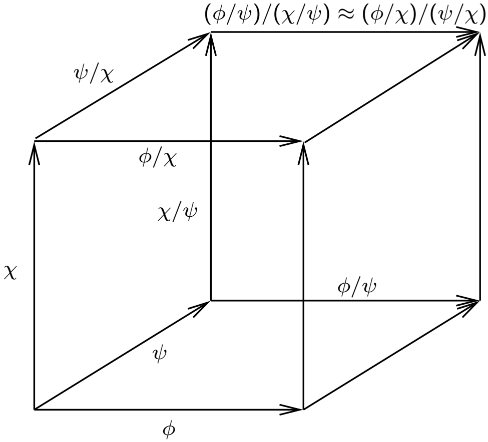
Exercise

show that third axiom is derivable

Cube axiom


$(\phi/\psi)/(\chi/\psi) \approx (\phi/\chi)/(\psi/\chi)$
ψ/χ
ϕ/χ
χ/ψ
χ
ϕ/ψ
ψ
ϕ

Residual system for orthogonal term rewrite systems

Definition

TRS is orthogonal if left-linear and non-overlapping

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Residual system for orthogonal term rewrite systems

Definition

TRS is orthogonal if left-linear and non-overlapping

- multi-steps as steps
- residual operation defined by induction on multi-steps

$$
\begin{aligned}
f(\phi_1,\ldots,\phi_n)/f(\psi_1,\ldots,\psi_n) &= f(\phi_1/\psi_1,\ldots,\phi_n/\psi_n)\\
\varrho(\phi_1,\ldots,\phi_n)/l(\psi_1,\ldots,\psi_n) &= \varrho(\phi_1/\psi_1,\ldots,\phi_n/\psi_n)\\
l(\phi_1,\ldots,\phi_n)/\varrho(\psi_1,\ldots,\psi_n) &= r(\phi_1/\psi_1,\ldots,\phi_n/\psi_n)\\
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\end{aligned}
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for every rule $\varrho(x_1,\ldots,x_n) : l(x_1,\ldots,x_n) \to r(x_1,\ldots,x_n)$

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for every rule $\varrho(x_1,\ldots,x_n) : \ l(x_1,\ldots,x_n) \to r(x_1,\ldots,x_n)$

Example

- $I(\iota(K))/\iota(IK) = \iota(K)$
- $SIK(\iota(K))/\varsigma(I,K,IK) = I(\iota(K))(K(\iota(K)))$

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Theorem

$\lesssim$ *is a quasi-order*

Proof.

- reflexivity: $\phi/\phi \approx 1$
- transitivity: if $\phi/\psi \approx 1$ and $\psi/\chi \approx 1$ then $\phi/\chi \approx 1$

$\square$

Residual order

Definition

$\phi \lesssim \psi$ if $\phi/\psi \approx 1$ (nothing remains)

Theorem

$\lesssim$ *is a quasi-order*

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Exercise

$\lesssim$ *is not necessarily a partial order (anti-symmetric)*

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$\lesssim$ is not necessarily a partial order (anti-symmetric)

Theorem

residual systems preserved by quotienting by $\lesssim \cap \gtrsim$.
yields a system having a residual order which is partial order.

From (multi-steps) to sequences

how to define residual system for sequences of (multi-)steps?

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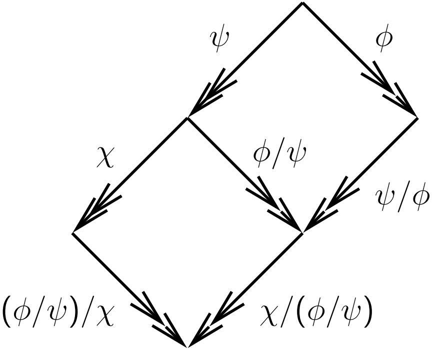

$$\phi/(\psi \circ \chi) \approx (\phi/\psi)/\chi$$

$$(\psi \circ \chi)/\phi \approx (\psi/\phi) \circ (\chi/(\phi/\psi))$$

take residuals (multi-)stepwise

Residual system with composition

extending residual operation to sequences generates:

Definition

Residual system with composition

- 1 the empty reduction
- $/$ the residual map from pairs of (co-initial) reductions to reductions
- $\circ$ the composition map on composable reductions
- $$\begin{array}{rcl} \phi/\phi & \approx & 1 \\ \phi/1 & \approx & \phi \\ 1/\phi & \approx & 1 \\ (\phi/\psi)/(\chi/\psi) & \approx & (\phi/\chi)/(\psi/\chi) \\ 1 \circ 1 & \approx & 1 \\ \chi/(\phi \circ \psi) & \approx & (\chi/\phi)/\psi \\ (\phi \circ \psi)/\chi & \approx & (\phi/\chi) \circ (\psi/(\chi/\phi)) \end{array}$$

Residual order gives greatest lower bound

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Example

- orthogonal TRSs

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- orthogonal higher-order term rewriting systems

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Example

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- ...

Non-standard examples of residual systems

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- sorting

Non-standard examples of residual systems

- sorting
- braids

Non-standard examples of residual systems

- sorting
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- self-distributivity

Non-standard examples of residual systems

- sorting
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- associativity

Non-standard examples of residual systems

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- ...

Sorting by swapping adjacent elements

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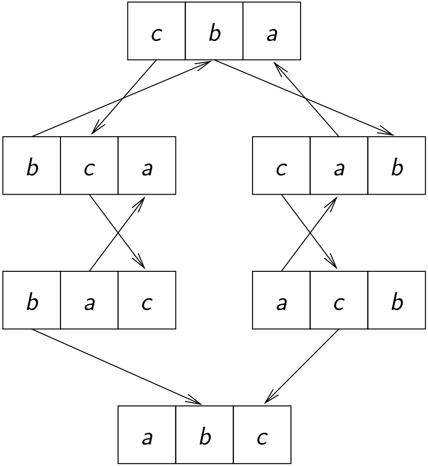
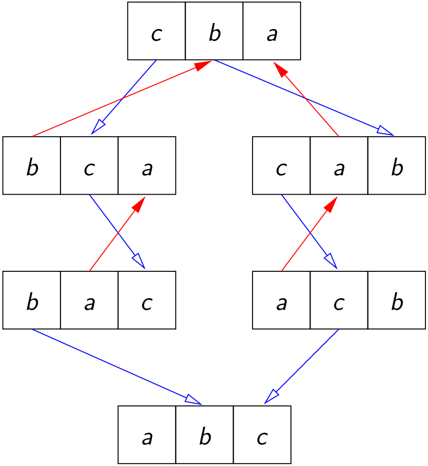


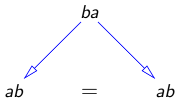
Reduction steps: arrows start at first element of swapped pair

Sorting by swapping adjacent elements

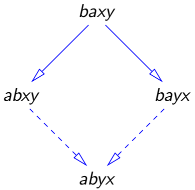


reduction steps: inversions in blue, anti-inversions in red

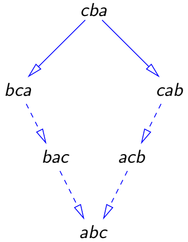
Inversion sort local confluence diagrams



same



independent



self-overlap

Residual system for inversion sort

- 1 the empty step
- / the residual map from pairs of steps to steps
- $$\begin{array}{rcl} \phi/\phi & \approx & 1 \\ \phi/1 & \approx & \phi \\ 1/\phi & \approx & 1 \\ (\phi/\psi)/(\chi/\psi) & \approx & (\phi/\chi)/(\psi/\chi) \end{array}$$

Residual system for inversion sort

Theorem

inversion sorting gives a residual system

Proof.

step ϕ from list ℓ is multi-inversion: relation $\widehat{\ }$ s.t. if $\widehat{ij}$

- out-of-order: $\ell = \ldots i \ldots j \ldots$ but $i > j$;
- transitive: if $\widehat{jk}$, then $\widehat{ik}$;
- scopic: if $\ell = \ldots i \ldots k \ldots j \ldots$, then either $\widehat{ik}$ or $\widehat{jk}$

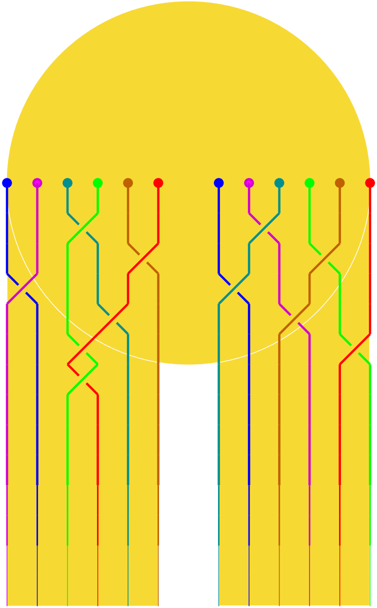
define 1 to be the empty relation,
define ϕ/ψ as $(\phi \cup \psi)^+ - \psi$. □

Example

$$(cba \to_{\widehat{c}ba} bca)/(cba \to_{\widehat{cba}} cab) = (cab \to_{\widehat{cab}} abc)$$

Braid problem

Braid problem



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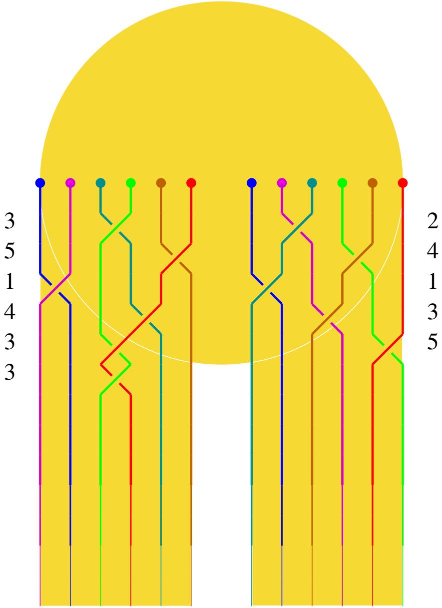
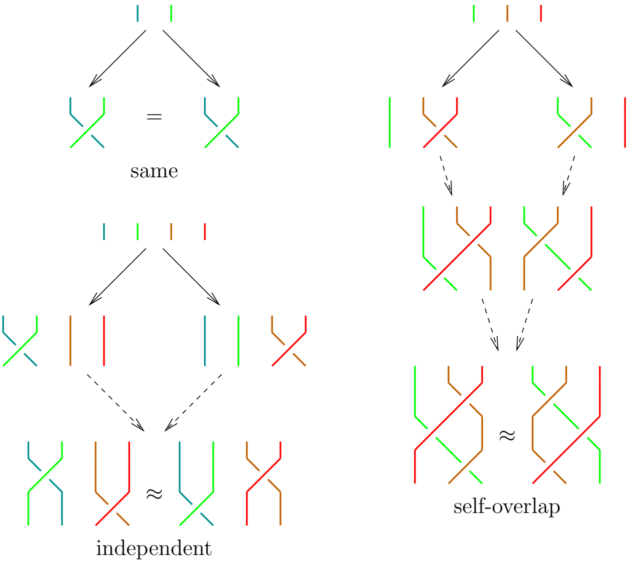
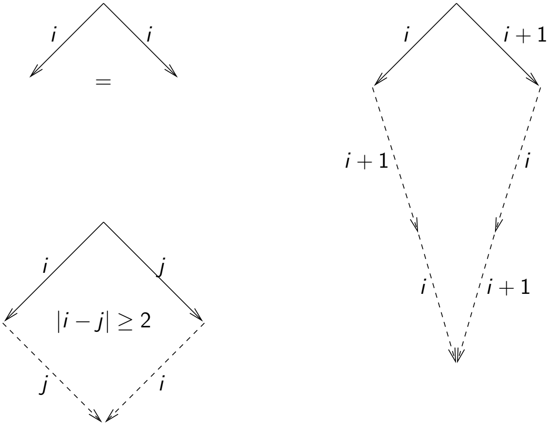


Braid confluence diagrams



reductions end in topologically equivalent (≈) braids

Braid confluence diagrams



reduction steps labelled by gap# of crossing
$ij \approx ji$ if $|i - j| \geq 2$ and $i(i+1)i \approx (i+1)i(i+1)$

Sorting vs. braiding

- sorting is braiding without crossing strands (inverting) twice

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Sorting vs. braiding

- sorting is braiding without crossing strands (inverting) twice
- model braids as ‘repeated sorting’
- model braids as reduction sequences of multi-inversions

Orthogonality of braids

Theorem

braiding gives a residual system with composition

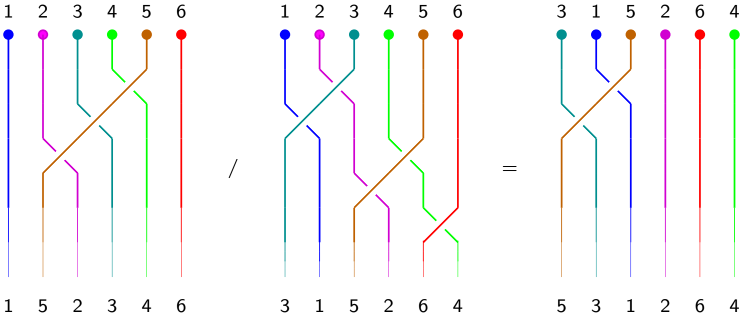
Proof.

- steps are sequences of multi-inversions
- without out-of-order restriction
- define $\circ$ to be formal composition
- $/$ on sequences defined via composition laws

$\square$

Orthogonality of braids

Example



Self-distributivity: $(x \cdot y) \cdot z \approx (x \cdot z) \cdot (y \cdot z)$

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Interpret as first projection

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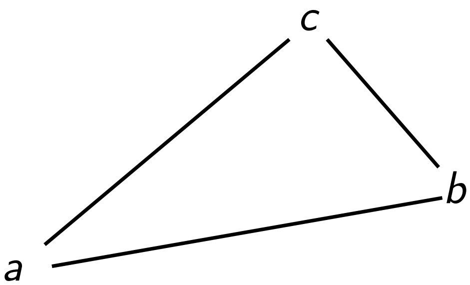
Interpret as an ACI-operation

$$
\begin{aligned}
(x \cdot y) \cdot z &=_A x \cdot (y \cdot z) \\
&=_I x \cdot (y \cdot (z \cdot z)) \\
&=_A x \cdot ((y \cdot z) \cdot z) \\
&=_C x \cdot (z \cdot (y \cdot z)) \\
&=_A (x \cdot z) \cdot (y \cdot z)
\end{aligned}
$$

Examples: disjunction/union, conjunction/intersection

Self-distributivity: $(x \cdot y) \cdot z \approx (x \cdot z) \cdot (y \cdot z)$

Interpret as 'middle'



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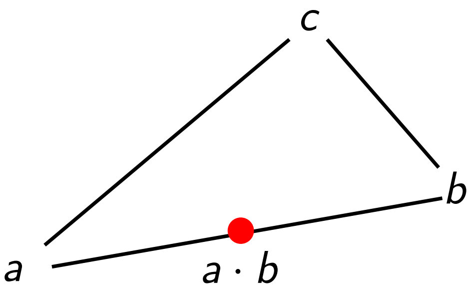

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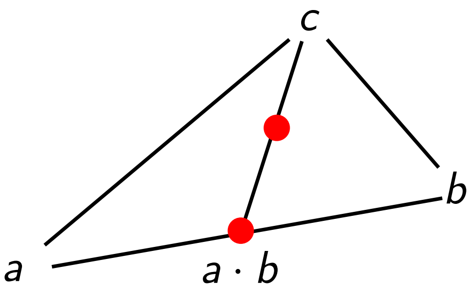

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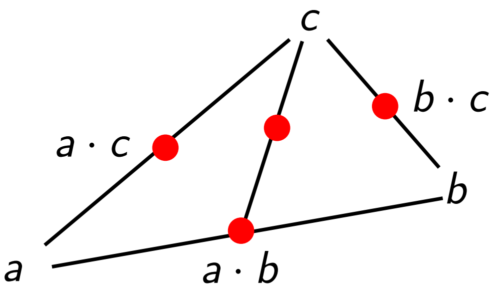

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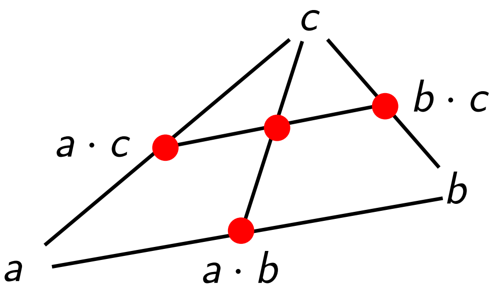

Self-distributivity rule: $xyz \longrightarrow xz(yz)$ critical pair

- applicative notation: $\cdot$ infix, associating to left

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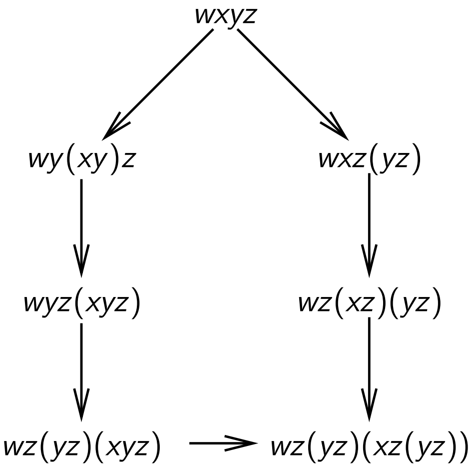

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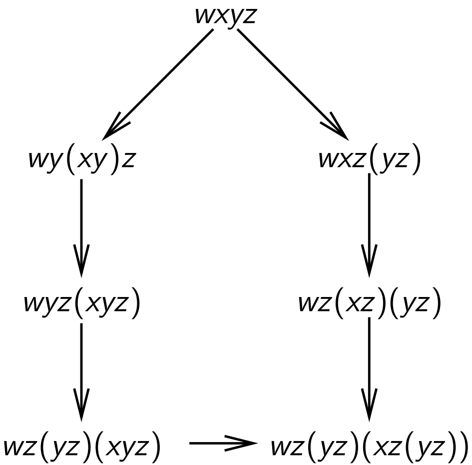
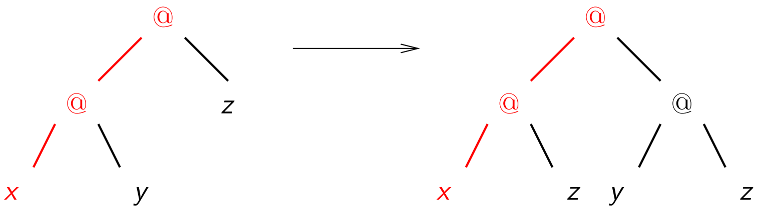


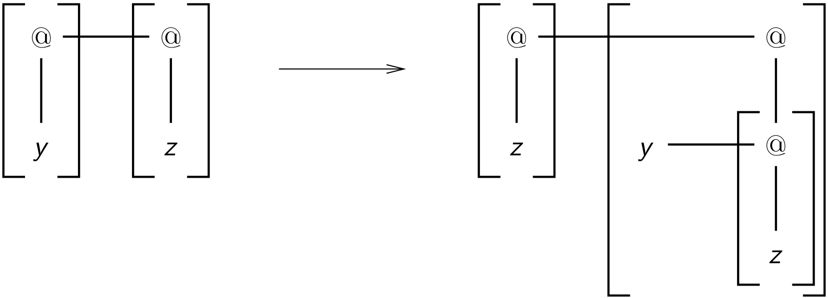
- w represents spine ...

Spine rectification



Spine is stable!

Spine rectification



If you don't have a spine, they can't break you

Self-distributivity rule: $[y][z] \to [z][y[z]]$

- elements on spine juxtaposed

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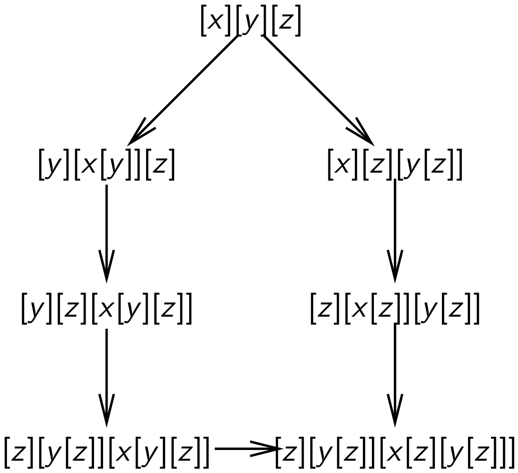

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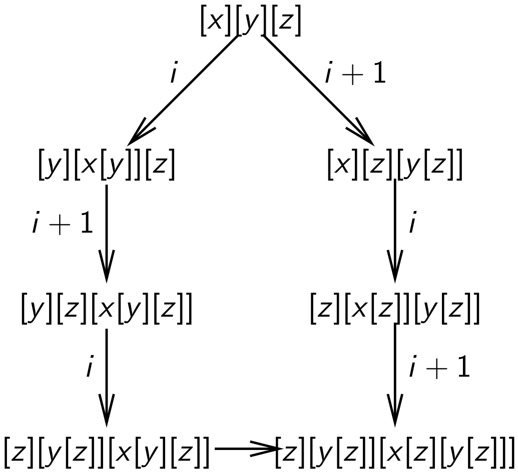

- almost braiding, but one extra step ...

Braiding vs. self-distributivity

- $[y][z] \to [z][y[z]]$ swaps z and y, remembering y crossed z...

Braiding vs. self-distributivity

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- extra step.

Orthogonality of self-distributivity

Theorem

self-distributivity gives a residual system

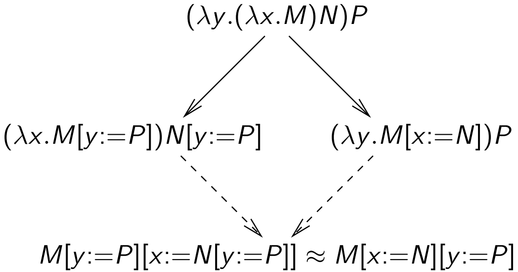
Idea.

Multi-distribution defined similar to multi-conversions, but

- relates positions in the (rectified) term
- may relate only to right-wing uncles; $(\widehat{piq)(pj})$ with $i < j$
- must be left-convex; $(\widehat{piq_1 q_2)(pj})$ implies $(\widehat{piq_1)(pj})$

/ as before; constructed by using standard residual theory to relate positions before and after the (non-linear) term rewrite step □

Substitution lemma of λ-calculus as self-distributivity



Substitution Lemma of the λ-calculus

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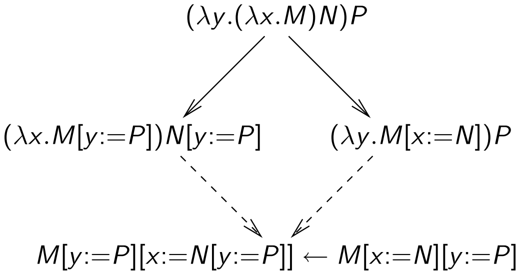
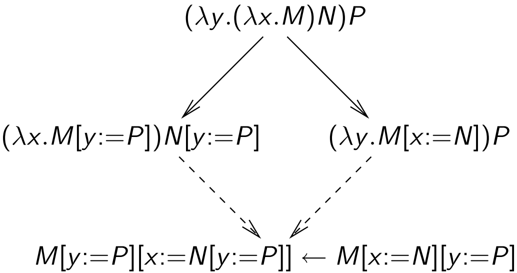


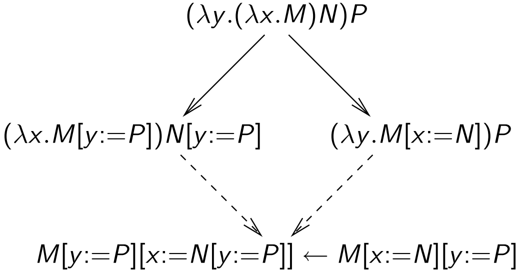
Critical pair for λ-calculus with explicit substitutions

Substitution lemma of λ-calculus as self-distributivity



Critical pair for λ-calculus with explicit substitutions
Is this rule in itself confluent? (left-to-right no)

Substitution lemma of λ-calculus as self-distributivity



Critical pair for λ-calculus with explicit substitutions
This is self-distributivity, so even orthogonal!

Confluification

Definition

confluification if local confluence completed by sequences, adjoin these to steps.

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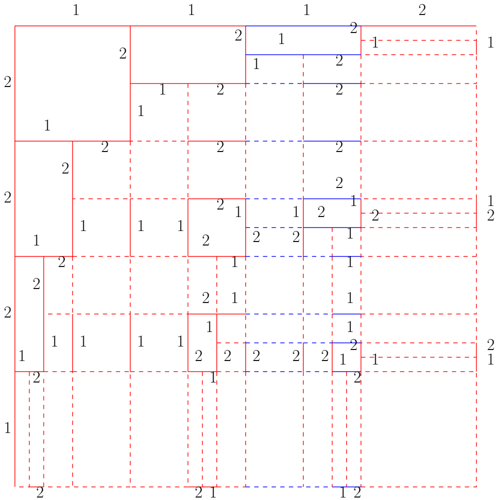

Confluification

Definition

confluification if local confluence completed by sequences, adjoin these to steps.

- for orthogonal term rewriting systems: parallel reductions
- for λ-calculus: developments

From residual systems with composition to algebras

Example

- multi-inversions in sorting

From residual systems with composition to algebras

Example

- multi-inversions in sorting
- braids

From residual systems with composition to algebras

Example

- multi-inversions in sorting
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From residual systems with composition to algebras

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From residual systems with composition to algebras

Example

- multi-inversions in sorting
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- orthogonal term rewriting systems (β-reduction, CL)
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- ...
- also many residual algebras (singleton carrier) ...

Residual algebras (with composition)

- natural numbers (as steps from object to itself)
- $\dot{-}$ (cut-off subtraction), 0 (zero), $+$ (addition);

$$
\begin{aligned}
n \dot{-} n &\approx 0 \\
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0 \dot{-} n &\approx 0 \\
(n \dot{-} m) \dot{-} (k \dot{-} m) &\approx (n \dot{-} k) \dot{-} (m \dot{-} k) \\
0 + 0 &\approx 0 \\
k \dot{-} (n + m) &\approx (k \dot{-} n) \dot{-} m \\
(n + m) \dot{-} k &\approx (n \dot{-} k) + (m \dot{-} (k \dot{-} n))
\end{aligned}
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Generated from its

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\end{aligned}
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Truth-values with reverse implication, false (no composition)

Positive natural numbers with cut-off division, 1, multiplication

Residual algebras (with composition)

- multisets over some set (as steps from object to itself)
- $-$ (multiset difference), $\emptyset$ (empty multiset), $\uplus$ (multiset sum);

$$
\begin{array}{rcl}
M - M & \approx & \emptyset \\
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(M - N) - (K - N) & \approx & (M - K) - (N - K) \\
\emptyset \uplus \emptyset & \approx & \emptyset \\
K - (M \uplus N) & \approx & (K - M) - N \\
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Sets with set-difference, $\emptyset$, disjoint union.

Residual algebras (with composition)

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K - (M \uplus N) &\approx (K - M) - N \\
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\end{aligned}
$$

all compositions are commutative

commutative residual algebras

Definition

commutative residual algebra with composition (CRAC) satisfies

$$\begin{array}{rcl} (\phi/\psi)/\phi & \approx & 1 \\ \phi/(\phi/\psi) & \approx & \psi/(\psi/\phi) \end{array}$$

(follows from computing $(\phi \circ \psi)/(\psi \circ \phi) \approx 1$!)

commutative residual algebras

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$$\begin{aligned}(\phi/\psi)/\phi &\approx 1\\ \phi/(\phi/\psi) &\approx \psi/(\psi/\phi)\end{aligned}$$

- 2nd equation states commutativity of intersection $\phi/(\phi/\psi)$

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- Very useful for equational reasoning about multisets in Coq.
- Iso to commutative BCK algebras with relative cancellation

commutative residual algebras

Definition

commutative residual algebra with composition (CRAC) satisfies

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- 2nd equation states commutativity of intersection $\phi/(\phi/\psi)$
- Very useful for equational reasoning about multisets in Coq.
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- every well-founded CRAC iso to multiset CRAC

Conclusion

- decreasing diagrams: well-founded indexing
- Z-property: bullet-function
- orthogonal systems: axiomatised residual operation