# Decreasing proof orders
## Interpreting conversions in involutive monoids


Vincent van Oostrom

Universiteit Utrecht


IWC, Nagoya, May 29, 2012

Decreasing tiles

Involutive proofs

French strings

Applications

# Alhambra

# Tiling puzzles (1964–73)

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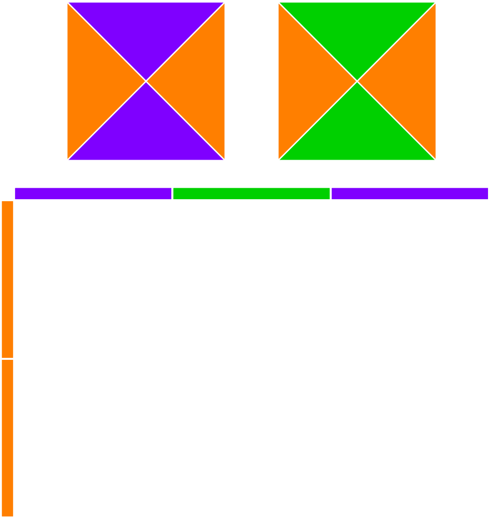

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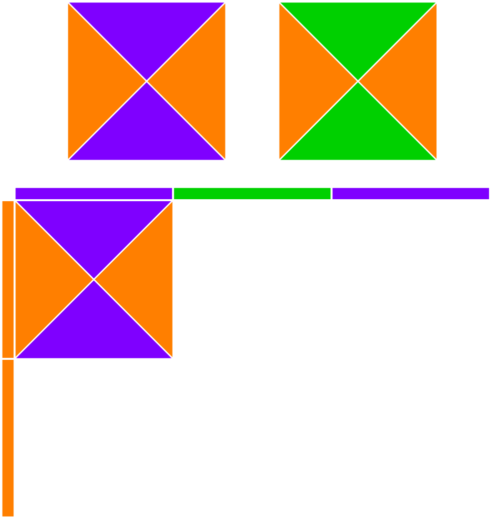



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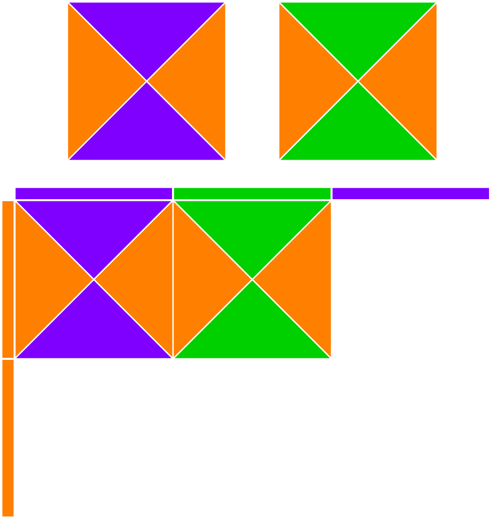

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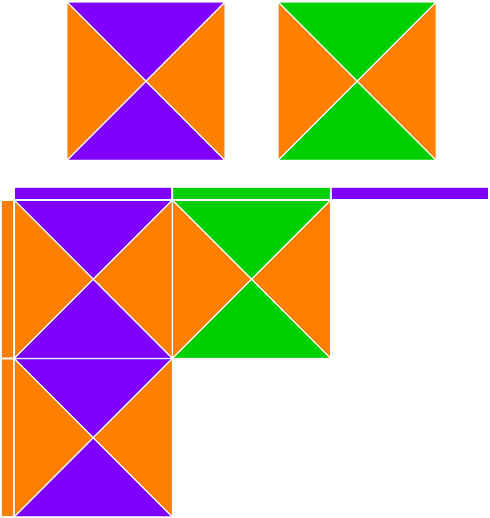

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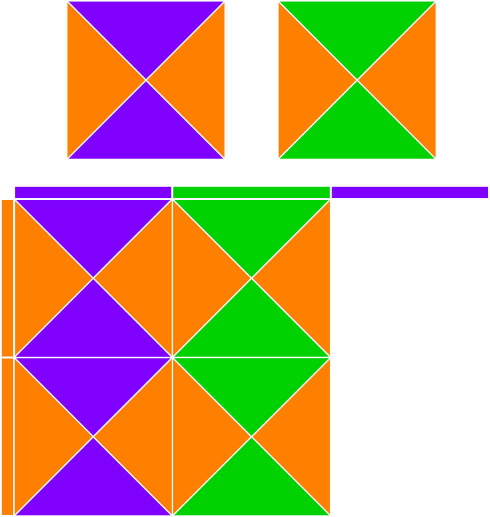

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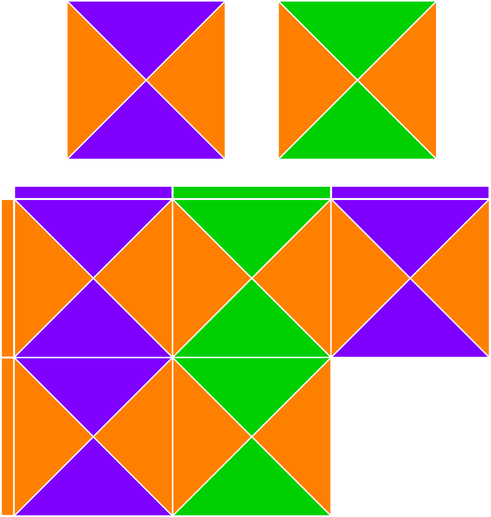

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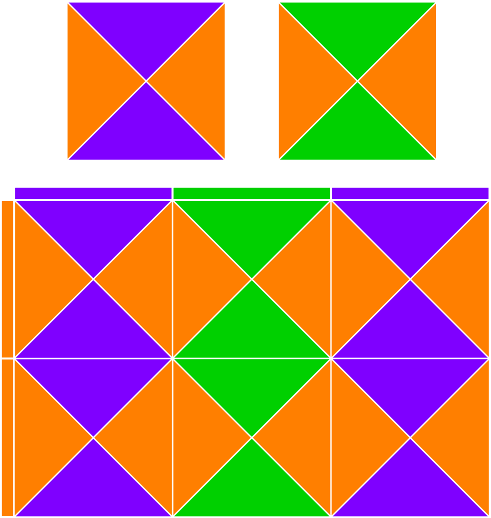

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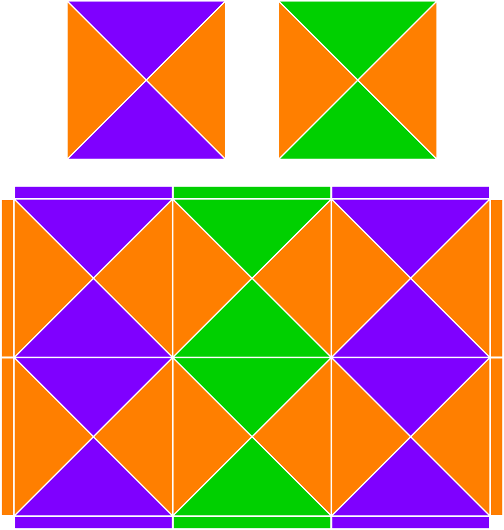

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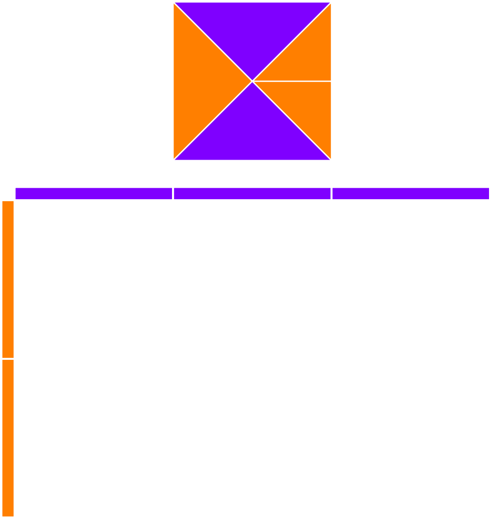

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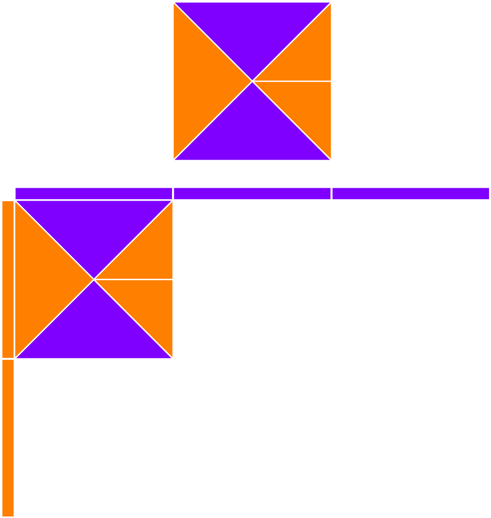

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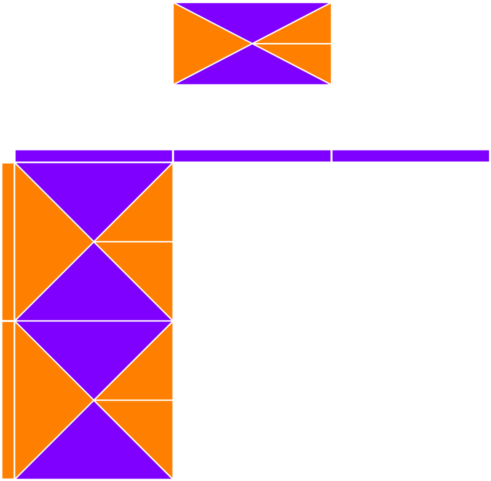

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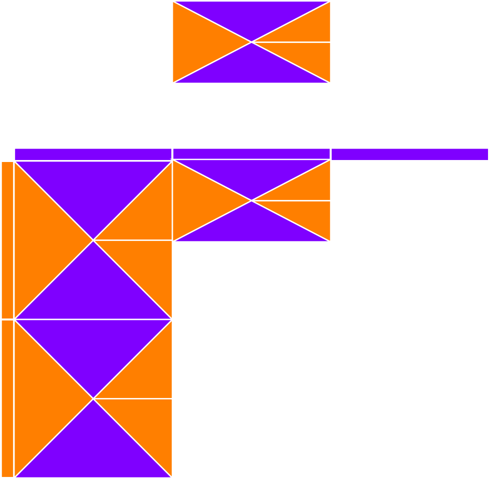



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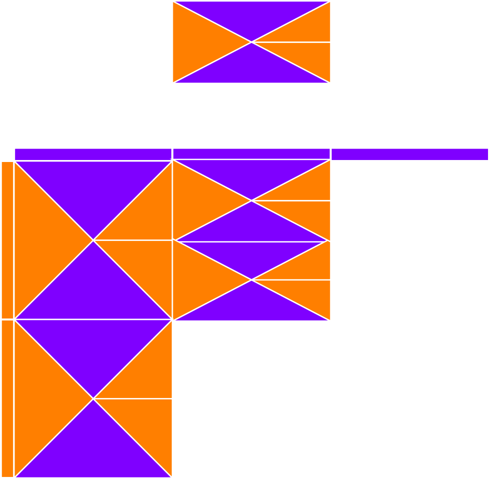

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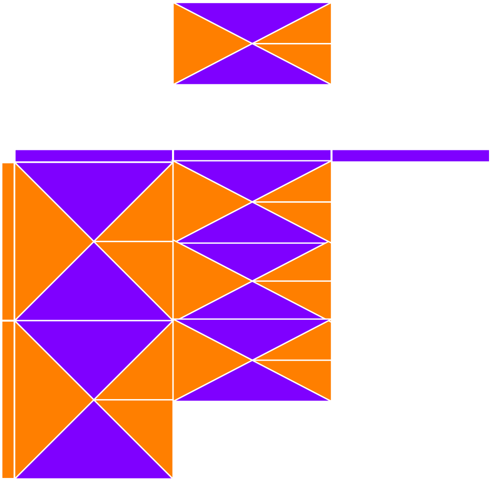

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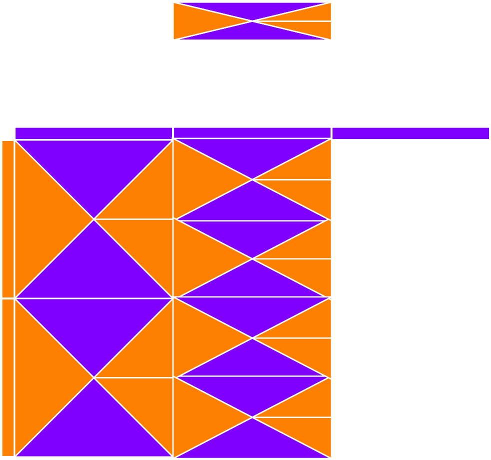

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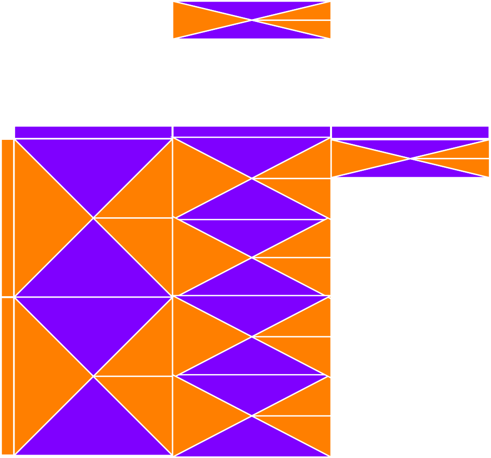

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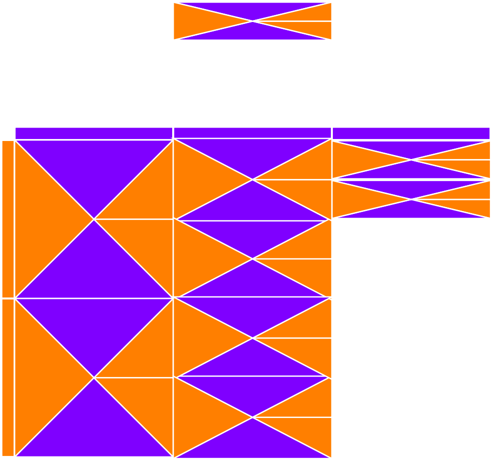

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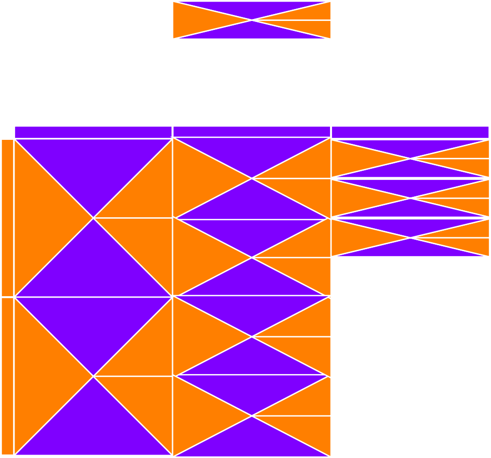

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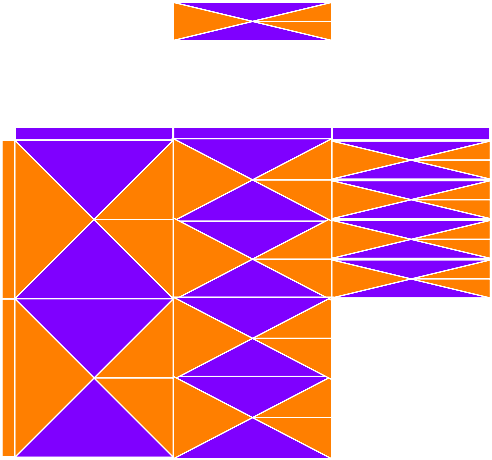

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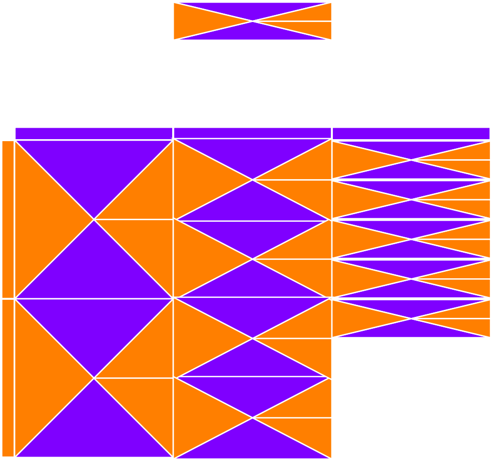

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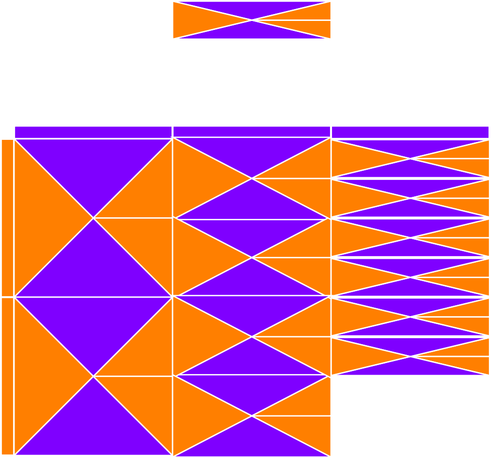

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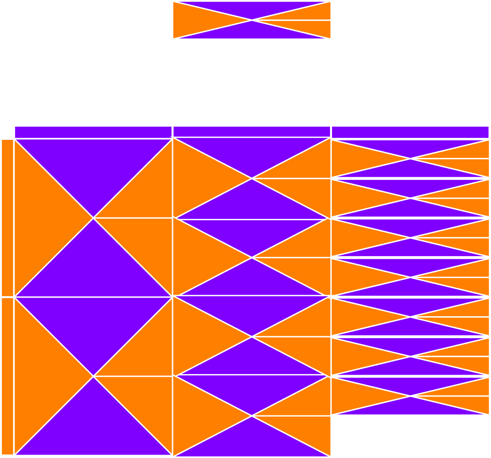

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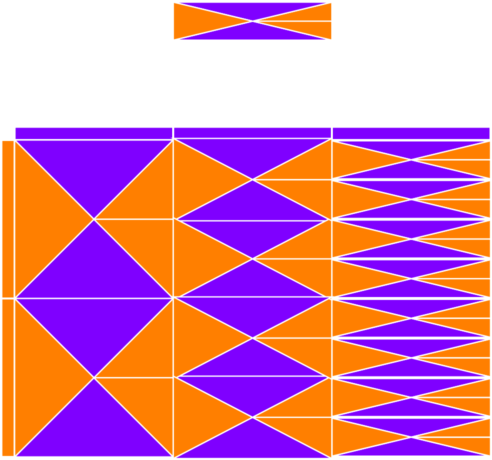



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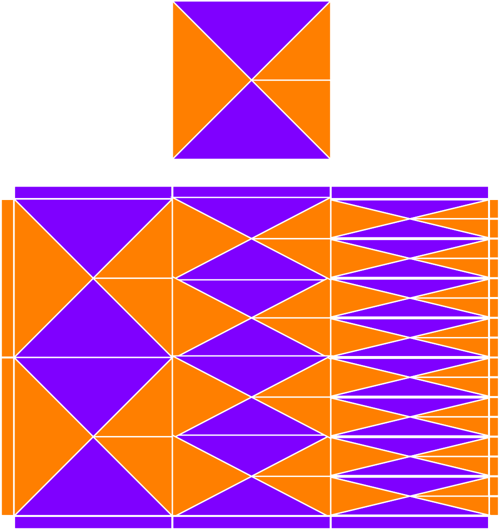

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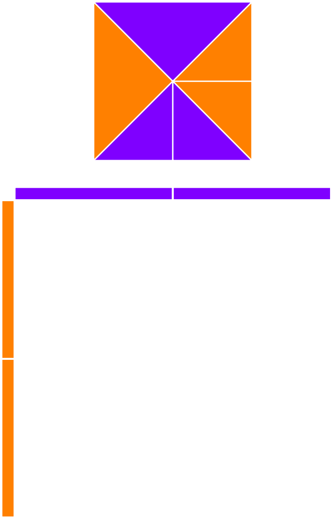

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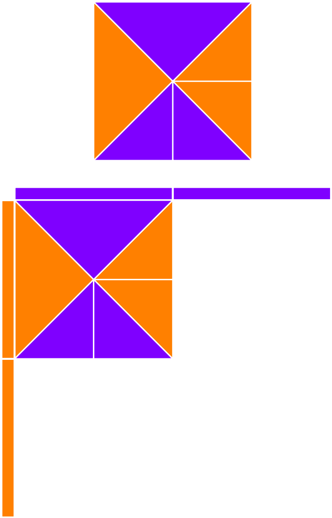

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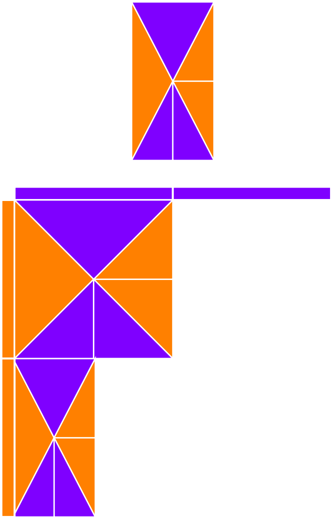

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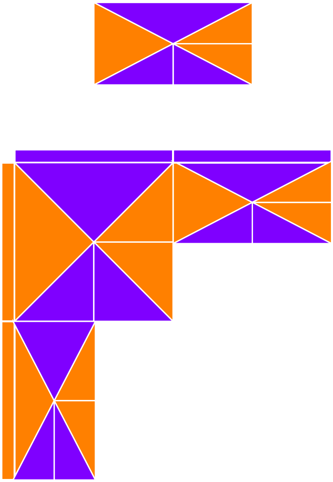



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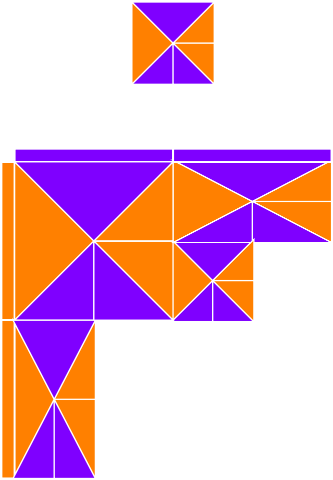

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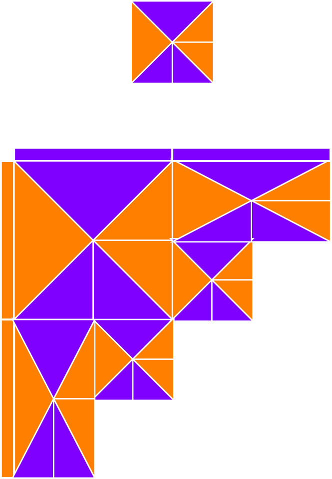



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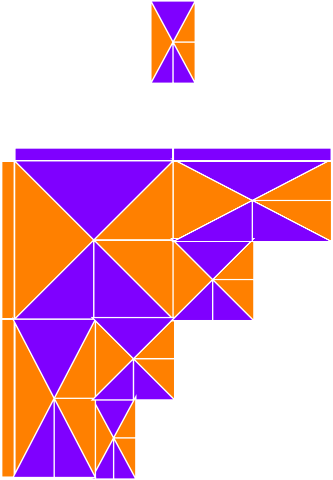

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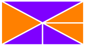

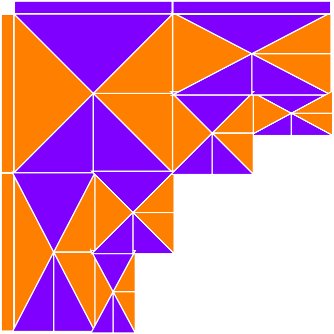

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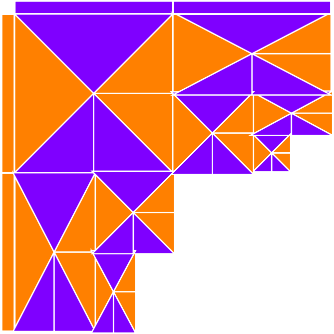



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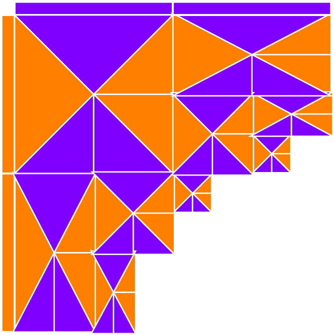

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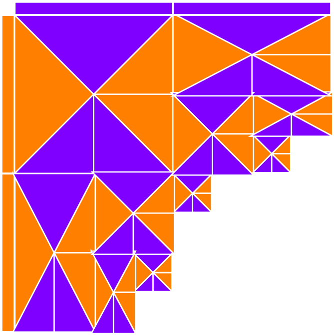



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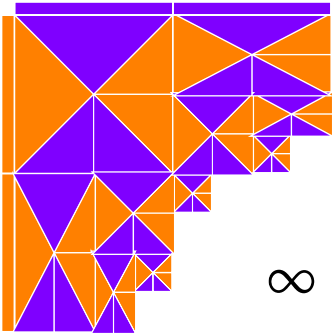

# Puzzling tiling questions



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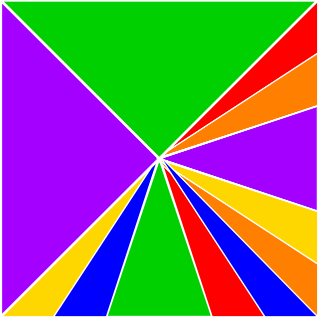
- Do all tiling strategies terminate?
- How many tiles are needed?

# Decreasing tiles (1978–94)



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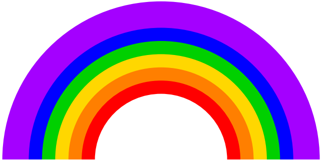

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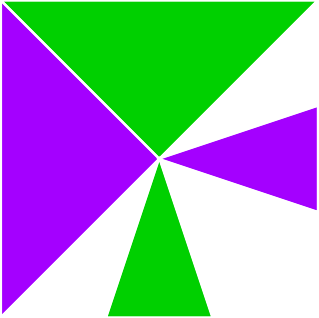

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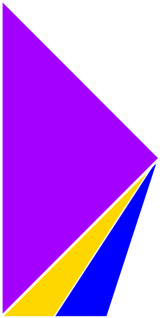

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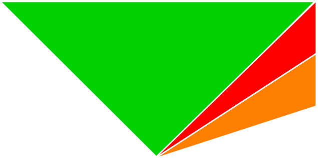

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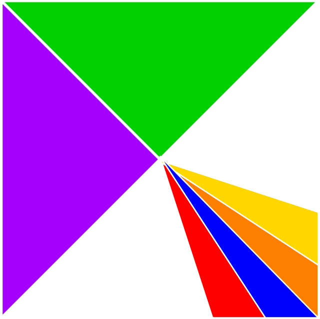

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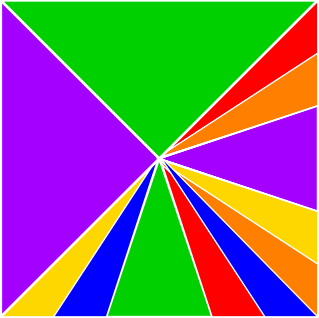

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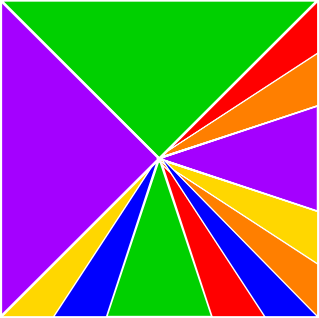



## Definition

set of such tiles decreasing if used colours well-ordered

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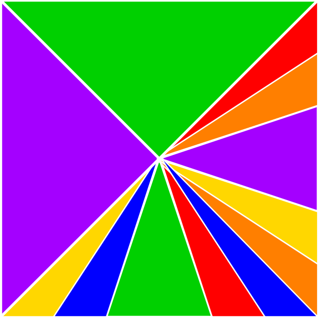

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# Terminating tiling strategy for decreasing tiles

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Issued August 1978.

A note on weak diamond properties.

by

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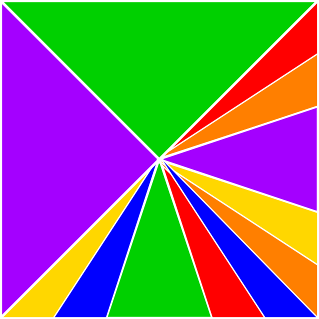
**Theorem**

*if tiles are decreasing, a tiling strategy exists that terminates*

# Decreasing rewrite systems





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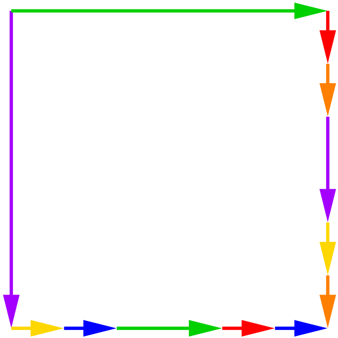



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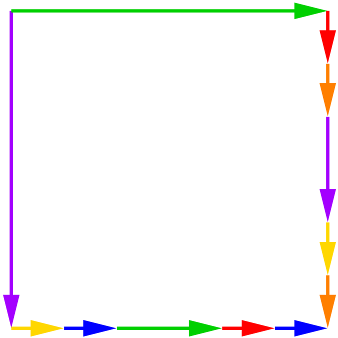



## Theorem

*if rewrite system decreasing, then confluent*

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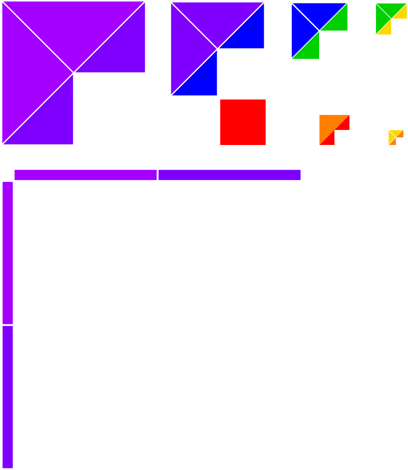

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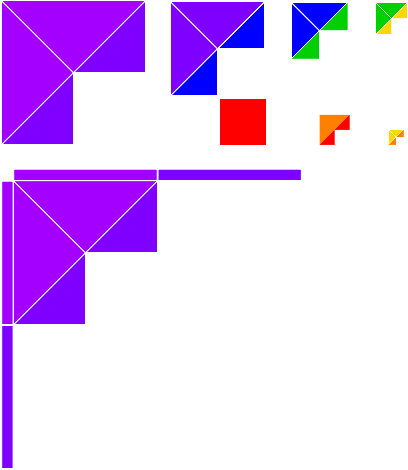

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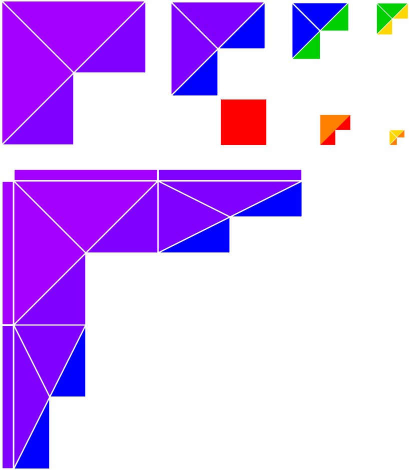

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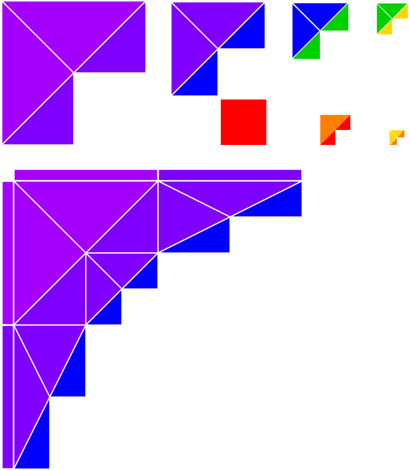

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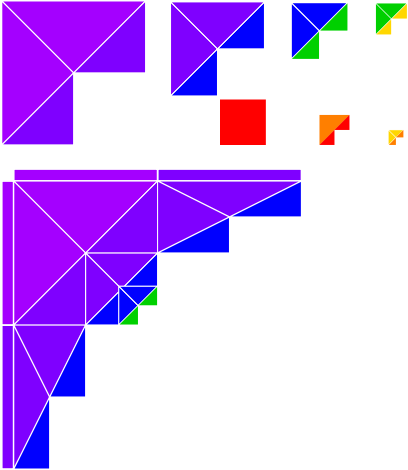



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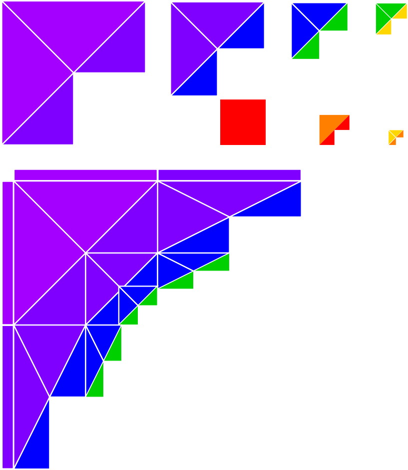

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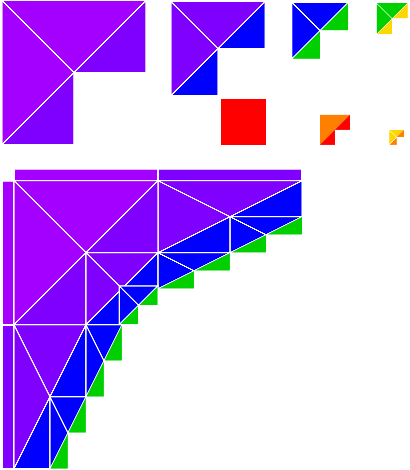

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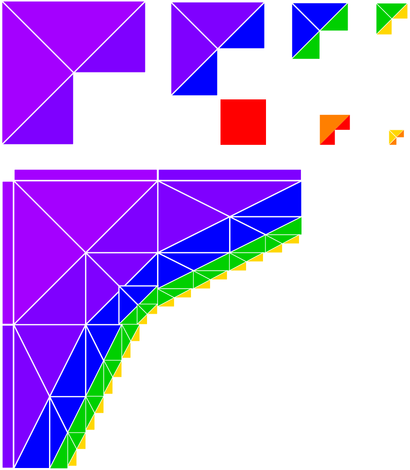



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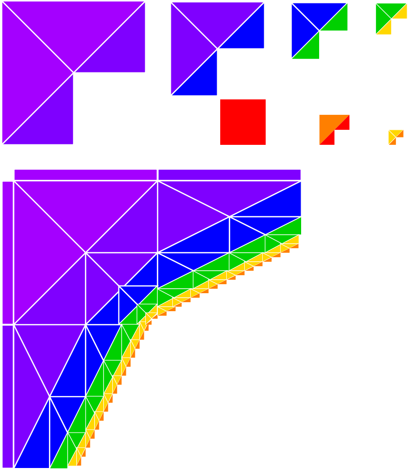

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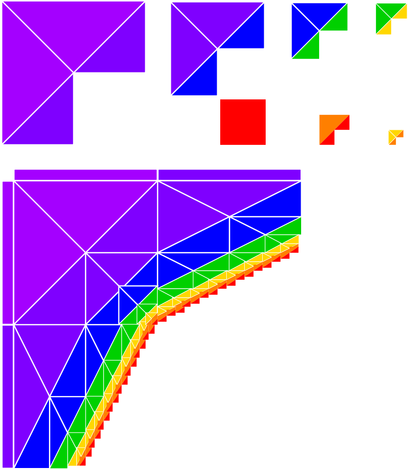

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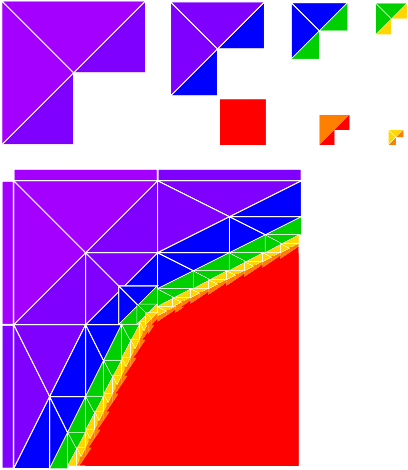
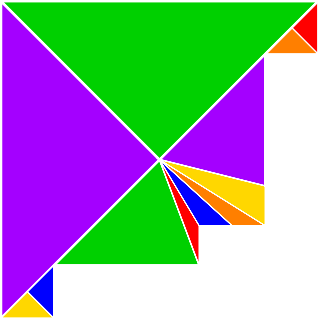


# Decreasing converted rewrite systems





**Theorem**

*if tiles decreasing converted, a tiling strategy exists that terminates*

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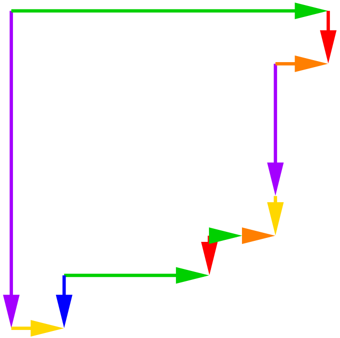

Theorem

*if rewrite system decreasing converted, then confluent*

## $\exists \rightarrow \forall$



Given set of decreasing tiles:

- Previous work: terminating tiling strategy exist

# $\exists \to \forall$

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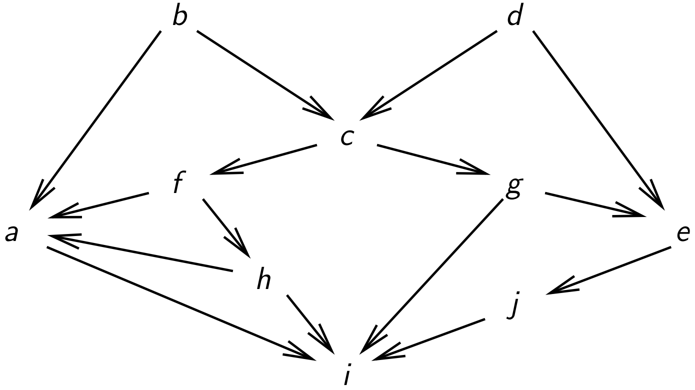

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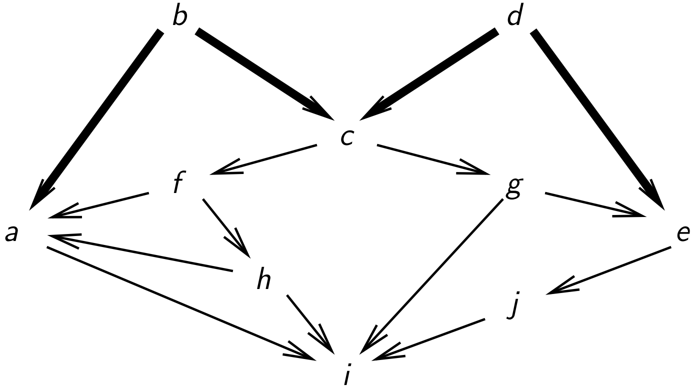


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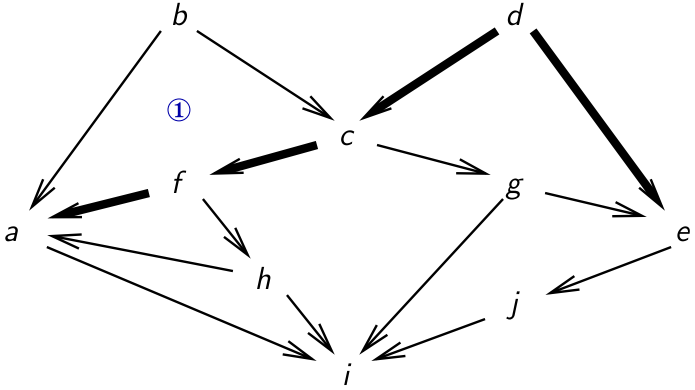


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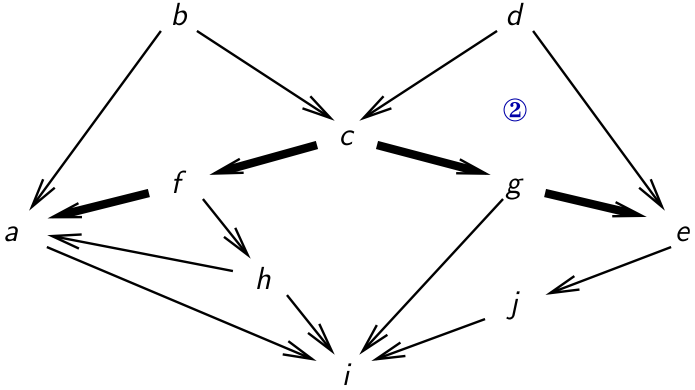
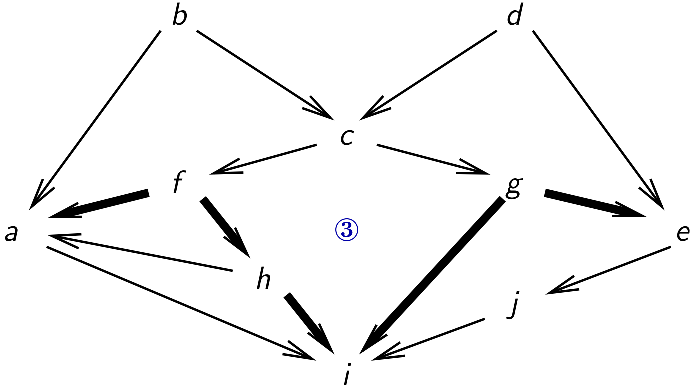
# Transforming conversions




$a$ convertible to $e$

# Transforming conversions

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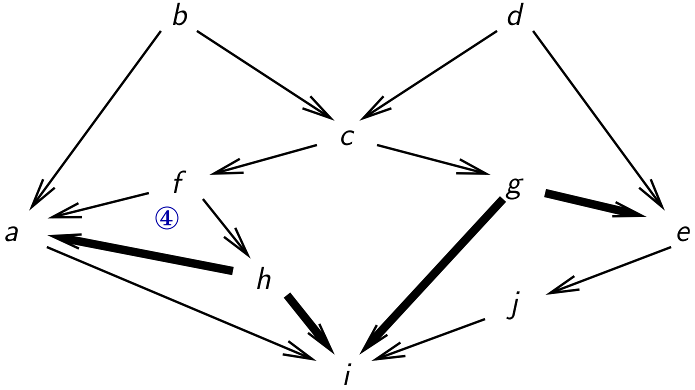


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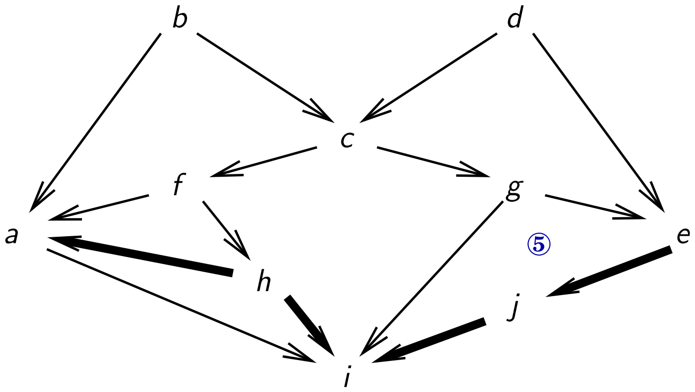


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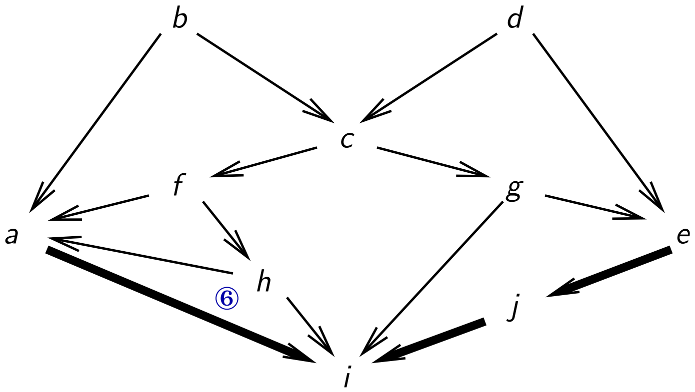
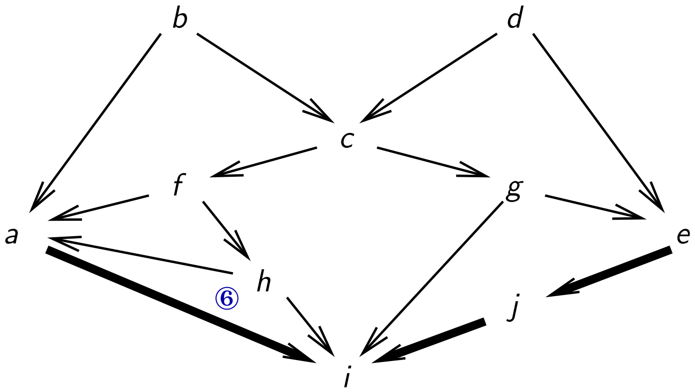



$a$ convertible to $e$ by rewrite proof

# Transforming conversions




why do these transformations terminate?

# Equational logic on nullary symbols (constants)

$$\frac{a \to b}{\mathsf{a}= \mathsf{b}}\,(\mathit{step}) \qquad \frac{}{\mathsf{a}= \mathsf{a}}\,(\mathsf{e}) \qquad \frac{\mathsf{a}= \mathsf{b}}{\mathsf{b}= \mathsf{a}}\,(\text{-}1) \qquad \frac{\mathsf{a}= \mathsf{b} \quad \mathsf{b}= \mathsf{c}}{\mathsf{a}= \mathsf{c}}\,(\cdot)$$

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no derivation rules for congruence or substitution

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Theorem ((sub)Birkhoff)

*abstract rewriting is logical, that is, = coincides with* $\leftrightarrow^*$

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- example: proof term $m^{-1} \cdot (\ell \cdot (k^{-1} \cdot m))$ represents conversion $a \leftarrow_m b \rightarrow_\ell c \leftarrow_k a \rightarrow_m b$
- equip proof terms with terminating rewrite relation compatible with decreasingness

# Conversions → proof terms → involutive monoid

## Definition

set with

- associative binary operation $\cdot$
- identity element $e$
- involutive anti-automorphism $^{-1}$

$$
\begin{array}{rcll}
(a \cdot b) \cdot c & = & a \cdot (b \cdot c) & \text{(associative)} \\
a \cdot e & = & a & \text{(right identity)} \\
e \cdot a & = & a & \text{(left identity)} \\
(a^{-1})^{-1} & = & a & \text{(involutive)} \\
(a \cdot b)^{-1} & = & b^{-1} \cdot a^{-1} & \text{(anti-automorphic)} \\
\\
\varepsilon^{-1} & = & \varepsilon & \text{(derived)}
\end{array}
$$

# Involutive monoid examples

- $\{*\}$ with binary, nullary, unary constant-$*$ map

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- integers with addition, zero, unary minus

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- $\{*\}$ with binary, nullary, unary constant-$*$ map
- positive rationals with multiplication, one, inverse

# Involutive monoid examples

- $\{*\}$ with binary, nullary, unary constant-$*$ map
- group

# Involutive monoid examples

- $\{*\}$ with binary, nullary, unary constant-$*$ map
- group (examples $(\mathbb{Z}, +, 0, -)$, $(\mathbb{Q}^+, \cdot, 1, \ ^{-1})$)
- natural numbers with addition, zero, identity map

# Involutive monoid examples

- $\{*\}$ with binary, nullary, unary constant-$*$ map
- group (examples $(\mathbb{Z}, +, 0, -)$, $(\mathbb{Q}^+, \cdot, 1, {}^{-1})$)
- multisets with multiset sum, empty multiset, identity map

# Involutive monoid examples

- $\{*\}$ with binary, nullary, unary constant-$*$ map
- group (examples $(\mathbb{Z}, +, 0, -)$, $(\mathbb{Q}^{+}, \cdot, 1, \ ^{-1})$)
- commutative monoid with identity map

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- diagrams of $\diagdown$ with gluing, point, mirroring in vertical axis

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- number pairs with pointwise addition, $(0, 0)$, swapping

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- diagrams of $\setminus$ with gluing, point, mirroring in vertical axis
- number triples with composition given by $(n_1, m_1, k_1) \cdot (n_2, m_2, k_2) = (n_1 + n_2, m_1 + k_1 \cdot n_2 + m_2, k_1 + k_2)$, zero $(0, 0, 0)$, involution $(n, m, k)^{-1} = (k, m, n)$

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$$
\begin{aligned}
& ((n_1, m_1, k_1) \cdot (n_2, m_2, k_2)) \cdot (n_3, m_3, k_3) \\
= \quad & (n_1 + n_2, m_1 + k_1 \cdot n_2 + m_2, k_1 + k_2) \cdot (n_3, m_3, k_3) \\
= \quad & (n_1 + n_2 + n_3, m_1 + k_1 \cdot n_2 + m_2 + (k_1 + k_2) \cdot n_3 + m_3, k_1 + k_2 + k_3) \\
= \quad & (n_1 + n_2 + n_3, m_1 + k_1 \cdot (n_2 + n_3) + m_2 + k_2 \cdot n_3 + m_3, k_1 + k_2 + k_3) \\
= \quad & (n_1, m_1, k_1) \cdot (n_2 + n_3, m_2 + k_2 \cdot n_3 + m_3, k_2 + k_3) \\
= \quad & (n_1, m_1, k_1) \cdot ((n_2, m_2, k_2) \cdot (n_3, m_3, k_3))
\end{aligned}
$$



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# Involutive monoid of French strings

## Definition

- French letter is an accented (acute or grave) letter

# Involutive monoid of French strings



## Definition

- French letter is an accented (acute or grave) letter
- juxtaposition ␣  èvèn juxtaposed to ḱńíḱté gives èvèǹḱńíḱté

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- $\widehat{L}$ set of French Strings on $L$ ($\hat{a}$ for either $\grave{a}$ or $\acute{a}$)

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letter markup (representation preserves length,prefix,suffix)

# Boustrophedon

Gortyn code, Crete, 5th century B.C. (wikipedia)

# Boustrophedon



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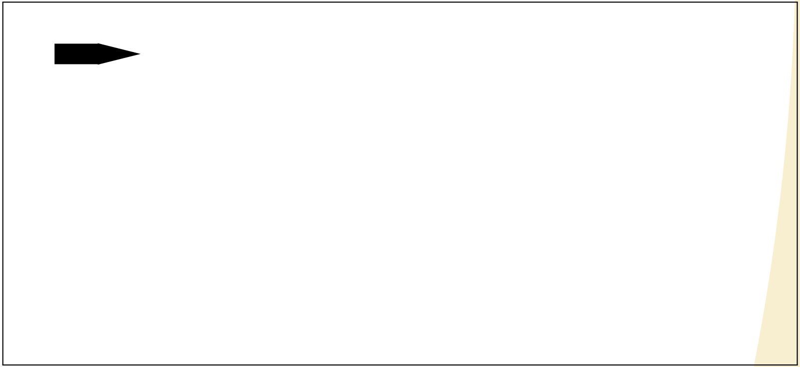
# Boustrophedon



how the cow ploughs



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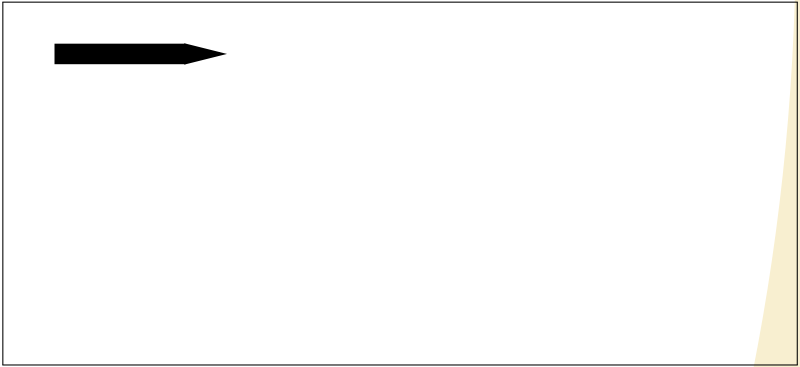
# Boustrophedon



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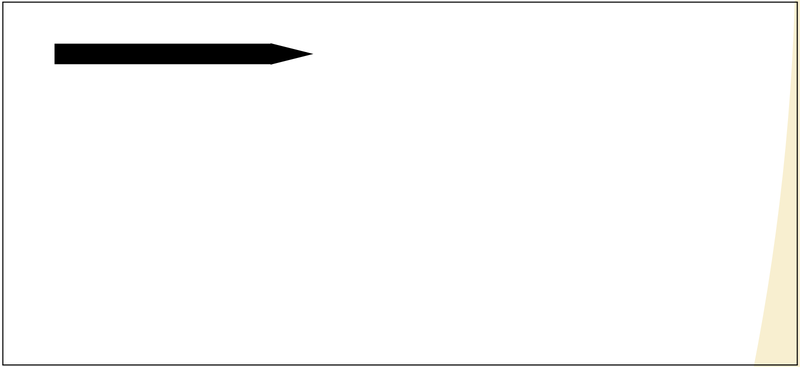

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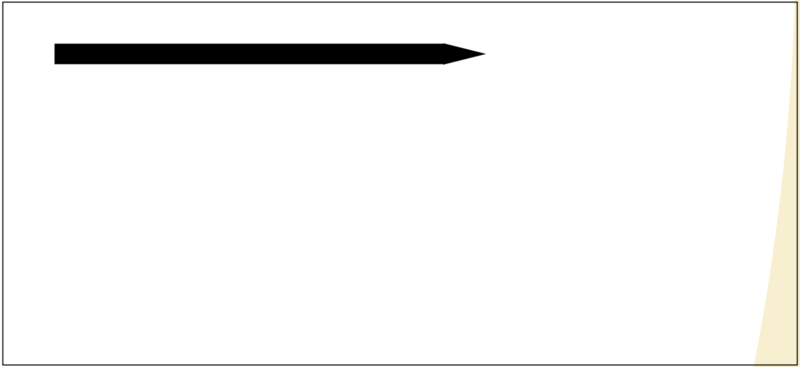

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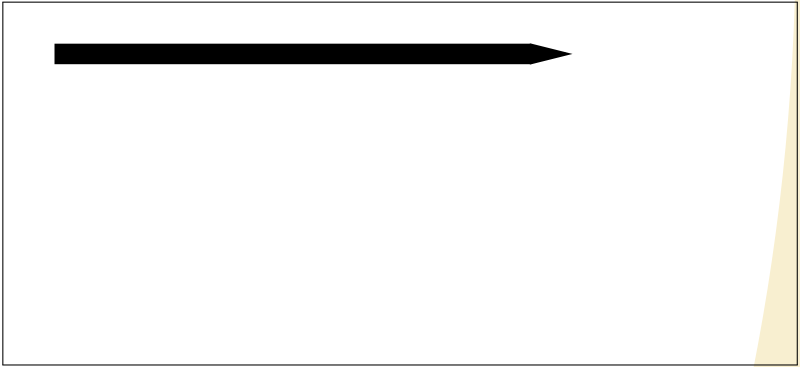

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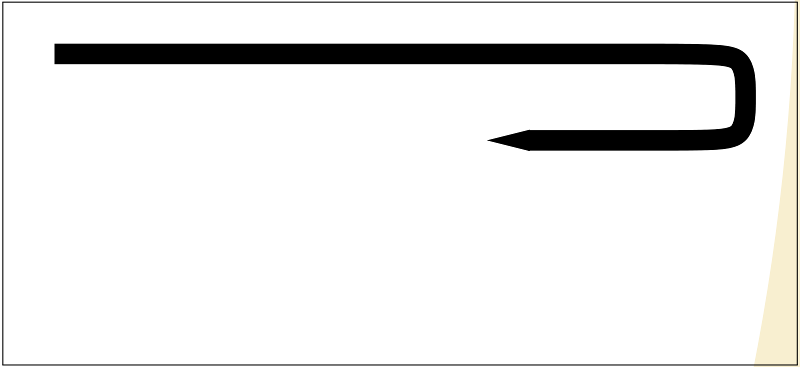

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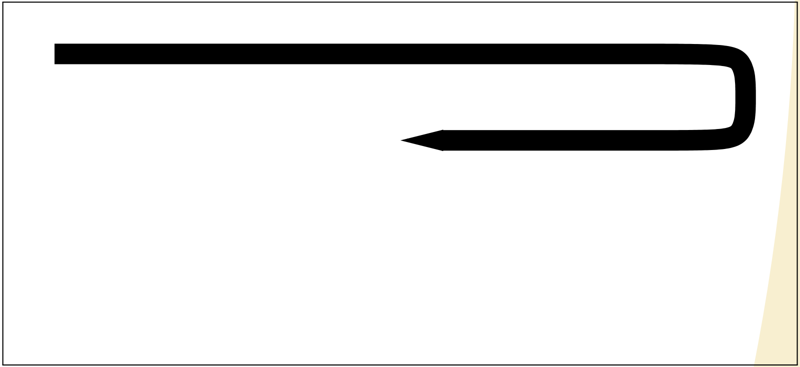

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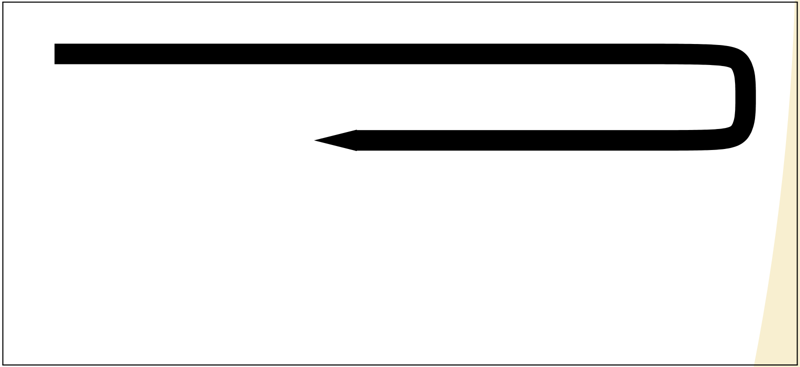

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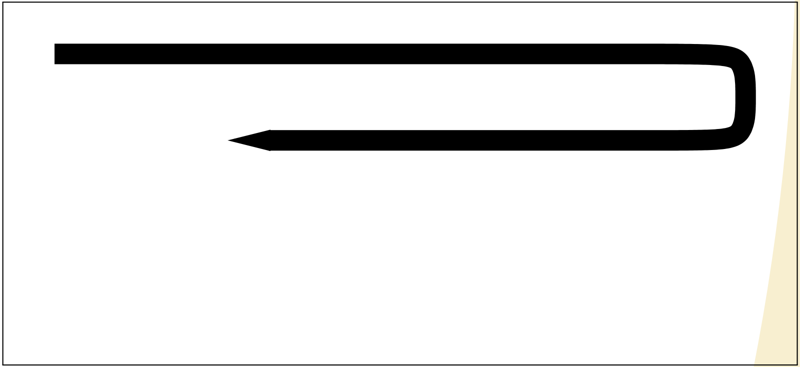

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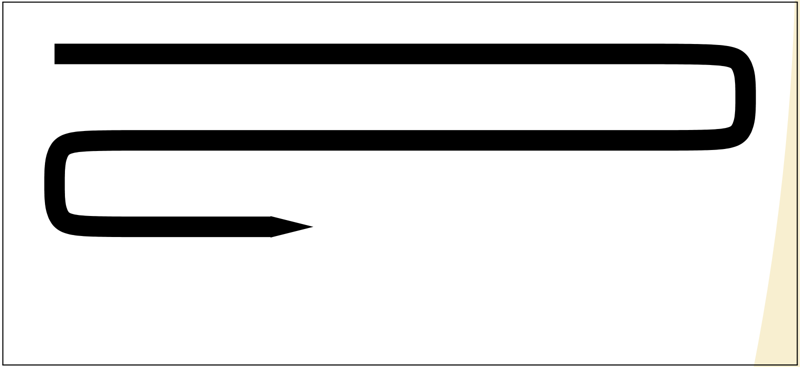

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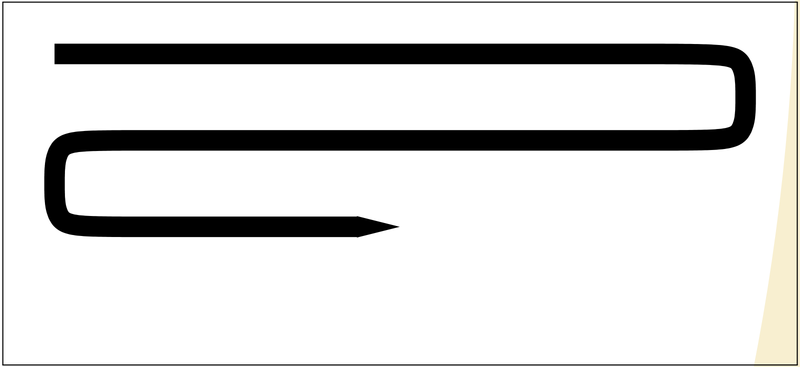

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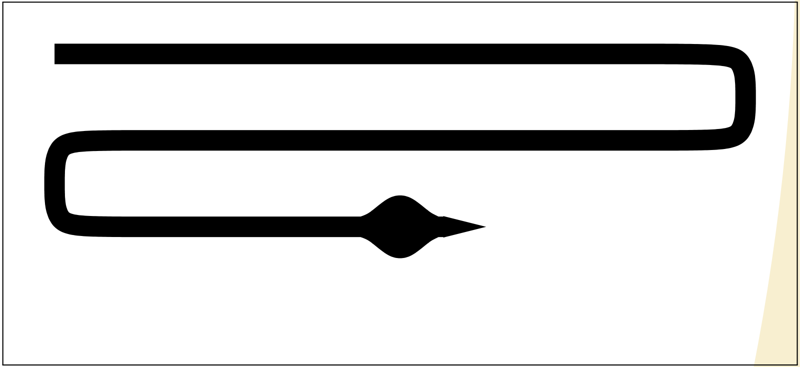

how the plow coughs

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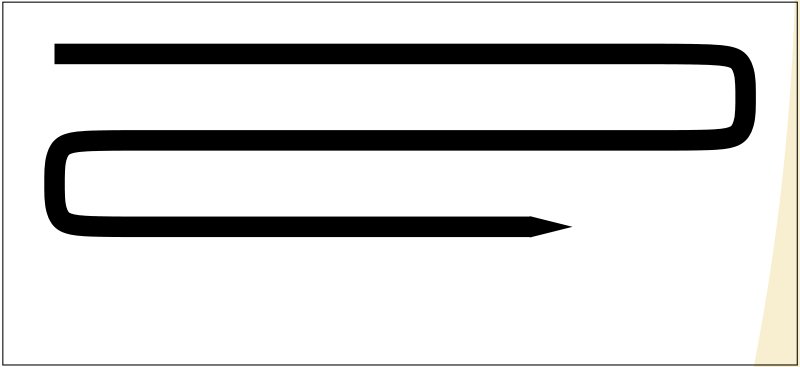

how the cow ploughs

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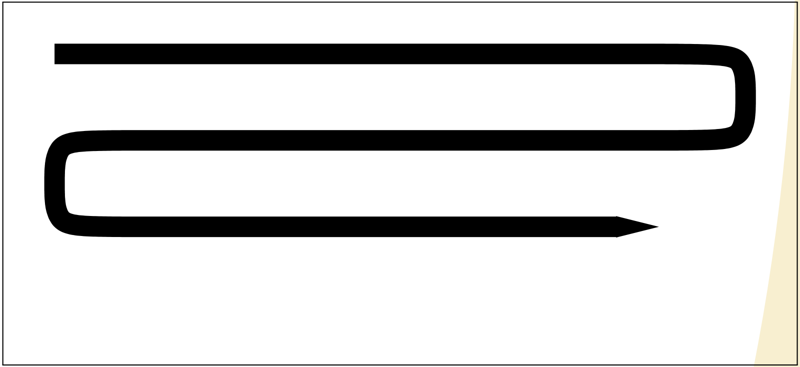

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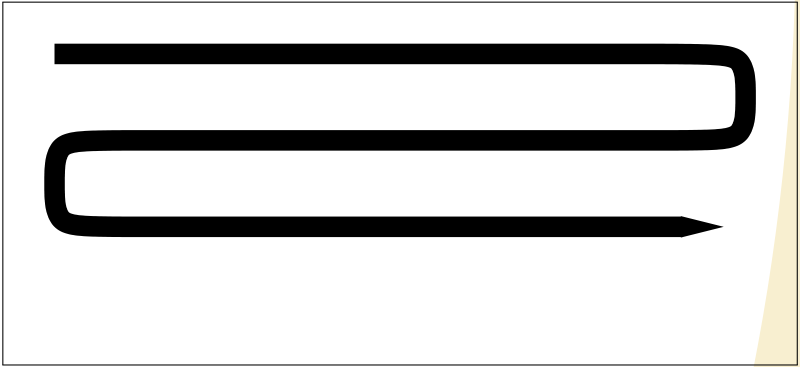

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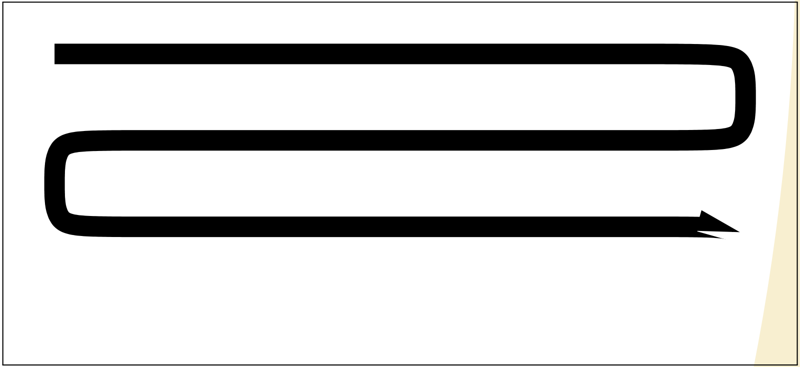

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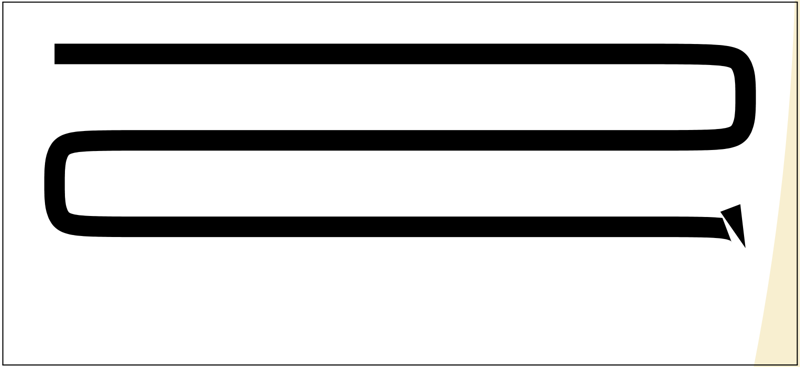

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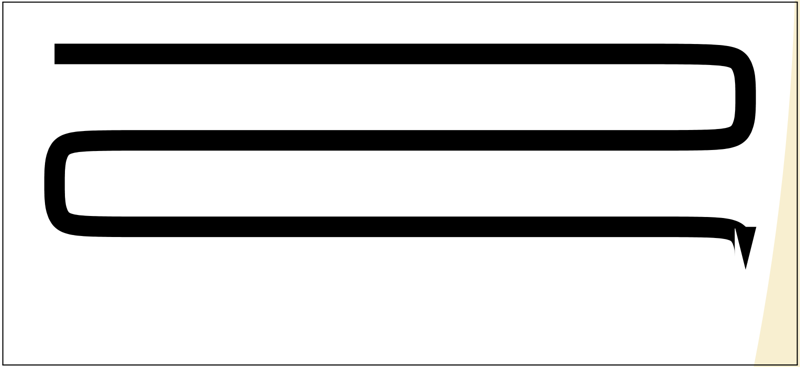

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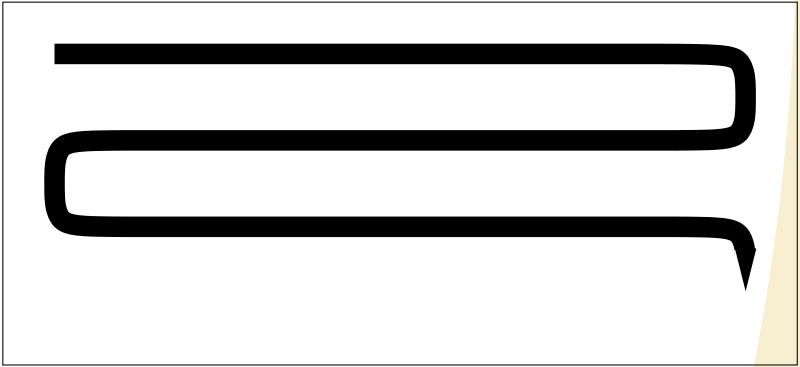

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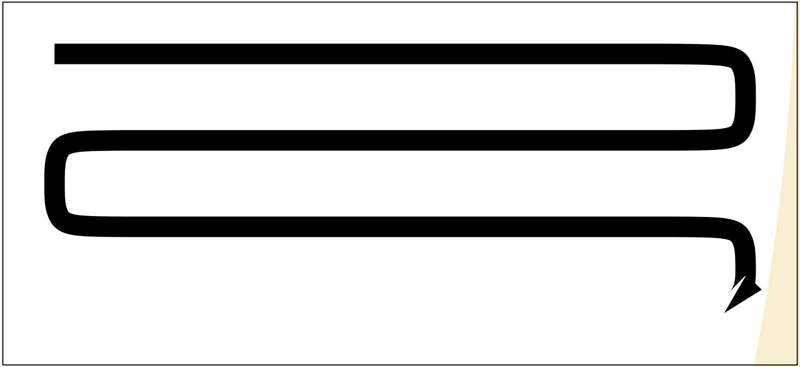

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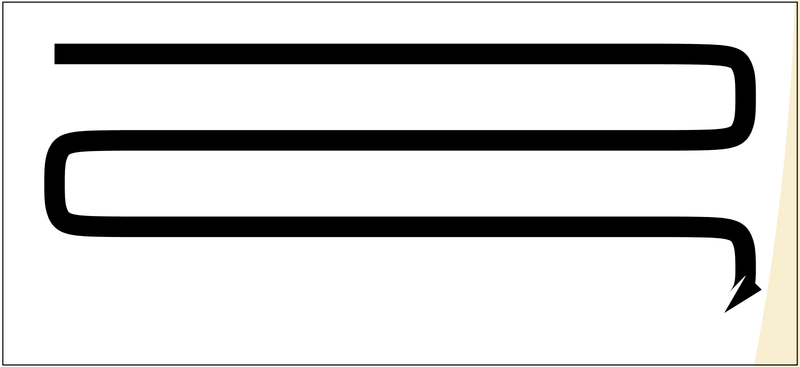

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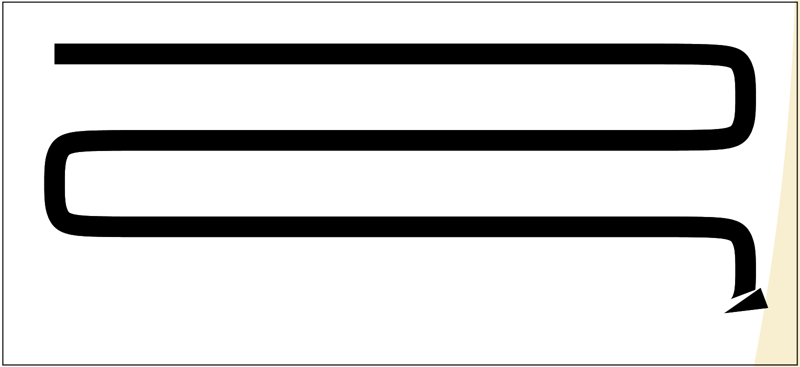

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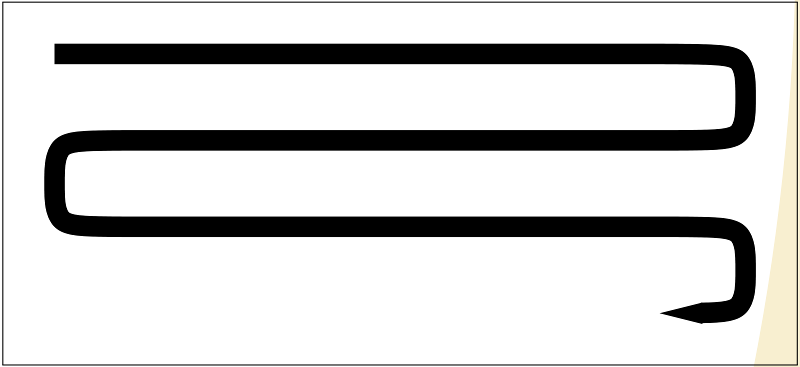

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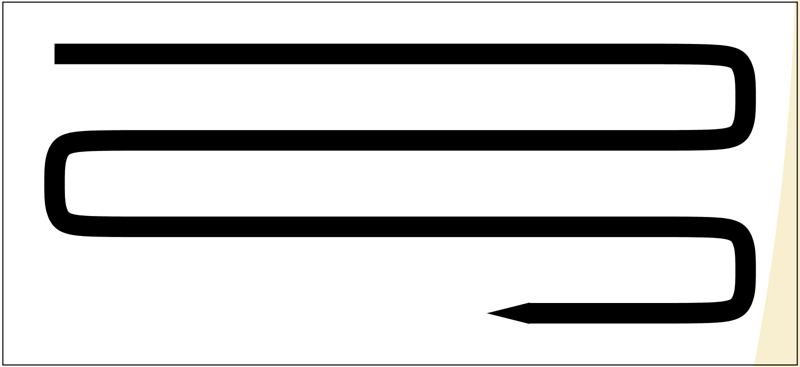

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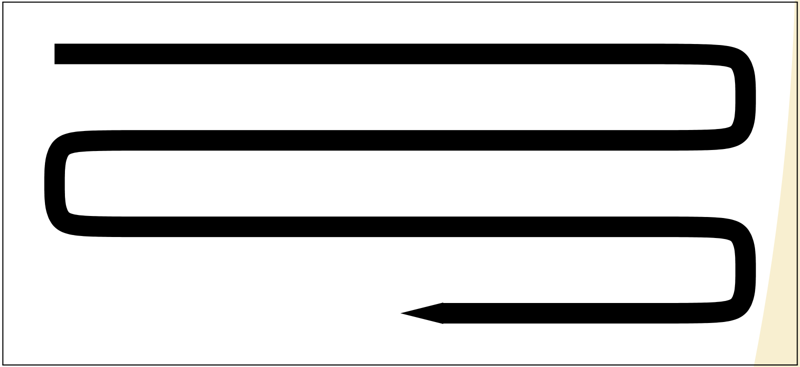

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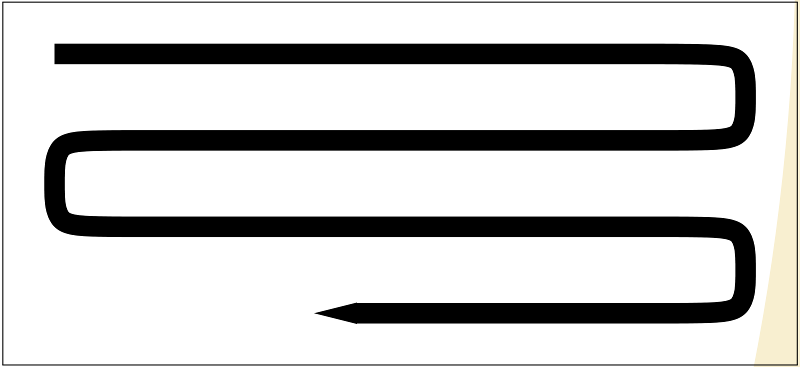

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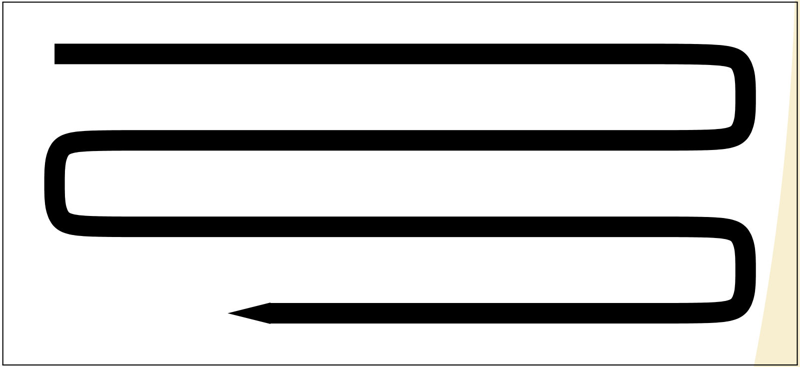

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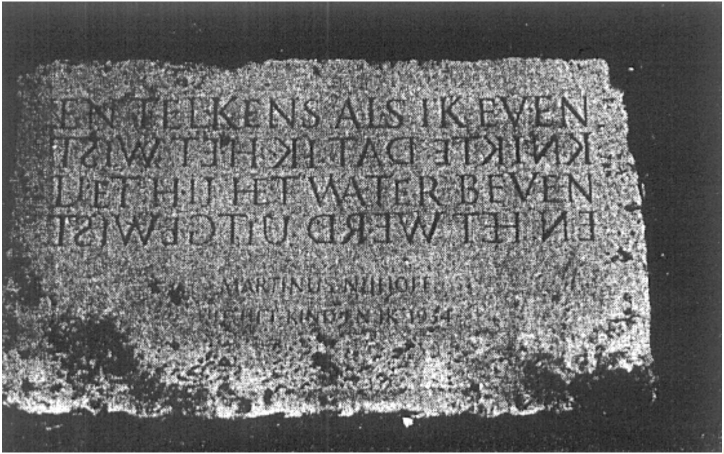
how the cow ploughs

# Boustrophedon






Martinus Nijhoff, Het kind en ik, Nieuwe Gedichten, 1934
(Hortus Botanicus, Universiteitsmuseum Utrecht, next to pond)

# Boustrophedon



EN TELKENS ALS IK EVEN
KNIKTE DAT IK HET WIST
LIET HIJ HET WATER BEVEN
EN HET WERD UITGEWIST

# Involutive monoid homomorphisms



Definition

homomorphism is map preserving operations

Examples

- involutive monoid to itself (identity)

# Involutive monoid homomorphisms



## Definition

homomorphism is map preserving operations

## Examples

- involutive monoid to itself (identity)
- French strings → number pairs (grave,acute)
  ćèńàr̀ $\mapsto (3, 2)$

# Involutive monoid homomorphisms



### Definition

homomorphism is map preserving operations

### Examples

- involutive monoid to itself (identity)
- number pairs → natural numbers (sum)
  $(3, 2) \mapsto 5$

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# Involutive monoid homomorphisms



## Definition

homomorphism is map preserving operations

## Examples

- involutive monoid to itself (identity)
- French strings → natural numbers (length)
  composition of previous two

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homomorphism is map preserving operations

### Examples

- involutive monoid to itself (identity)
- French strings → natural numbers (length)
- French strings → multisets (letters)
  $\acute{b}\acute{a}\acute{r}\grave{b}\grave{a}\acute{r}\acute{o} \mapsto [a, a, b, b, o, r, r]$

# Involutive monoid homomorphisms

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## Examples

- involutive monoid to itself (identity)
- French strings → natural numbers (length)
- French strings → multisets (letters)
- French strings → diagrams

ćèńàr̀ ↦ 

# Involutive monoid homomorphisms



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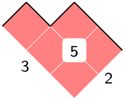
- involutive monoid to itself (identity)
- French strings → natural numbers (length)
- French strings → multisets (letters)
- diagrams → triples

 $\mapsto (3, 5, 2)$ cf. 

# Involutive monoid homomorphisms



## Definition

homomorphism is map preserving operations

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- involutive monoid to itself (identity)
- French strings → natural numbers (length)
- French strings → multisets (letters)
- French strings → triples (area)
  composition of previous two

# Free involutive monoid on generators



**Theorem**

*French strings on $L$ give free involutive monoid on $L$*

# Free involutive monoid on generators



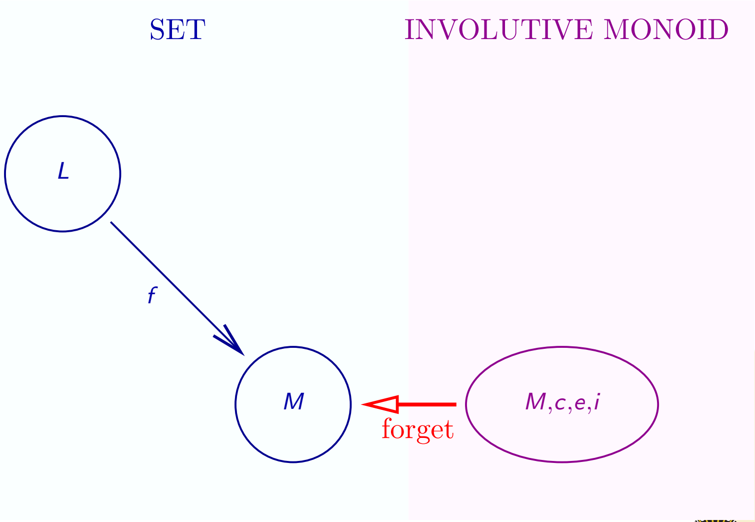
**Theorem**
*French strings on $L$ give free involutive monoid on $L$*

French string : conversion = string : reduction

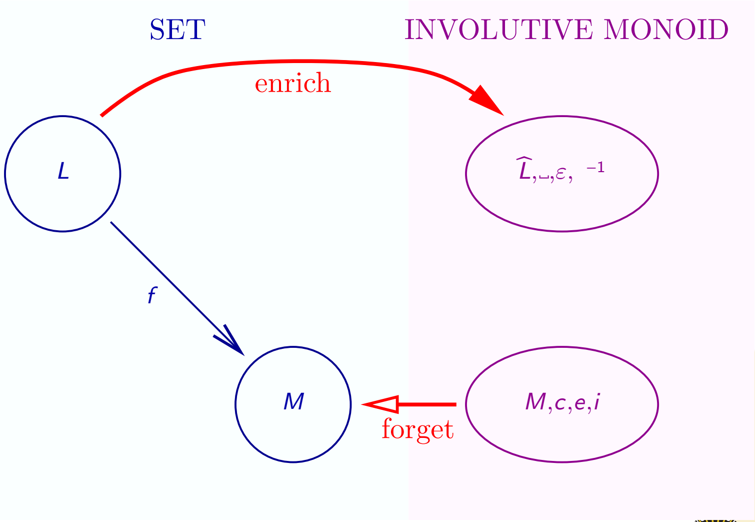
# Freeness of involutive monoid of French Strings

$$\begin{tikzcd} L \arrow[dr, "f"'] & \\ & M \end{tikzcd}$$

# Freeness of involutive monoid of French Strings



# Freeness of involutive monoid of French Strings



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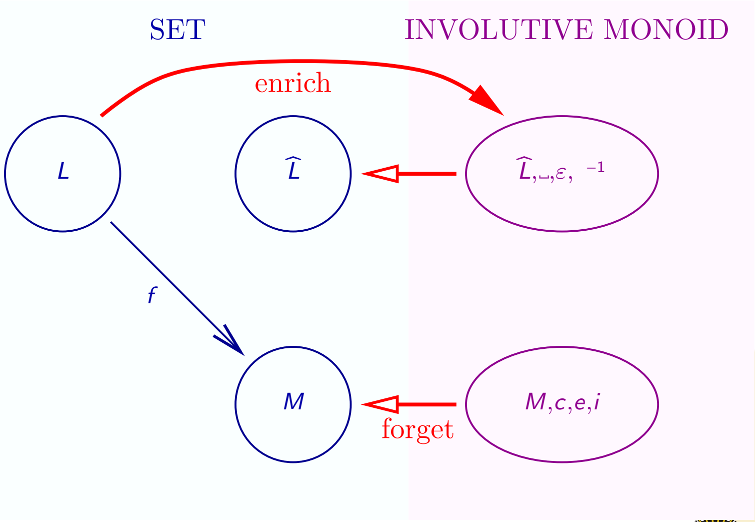
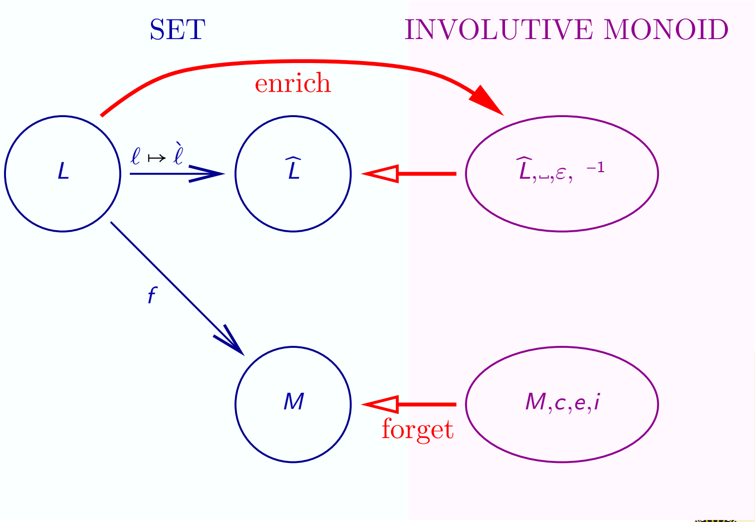
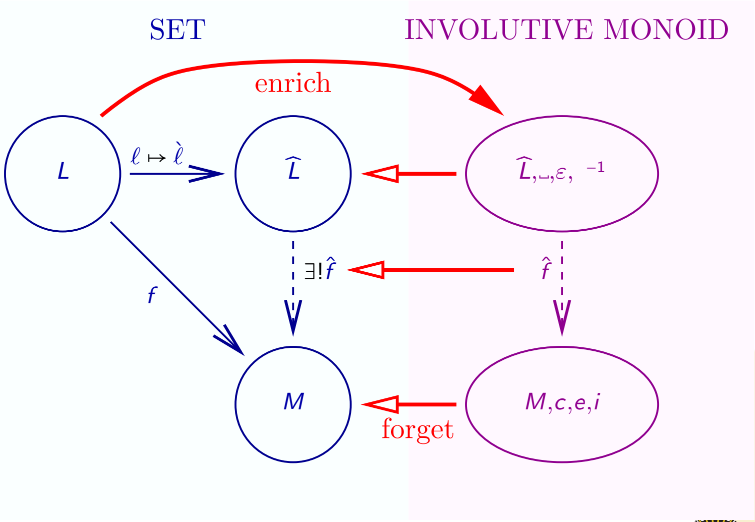
# Freeness of involutive monoid of French Strings



# Freeness of involutive monoid of French Strings



# Freeness of involutive monoid of French Strings



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# Free involutive monoid on generators

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# Free involutive monoid on generators

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## Proof.

$\widehat{L}$ in bijection via $\grave{\ell} \mapsto \ell$, with union of $\{e\}$ and

$$N ::= \ell \mid i(\ell) \mid c(\ell, N) \mid c(i(\ell), N)$$

# Free involutive monoid on generators

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*French strings on $L$ give free involutive monoid on $L$*

Proof.

$\widehat{L}$ in bijection via $\grave{\ell} \mapsto \ell$, with union of $\{e\}$ and

$$N ::= \ell \mid i(\ell) \mid c(\ell, N) \mid c(i(\ell), N)$$

$N$ set of normal forms on $L$ for TRS completing axioms

$$\begin{aligned} c(c(x,y),z) &\to c(x,c(y,z)) \\ c(x,e) &\to x \\ c(e,x) &\to x \\ i(i(x)) &\to x \\ i(c(x,y)) &\to c(i(y),i(x)) \\ i(e) &\to e \end{aligned}$$

# Involutive monoid on French terms $L^\sharp$

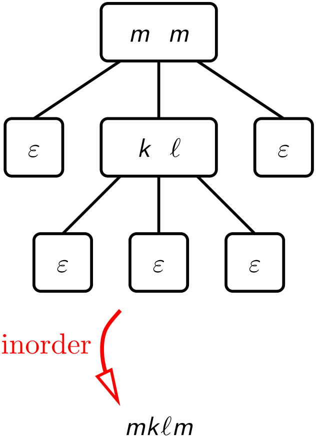
## Definition

certain terms on certain French strings

# Involutive monoid on French terms $L^\sharp$

## Definition

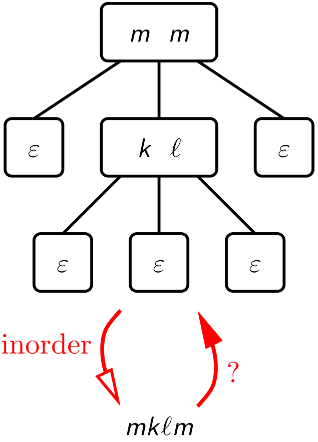
terms on strings



# Involutive monoid on French terms $L^\sharp$

## Definition

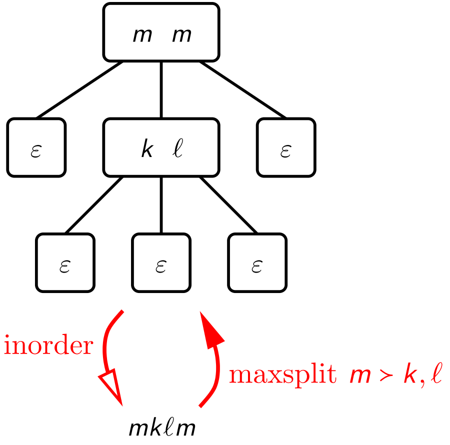
terms on strings



# Involutive monoid on French terms $L^\sharp$

## Definition

terms on strings on $>$-ordered letters



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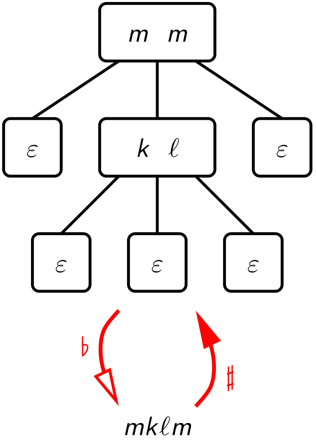
# Involutive monoid on French terms $L^\sharp$

## Definition

terms on strings on $>$-ordered letters



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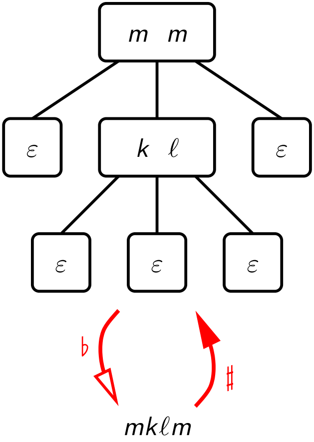
## Definition

terms on strings on $\succ$-ordered letters where $\flat \circ \sharp$ identity



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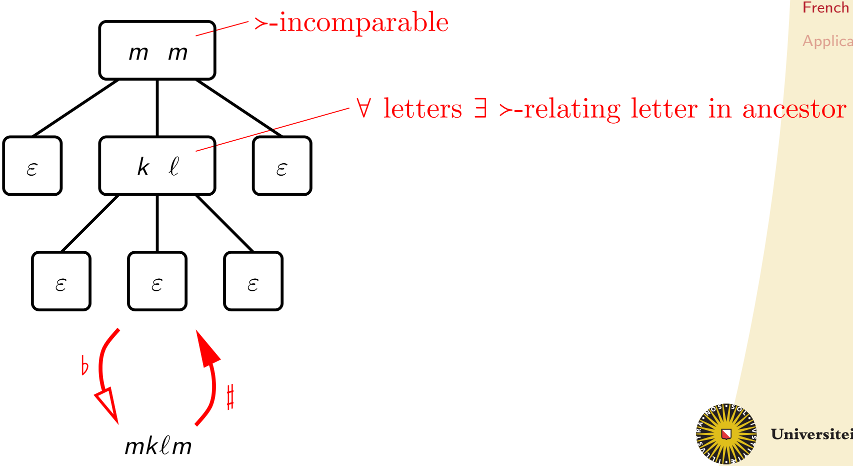

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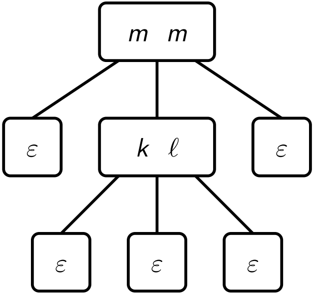
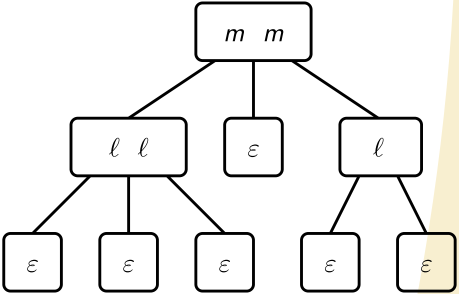


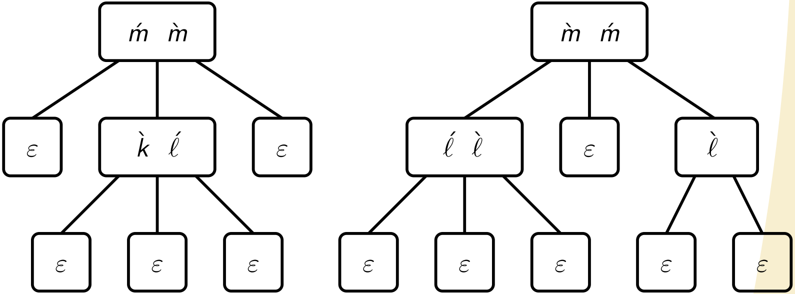




$mk\ell m$ $\qquad$ $\ell\ell mm\ell$

# Involutive monoid on French terms $L^\sharp$

## Definition

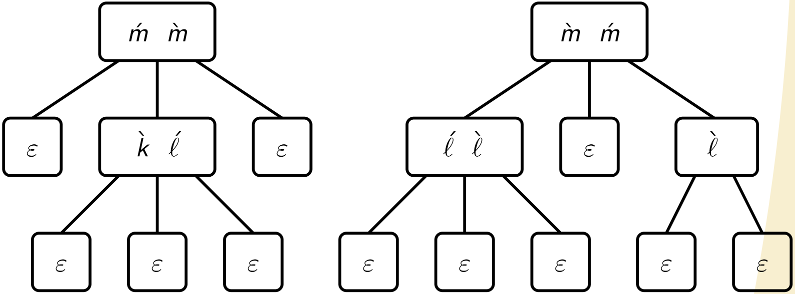
terms on French strings on $\succ$-ordered letters where $\flat \circ \sharp$ identity
operations on $L^\sharp$ defined via $\widehat{L}$, e.g. $t \cdot u = (t^\flat u^\flat)^\sharp$




$\acute{m}\grave{k}\acute{\ell}\grave{m}$ $\acute{\ell}\grave{\ell}\,\grave{m}\acute{m}\grave{\ell}$

# A well-founded order on French terms

- (iterative) lexicographic path order based on $>$




$\acute{m}\grave{k}\acute{\ell}\grave{m}$ $\qquad$ $\acute{\ell}\grave{\ell}\,\grave{m}\acute{m}\grave{\ell}$

# A well-founded order on French terms

- (iterative) lexicographic path order based on $>$
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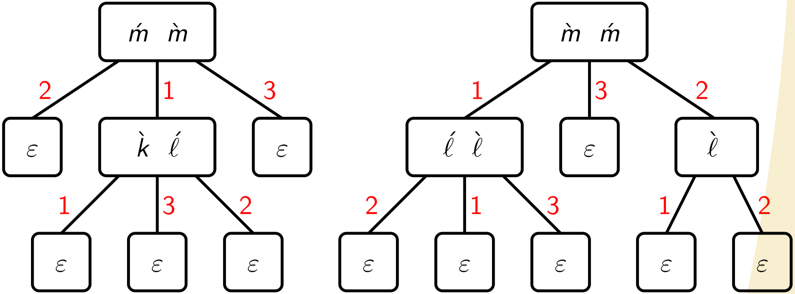


$\acute{m}\grave{k}\acute{\ell}\grave{m}$ $\acute{\ell}\grave{\ell}\ \grave{m}\acute{m}\grave{\ell}$

# A well-founded order on French terms

- (iterative) lexicographic path order based on $>$
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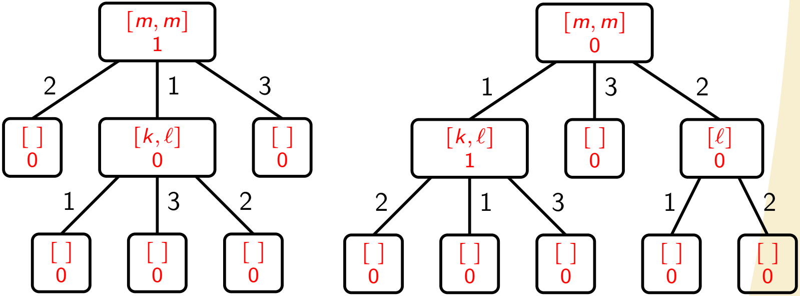
- signature ordered by $\succ = \binom{\succ_{mul}}{>}$ via $\binom{\text{multiset}}{\text{area}}$




$\acute{m}\grave{k}\acute{\ell}\grave{m}$ $\acute{\ell}\grave{\ell}\,\grave{m}\acute{m}\grave{\ell}$

# A well-founded order on French strings/terms

- (iterative) lexicographic path order based on $\succ$
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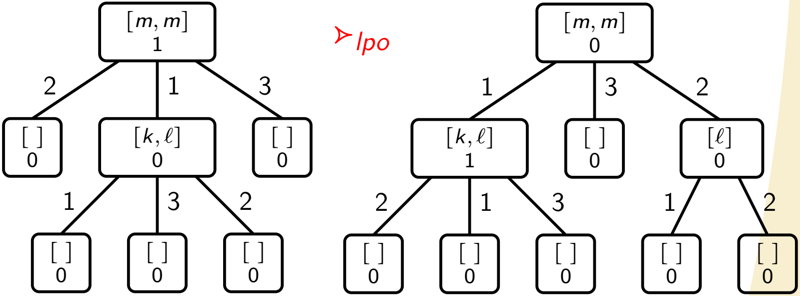
- signature ordered by $\succ = \binom{\succ_{mul}}{>}$ via $\binom{\text{multiset}}{\text{area}}$






$\acute{m}\grave{k}\acute{\ell}\grave{m} \succ_{lpo} \acute{\ell}\grave{\ell}\,\grave{m}\acute{m}\grave{\ell}$

# Properties of $\succ_{lpo}$

- head of term $\succ$-related to heads of all subterms

# Properties of $\succ_{lpo}$

- head of term $\succ$-related to heads of all subterms
- $\succ_{lpo}$ not an ordered monoid: $\grave{k}\grave{\ell} \succ_{lpo} \acute{\ell}$ but $\grave{k}\grave{\ell}\grave{\ell} \not\succ_{lpo} \acute{\ell}\grave{\ell}$

# Properties of $\succ_{lpo}$

- head of term $\succ$-related to heads of all subterms
- $\succ_{lpo}$ not an ordered monoid
- $s\hat{\ell}r \succ_{lpo} s\{\ell\gt\}r$ (in EBNF { } is arbitrary repetition)

# Properties of $\succ_{lpo}$

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- $s\hat{\ell}r \succ_{lpo} s\{\ell>\}r$

Proof.

induction on length $sr$, cases whether $\ell$ is $\succ$-maximal in $s\hat{\ell}r$

yes decrease in multiset of head

no induction on substring/term $\hat{\ell}$ is in □

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- $s\acute{\ell}\grave{m}r \succ_{lpo} s\{\ell\succ\}[\grave{m}]\{\ell, m\succ\}[\acute{\ell}]\{m\succ\}r$ ([ ] is option)

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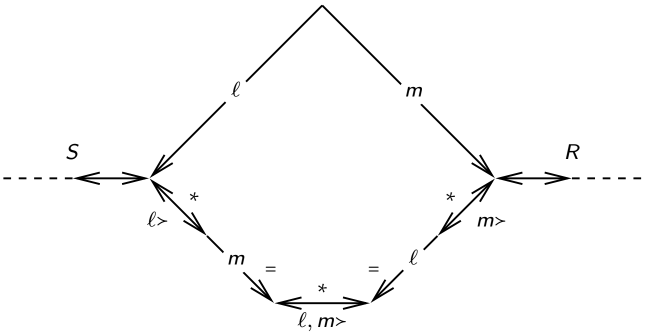
- $s\acute{\ell}\grave{m}r \succ_{lpo} s\{\ell\succ\}[\grave{m}]\{\ell, m\succ\}[\acute{\ell}]\{m\succ\}r$

Proof.
induction on length $sr$, cases whether $\ell, m$ are $\succ$-maximal in $s\acute{\ell}\grave{m}r$

both decrease in area of head
$\acute{\ell}$ decrease in the substring/term to the right of $\acute{\ell}$
$\grave{m}$ decrease in the substring/term to the left of $\grave{m}$
neither induction on substring/term $\acute{\ell}\grave{m}$ is in $\square$

# Filling in locally decreasing diagram decreases

## Theorem



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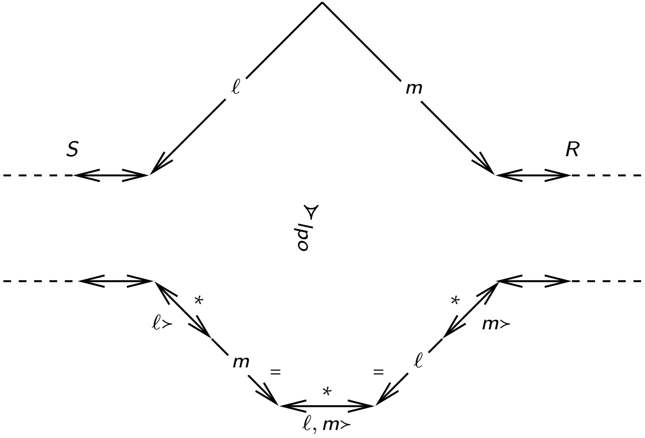
# Filling in locally decreasing diagram decreases

## Theorem



# Filling in locally decreasing diagram decreases

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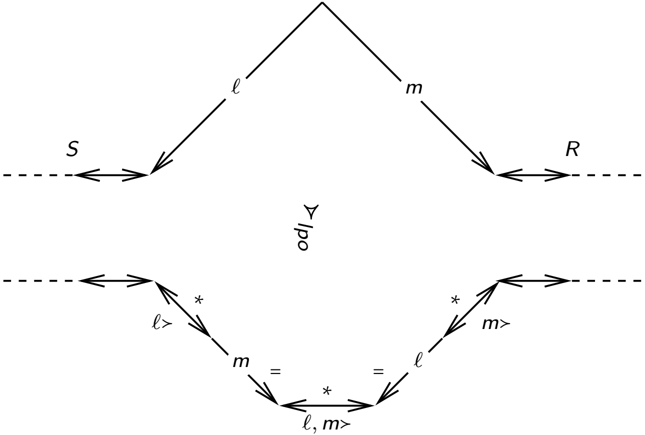
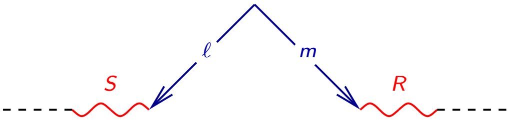
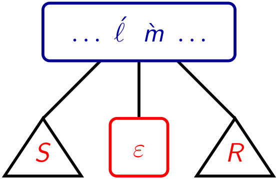



## Proof.

$s\acute{\ell}\grave{m}r \succ_{lpo} s\{\ell\succ\}[\grave{m}]\{\ell, m\succ\}[\acute{\ell}]\{m\succ\}r$

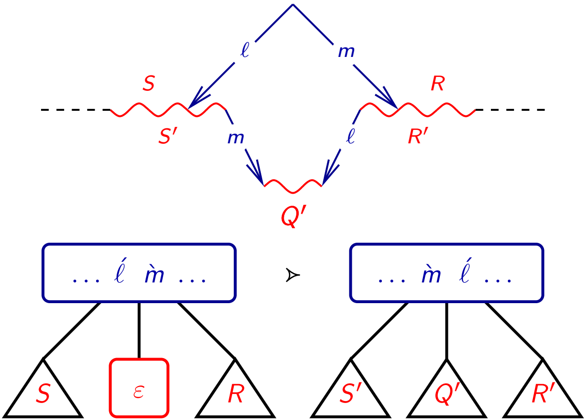
# Idea: $\succ$-maximal steps modulo non-$\succ$-maximal steps




case 1: local confluence peak of $\succ$-maximal steps

# Idea: $\succ$-maximal steps modulo non-$\succ$-maximal steps




area decrease



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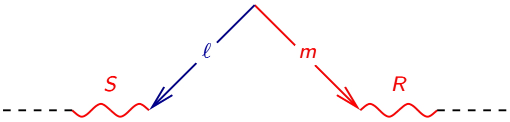
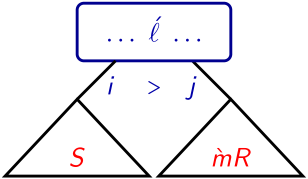
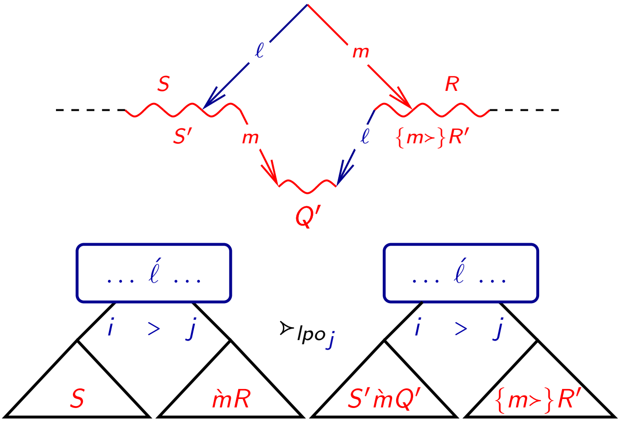
# Idea: ≻-maximal steps modulo non-≻-maximal steps



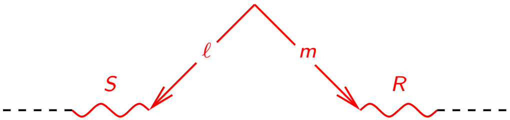

case 2: local coherence peak of ≻-maximal and non-≻-maximal step

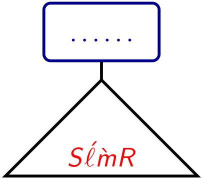
# Idea: $\succ$-maximal steps modulo non-$\succ$-maximal steps




decrease in $j$th argument, lexicographically before $i$th

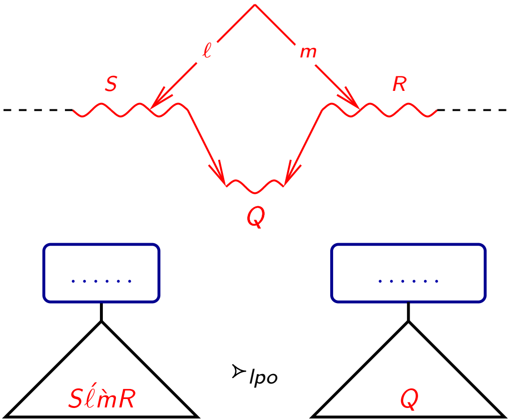
# Idea: ≻-maximal steps modulo non-≻-maximal steps



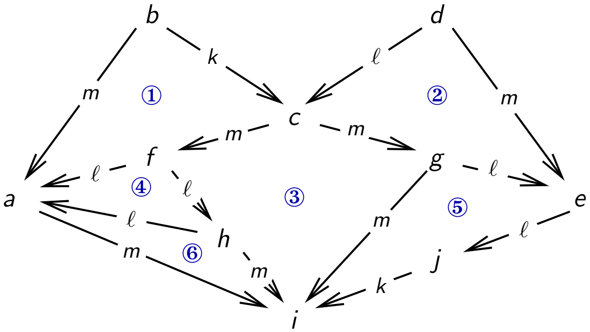

case 3: local modulo peak of non-≻-maximal steps

# Idea: $\succ$-maximal steps modulo non-$\succ$-maximal steps



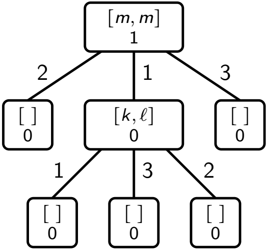

decrease in argument both steps are in

# $\succ_{lpo}$ at work



# Filling in local diagrams ①

$$\begin{tikzcd}[column sep=small]
 & b \arrow[dl, "m"'] \arrow[dr, "k"] & & d \arrow[dl, "\ell"'] \arrow[dr, "m"] & \\
a & & c & & e
\end{tikzcd}$$



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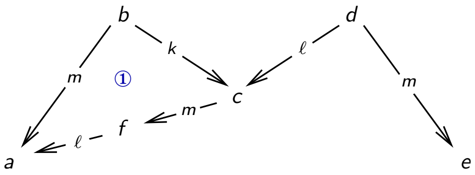


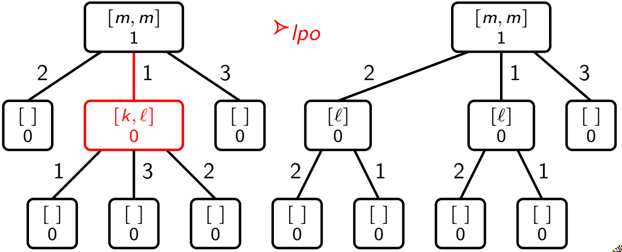

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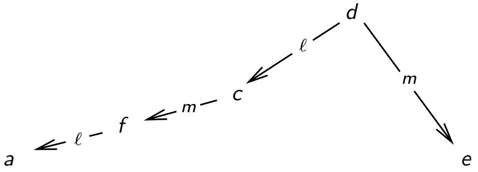


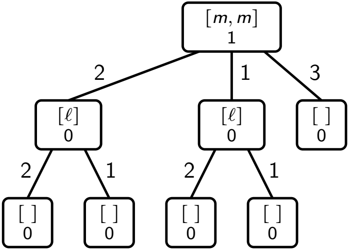




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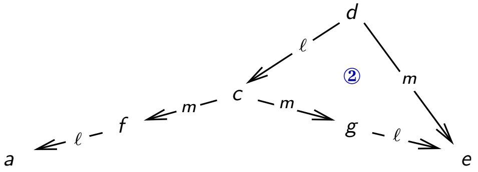
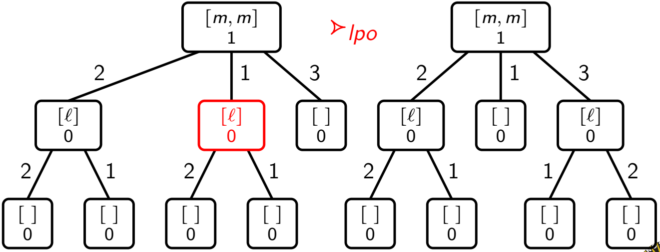
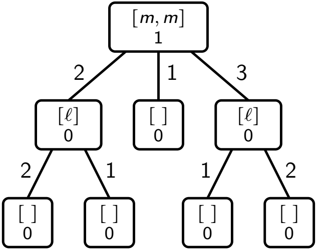


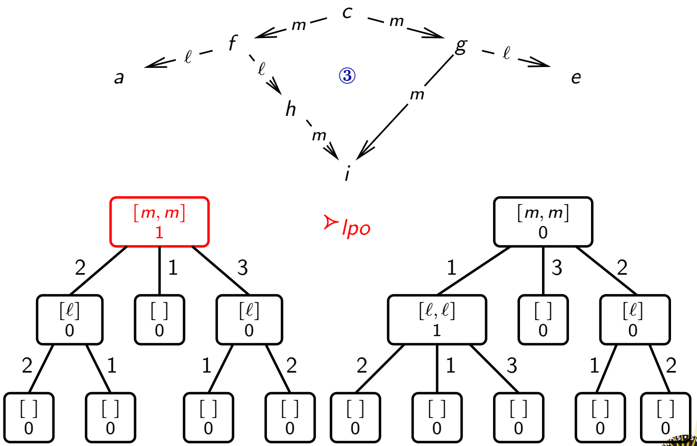



# Filling in local diagrams ③

$$a \xleftarrow{\ell} f \xleftarrow{m} c \xrightarrow{m} g \xrightarrow{\ell} e$$



# Filling in local diagrams ③



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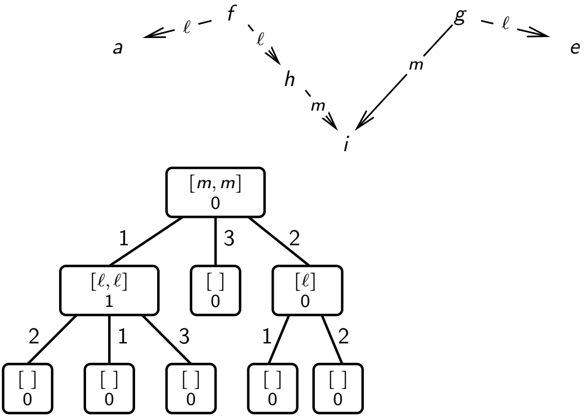

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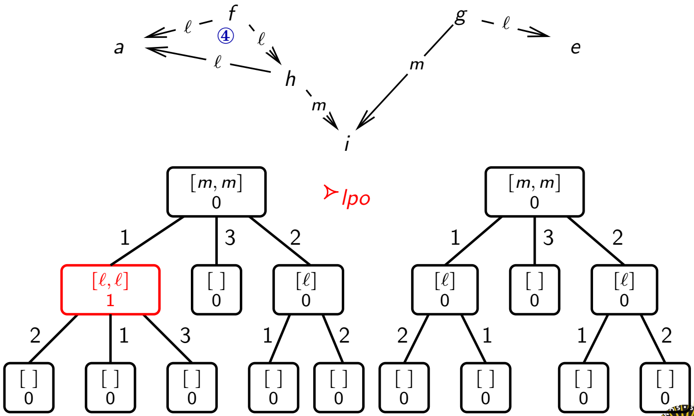




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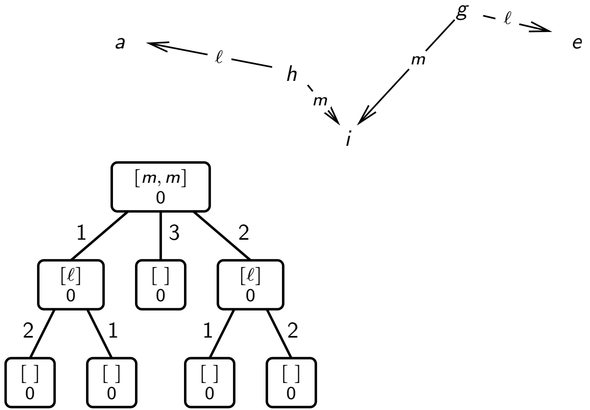
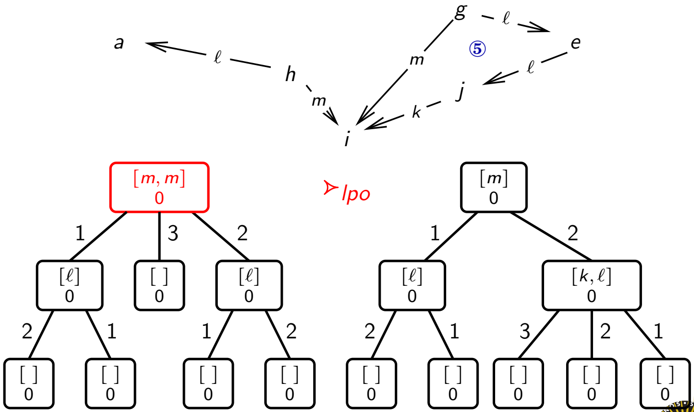
# Filling in local diagrams ⑤



# Filling in local diagrams ⑤



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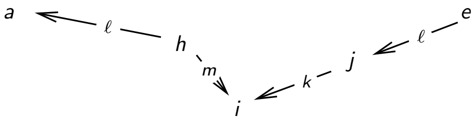


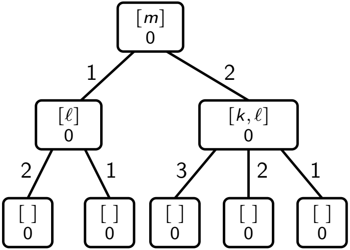

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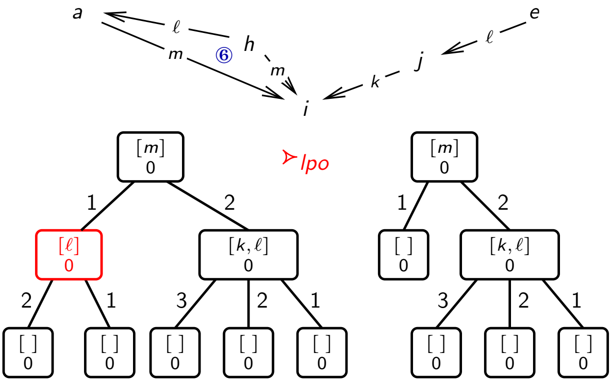

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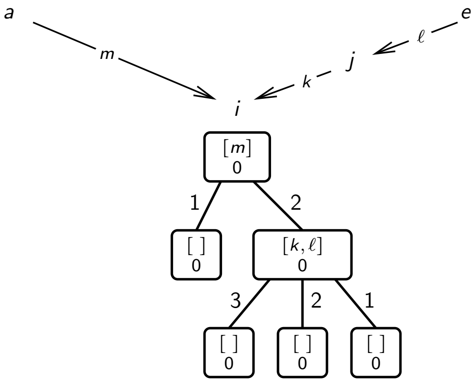

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Adaptations:

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- decreasing diagrams modulo: involutive letters $\dot{\ell}$, i.e. $\dot{\ell}^{-1} = \dot{\ell}$

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- application to factorisation theorems (factorisation is commutation with the inverse, RTA 2012, Beniamino Accattoli)

# Conclusion

- alternative correctness proof of decreasing diagrams (De Bruijn,vO,Klop,de Vrijer,Bezem,Jouannaud)

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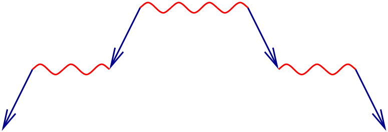

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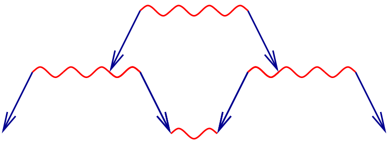

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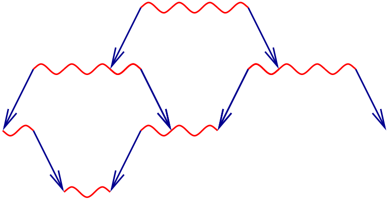

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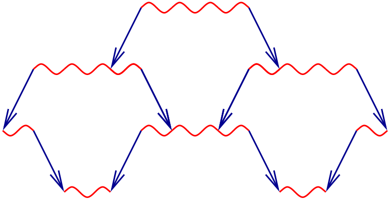

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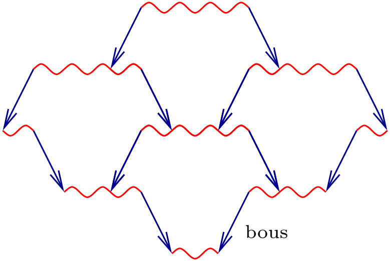




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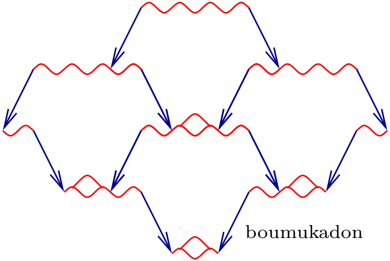

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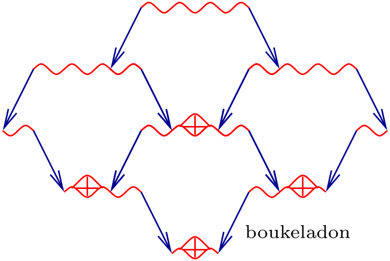




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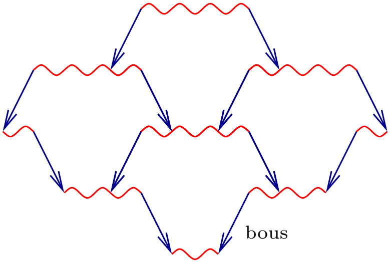
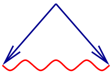
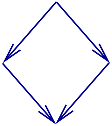



- Newman's Lemma (multiset)+Lemma of Hindley–Rosen (area)





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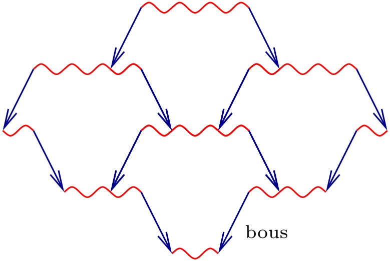


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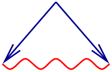

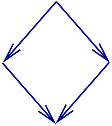



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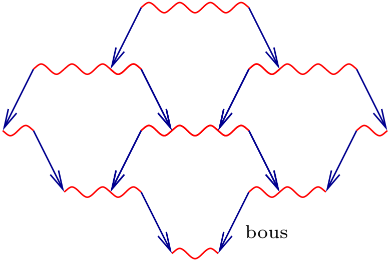


- Newman's Lemma+Lemma of Hindley–Rosen

- flexible



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# *Het kind en ik*

*Ik zou een dag uit vissen,*
*ik voelde mij moedeloos.*
*Ik maakte tussen de lissen*
*met de hand een wak in het kroos.*

*Er steeg licht op van beneden*
*uit de zwarte spiegelgrond.*
*Ik zag een tuin onbetreden*
*en een kind dat daar stond.*

*Het stond aan zijn schrijftafel*
*te schrijven op een lei.*
*Het woord onder de griffel*
*herkende ik, was van mij.*

*Maar toen heeft het geschreven,*
*zonder haast en zonder schroom,*
*al wat ik van mijn leven*
*nog ooit te schrijven droom.*

*En telkens als ik even*
*knikte dat ik het wist,*
*liet hij het water beven*
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