R
Termination — a selection
A

Aart Middeldorp

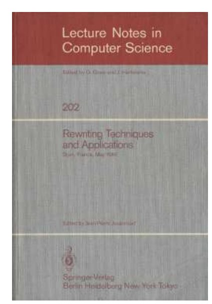
University of Innsbruck

past

Overview

- ➜ past
- ➜ present
- ➜ future

RTA 1985



➜ Termination (invited paper)
Dershowitz

Termination (journal version JSC 1987)
Nachum Dershowitz

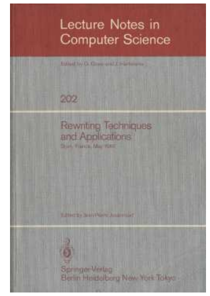
- very influential
- state of the art
  - Knuth-Bendix order — Knuth and Bendix 1970
  - polynomial interpretations — Lankford 1978
  - recursive path order — Dershowitz 1982
  - simplification orders — Dershowitz 1979
  - semantic path order — Kamin and Lévy 1980
  - (un)decidability
  - combined termination — Dershowitz ICALP 1981

$$\mathsf{b} \to \mathsf{a}$$

Toyama IPL 1987

$$\left\{ \begin{array}{l} \mathsf{f}(\mathsf{a},\mathsf{b},x) \to \mathsf{f}(x,x,x) \\ \mathsf{g}(x,y) \to x \\ \mathsf{g}(x,y) \to y \end{array} \right.$$

RTA 1985



- Termination (invited paper)
  Dershowitz
- Path of Subterms Ordering and Recursive Decomposition Ordering Revisited
  Rusinowitch
- Associative Path Orderings
  Bachmair and Plaisted
- A Procedure for Automatically Proving the Termination of a Set of Rewrite Rules
  Detlefs and Forgaard

DEFINITION

- simplification order is strict order on terms, closed under contexts and substitutions, with subterm property
- TRS is simply terminating if its rewrite rules are oriented by simplification order
- $\mathcal{E}$mb consists of all possible projection rules

$$\mathsf{f}(x_1,\ldots,x_n) \to x_i$$

LEMMA

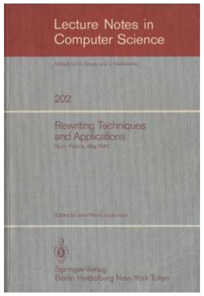
TRS $\mathcal{R}$ is simply terminating $\iff$ $\mathcal{R} \cup \mathcal{E}\mathsf{mb}$ is terminating

A Procedure for Automatically Proving the Termination of a Set of Rewrite Rules
David Detlefs and Randy Forgaard

"In this paper, we present an algorithm that can automatically prove the termination of many sets of rewrite rules. Previous techniques for proving termination of sets of rewrite rules have either required user help in guiding the proof, or have been too restrictive to be generally applicable."

dependency pairs	?	no
MSPO	?	no
EPOS	!?	"a new higher-order ordering" variant of (first-order) RPOS

RTA 1985



- ➜ Termination (invited paper)
  Dershowitz
- ➜ Path of Subterms Ordering and Recursive Decomposition Ordering Revisited
  Rusinowitch
- ➜ Associative Path Orderings
  Bachmair and Plaisted
- ➜ A Procedure for Automatically Proving the Termination of a Set of Rewrite Rules
  Detlefs and Forgaard
- ➜ Fairness in Term Rewriting Systems
  Porat and Francez

RTA 1987

?

- ➜ How to Choose Weights in the Knuth-Bendix Ordering
  Martin
- ➜ Detecting Looping Simplifications
  Purdom Jr.

Fairness in Term Rewriting Systems
Sara Porat and Nissim Francez

DEFINITION

TRS is fairly terminating if there are no infinite rewrite sequences in which a rewrite rule that is infinitely often enabled is used only finitely many times

no	a → f(a) a → b	a → f(a) b → c	yes

QUESTION

is fair termination decidable for ground TRSs ?

ANSWER

yes Fair Termination is Decidable for Ground Systems
Sophie Tison RTA 1989

How to Choose Weights in the Knuth-Bendix Ordering
Ursula Martin

Automating the Knuth Bendix Ordering (journal version AI 1990)
Jeremy Dick, John Kalmus and Ursula Martin

$s >_{\text{kbo}} t$ if $|s|_x \geqslant |t|_x$ for all $x \in \mathcal{V}$ and either

1. $w(s) > w(t)$
2. $w(s) = w(t)$ and either
   - ① $\exists\, x \in \mathcal{V}\ \exists\, n > 0 \quad s = f^n(x)$ and $t = x$
   - ② $s = f(s_1, \ldots, s_n)$, $t = f(t_1, \ldots, t_n)$ and $(s_1, \ldots, s_n) >^{\text{lex}}_{\text{kbo}} (t_1, \ldots, t_n)$
   - ③ $s = f(s_1, \ldots, s_n)$, $t = g(t_1, \ldots, t_m)$ and $f > g$

- ➜ based on linear programming techniques
- ➜ polynomial-time algorithm:

  Verifying Orientability of Rewrite Rules using the Knuth-Bendix Order
  Konstantin Korovin and Andrei Voronkov RTA 2001

➜ interesting example

$$
\begin{array}{rclrcl}
1p + 1p & \to & 2p & 1p + (1p + x) & \to & 2p + x \\
2p + 1p & \to & 1p + 2p & 2p + (1p + x) & \to & 1p + (2p + x) \\
5p + 1p & \to & 1p + 5p & 5p + (1p + x) & \to & 1p + (5p + x) \\
5p + 2p & \to & 2p + 5p & 5p + (2p + x) & \to & 2p + (5p + x) \\
5p + 5p & \to & 10p & 5p + (5p + x) & \to & 10p + x \\
10p + 1p & \to & 1p + 10p & 10p + (1p + x) & \to & 1p + (10p + x) \\
10p + 2p & \to & 2p + 10p & 10p + (2p + x) & \to & 2p + (10p + x) \\
10p + 5p & \to & 5p + 10p & 10p + (5p + x) & \to & 5p + (10p + x) \\
1p + (2p + 2p) & \to & 5p & 1p + (2p + (2p + x)) & \to & 5p + x \\
2p + (2p + 2p) & \to & 1p + 5p & 2p + (2p + (2p + x)) & \to & 1p + (5p + x) \\
(x + y) + z & \to & x + (y + z) & & &
\end{array}
$$

weight function $1p, 2p, 5p, 10p: 1 \qquad +: 0$

precedence $10p > 5p > 2p > 1p$

"Clearly, any rule whose left- and right-hand sides contain identical symbols, cannot be ordered by weights alone (since both sides balance), but will require an ordering on operators."

Embedding with Patterns and Associated Recursive Path Ordering

Laurence Puel

➜ new ordering (SRPO) to prove termination of TRSs that cannot be handled by simplification orders

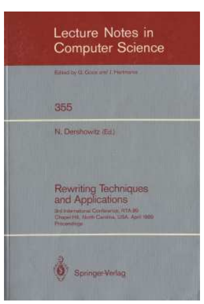
$$
\begin{array}{c}
f(s(x)) \to s(x) \times f(p(s(x))) \\
p(s(x)) \to x
\end{array}
$$

$$
\begin{array}{ccc}
x \times y \to x & s(x) \to x & \cancel{p(x) \to x} \\
x \times y \to y & f(x) \to x & p(p(x)) \to x
\end{array}
$$

➜ based on extension of Kruskal's Tree Theorem with unavoidable patterns

➜ problem: SRPO is not closed under contexts

RTA 1989



➜ Simulation of Turing Machines by a Left-Linear Rewrite Rule
Dauchet

➜ Termination Proofs and the Length of Derivations
Hofbauer and Lautemann

➜ A Local Termination Property for Term Rewriting Systems
Latch and Sigal

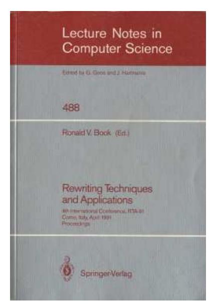
➜ Embedding with Patterns and Associated Recursive Path Ordering
Puel

➜ Extensions and Comparison of Simplification Orderings
Steinbach

➜ Fair Termination is Decidable for Ground Systems
Tison

➜ Termination for the Direct Sum of Left-Linear Term Rewriting Systems
Toyama, Klop and Barendregt

RTA 1991



➜ Transfinite Reductions in Orthogonal Term Rewriting Systems
Kennaway, Klop, Sleep, de Vries

➜ Incremental Termination Proofs and the Length of Derivations
Drewes and Lautemann

➜ Time Bounded Rewrite Systems and Termination Proofs by Generalized Embedding
Hofbauer

➜ An Efficient Representation of Arithmetic for Term Rewriting
Cohen and Watson

➜ Decidability of Confluence and Termination of Monadic Term Rewriting Systems
Salomaa

An Efficient Representation of Arithmetic for Term Rewriting

Dave Cohen and Phil Watson

- ➜ challenging termination problem (RTA open problem #65)

$$
\begin{array}{rcl}
0 \times x & \to & 0 \\
1 \times x & \to & x \\
2 \times 2 & \to & 1 \cdot 0 \\
3 \times x & \to & x \cdot (\mathsf{MIN} \times x) \\
\mathsf{MIN} \times \mathsf{MIN} & \to & 1 \\
2 \times \mathsf{MIN} & \to & \mathsf{MIN} \cdot 2 \\
(x \cdot y) \times z & \to & (x \times z) \cdot (y \times z) \\
(y + z) \times x & \to & (x \times y) + (x \times z) \\
\\
\mathsf{MIN} \cdot 3 & \to & \mathsf{MIN} \\
x \cdot \mathsf{MIN} & \to & (\mathsf{MIN} + x) \cdot 3 \\
0 \cdot x & \to & x \\
x \cdot (y \cdot z) & \to & (x + y) \cdot z
\end{array}
\qquad
\begin{array}{rcl}
0 + x & \to & x \\
x + x & \to & 2 \times x \\
1 + 2 & \to & 3 \\
1 + \mathsf{MIN} & \to & 0 \\
2 + \mathsf{MIN} & \to & 1 \\
3 + x & \to & 1 \cdot (\mathsf{MIN} + x) \\
(x \cdot y) + z & \to & x \cdot (y + z) \\
(2 \times x) + x & \to & 3 \times x \\
(\mathsf{MIN} \times x) + x & \to & 0 \\
(2 \times x) + (\mathsf{MIN} \times x) & \to & x
\end{array}
$$

Polynomial Time Termination and Constraint Satisfaction Tests

David A. Plaisted

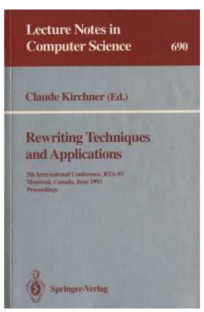
- ➜ termination of ground TRSs is decidable in polynomial time
- ➜ simple termination of ground TRSs is decidable in polynomial time

QUESTION

what about right-ground TRSs ?

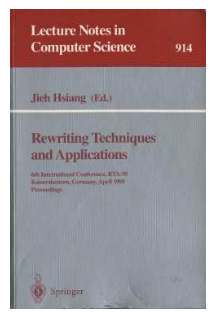
- ➜ termination of right-ground TRSs is decidable
  Dershowitz ICALP 1981
- ➜ confluence of right-ground TRSs is decidable (EXPTIME hard)
  Godoy, Tiwari and Verma AAECC 2004
- ➜ confluence of ground TRSs is PTIME complete
  Comon, Godoy and Nieuwenhuis FOCS 2001

RTA 1993



- ➜ Topics in Termination
  Dershowitz and Hoot
- ➜ Total Termination of Term Rewriting
  Ferreira and Zantema
- ➜ Simple Termination is Difficult
  Middeldorp and Gramlich
- ➜ Polynomial Time Termination and Constraint Satisfaction Tests
  Plaisted
- ➜ Linear Interpretations by Counting Patterns
  Martin
- ➜ Some Undecidable Termination Problems for Semi-Thue Systems
  Sénizergues

RTA 1995



- ➜ Automatic Termination Proofs With Transformation Orderings
  Steinbach
- ➜ A Termination Ordering for Higher Order Rewrite System
  Lysne and Piris
- ➜ A Complete Characterization of Termination of $0^p 1^q \to 1^r 0^s$
  Geser and Zantema
- ➜ Towards a Domain Theory for Termination Proofs
  Kahrs
- ⋗ Generating Polynomial Orderings for Termination Proofs
  Giesl

Automatic Termination Proofs With Transformation Orderings

Joachim Steinbach

- ➜ algorithm (heuristic) for automating transformation orderings

  Bellegarde and Lescanne AAECC 1990

"We have applied Algorithm 3 to 60 TRSs (which are *not* orientable with simplification orderings) randomly chosen from the literature. For 47 TRSs (78.3%) the method succeeds."

$$
\begin{aligned}
\mathsf{fac}(\mathsf{s}(x)) &\to \mathsf{fac}(\mathsf{p}(\mathsf{s}(x))) \times \mathsf{s}(x)\\
\mathsf{p}(\mathsf{s}(0)) &\to 0\\
\mathsf{p}(\mathsf{s}(\mathsf{s}(x))) &\to \mathsf{s}(\mathsf{p}(\mathsf{s}(x)))
\end{aligned}
$$

Termination of Constructor Systems

Thomas Arts and Jürgen Giesl

- ➜ first paper on dependency pairs

CAAP	1997	FroCoS	1998
CADE	2000, 2003	IJCAR	2001, 2004
CSL	1999	LPAR	2003
RTA	1997, 1998, 2000, 2001, 2003, 2004		

- ➜ dependency pairs are used in most current termination provers

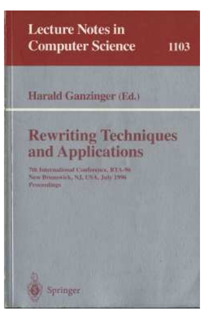
  AProVE CiME TORPA T$_{\mathsf{T}}$T

- ➜ Google

  Using Syntactic Dependency-Pairs Conflation to Improve Retrieval Performance in Spanish

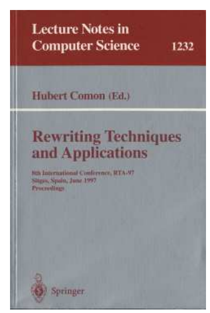
  Vilares, Barcala and Alonso CICLing 2002

RTA 1996



- ➜ Termination of Constructor Systems
  Arts and Giesl
- ➜ On Proving Termination by Innermost Termination
  Gramlich
- ➜ A Recursive Path Ordering for Higher-Order Terms in η-Long β-Normal Form
  Jouannaud and Rubio
- ➜ Modularity of Termination in Term Graph Rewriting
  Krishna Rao
- ➜ On the Termination Problem for One-Rule Semi-Thue Systems
  Sénizergues

RTA 1997



- ➜ A Total Ground Path Ordering for Proving Termination of AC Rewrite Systems
  Kapur and Sivakumar
- ➜ Proving Innermost Normalisation Automatically
  Arts and Giesl
- ➜ Termination of Context-Sensitive Rewriting
  Zantema
- ➜ Innocuous Constructor-Sharing Combinations
  Dershowitz
- ➜ Scott's Conjecture is True, Positive Sensitive Weights
  Perlo-Freeman and Pröhle

Termination of Context-Sensitive Rewriting
Hans Zantema

- transformation to prove termination of context-sensitive rewrite systems

  Lucas ICALP 1996

$$\begin{aligned} \mathsf{from}(x) &\to x : \mathsf{from}(\mathsf{s}(x)) \\ \mathsf{sel}(0, y : z) &\to y \\ \mathsf{sel}(\mathsf{s}(x), y : z) &\to \mathsf{sel}(x, z) \end{aligned}$$

avoid non-termination by forbidding rewriting in second argument of :

$$\begin{aligned} \mathsf{from}(x) &\to x : \underline{\mathsf{from}}(\mathsf{s}(x)) \\ \mathsf{sel}(0, y : z) &\to y & \mathsf{from}(x) &\to \underline{\mathsf{from}}(x) \\ \mathsf{sel}(\mathsf{s}(x), y : z) &\to \mathsf{sel}(x, \mathsf{a}(z)) & \mathsf{a}(\underline{\mathsf{from}}(x)) &\to \mathsf{from}(x) \end{aligned}$$

Normalization of S-Terms is Decidable
Johannes Waldmann

- normalization of ground terms with respect to

  $$\mathsf{S}\, x\, y\, z \to x\, z\, (y\, z)$$

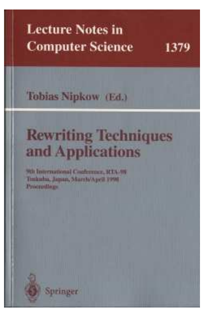
  is decidable

- normalization coincides with termination because S combinator rule is orthogonal and non-erasing

  O'Donnell 1977

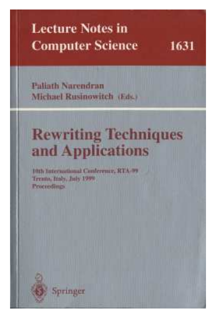
- decidability proof relies on properties of regular tree languages

RTA 1998



- Normalization of S-Terms is Decidable
  Waldmann
- Modularity of Termination Using Dependency Pairs
  Arts and Giesl
- Termination of Associative-Commutative Rewriting by Dependency Pairs
  Marché and Urbain
- Termination Transformation by Tree Lifting Ordering
  Aoto and Toyama
- Towards Automated Termination Proofs through "Freezing"
  Xi
- SN Combinators and Partial Combinatory Algebras
  Akama

RTA 1999



- On the Strong Normalisation of Natural Deduction with Permutation-Conversions
  de Groote
- Strong Normalization of Proof Nets Modulo Structural Congruences
  Di Cosmo and Guerrini
- A Fully Syntactic AC-RPO
  Rubio
- Theory Path Orderings
  Stuber
- Decidability for Left-Linear Growing Term Rewriting Systems
  Nagaya and Toyama
- Transforming Context-Sensitive Rewrite Systems
  Giesl and Middeldorp

Decidability for Left-Linear Growing Term Rewriting Systems
Takashi Nagaya and Yoshihito Toyama

- ➜ termination of almost orthogonal growing TRSs is decidable

$$x \wedge \mathsf{T} \to x$$
$$\mathsf{T} \wedge y \to y$$
$$\mathsf{F} \wedge \mathsf{F} \to \mathsf{F}$$

- ➜ left-linear
- ➜ one trivial critical pair at the root
- ➜ $\forall\, l \to r \quad \forall\, x \in \mathsf{var}(l) \cap \mathsf{var}(r) \quad x$ occurs in l always at depth 1

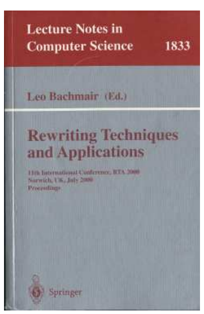
The Dependency Pair Method
Thomas Arts

- ➜ second implementation of the dependency pair method (CiME was first)

"For a statement about performance a measure on number of lines in a TRS or number of function symbols times their arity is insufficient, since it does not relate to the number of dependency pairs, cycles, necessary narrowing steps and such. By unavailability of a satisfactory measure, only examples may demonstrate the usefulness of the application."

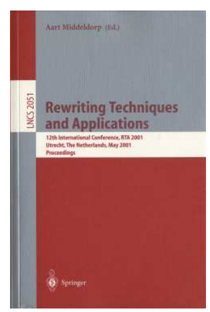
- ➜ no benchmarks

RTA 2000



- ➜ Termination and Confluence of Higher-Order Rewrite Systems
  Blanqui
- ⋗ The Dependency Pair Method
  Arts
- ⋗ TALP: A Tool for the Termination Analysis of Logic Programs
  Ohlebusch, Claves and Marché

RTA 2001



- ➜ Dependency Pairs for Equational Rewriting
  Giesl and Kapur
- ➜ Termination Proofs by Context-Dependent Interpretations
  Hofbauer
- ➜ Verifying Orientability of Rewrite Rules Using the Knuth-Bendix Order
  Korovin and Voronkov
- ➜ On Termination of Higher-Order Rewriting
  van Raamsdonk
- ➜ Confluence and Termination of Simply Typed Term Rewriting Systems
  Yamada

Termination Proofs by Context-Dependent Interpretations
Hofbauer

- strictly monotone interpretation into $\mathbb{N}$ provides poor upper bound on derivation height

$$\mathsf{dh}(t) = \max\{n \in \mathbb{N} \mid \exists\, u \colon t \xrightarrow{n} u\}$$

$$\mathsf{a}(\mathsf{b}(x)) \to \mathsf{b}(\mathsf{a}(x)) \qquad \mathsf{a}(\mathsf{a}(\mathsf{a}(\mathsf{b}(\mathsf{b}(\mathsf{b}(\mathsf{c}))))))$$

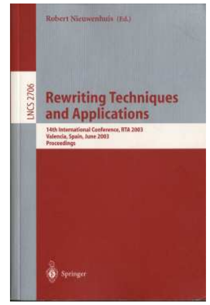
$$\mathsf{a}_{\mathbb{N}}(x) = 2x \qquad \mathsf{b}_{\mathbb{N}}(x) = x + 1 \qquad \mathsf{c}_{\mathbb{N}} = 0$$

$$\mathsf{a}(\mathsf{a}(\mathsf{a}(\mathsf{b}(\mathsf{b}(\mathsf{b}(\mathsf{c}))))))_{\mathbb{N}} = 24 \qquad \mathsf{dh}(\mathsf{a}(\mathsf{a}(\mathsf{a}(\mathsf{b}(\mathsf{b}(\mathsf{b}(\mathsf{c}))))))) = 9$$

$$\mathsf{a}^m(\mathsf{b}^n(\mathsf{c}))_{\mathbb{N}} = 2^m n \qquad \mathsf{dh}(\mathsf{a}^m(\mathsf{b}^n(\mathsf{c}))) = mn$$

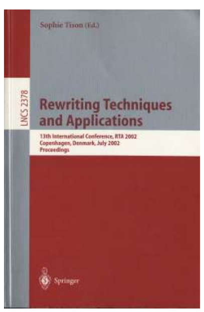
- context-dependent interpretation into $\mathbb{R}_0^+$ can provide tight upper bound on derivation height

RTA 2003



- Size-Change Termination for Term Rewriting
  Thiemann and Giesl
- Relating Derivation Lengths with the Slow-Growing Hierarchy Directly
  Moser and Weiermann
- ⋗ Tsukuba Termination Tool
  Hirokawa and Middeldorp
- Termination of Simply Typed Term Rewriting Systems by Translation and Labelling
  Aoto and Yamada
- Termination of String Rewriting Rules That Have One Pair of Overlaps
  Geser

RTA 2002



- Loops of Superexponential Lengths in One-Rule String Rewriting
  Geser
- Recursive Derivational Length Bounds for Confluent Term Rewrite Systems
  Tahhan-Bittar
- Termination of (Canonical) Context-Sensitive Rewriting
  Lucas

present

RTA 2004

?

- Dependency Pairs Revisited (invited paper)
  Hirokawa and Middeldorp
- ⋗ Automated Termination Proofs with AProVE
  Giesl, Thiemann, Schneider-Kamp and Falke
- Termination of S-Expression Rewriting Systems: Lexicographic Path Ordering for Higher-Order Terms
  Toyama
- ⋗ TORPA: Termination of Rewriting Proved Automatically
  Zantema
- ⋗ Matchbox: a Tool for Match-Bounded String Rewriting
  Waldmann

Dependency Pairs Revisited

Nao Hirokawa   Aart Middeldorp

University of Innsbruck

FACT

$\forall$ non-terminating TRS $\exists$ $\underbrace{\text{minimal}}_{\text{all proper subterms are terminating}}$ non-terminating term

DEFINITION

$\mathcal{T}_\infty$ is set of all minimal non-terminating terms

LEMMA

$\forall\, t \in \mathcal{T}_\infty \quad \exists\, l \to r \in \mathcal{R} \quad \exists\, \sigma \quad \exists$ non-variable subterm u of r

$$t \xrightarrow{>\epsilon}{}^{*} l\sigma \xrightarrow{\epsilon} r\sigma \trianglerighteq u\sigma \in \mathcal{T}_\infty$$

COROLLARY

every term in $\mathcal{T}_\infty$ has defined root symbol

Leading Example

$$\neg\neg x \to x$$
$$\neg(x \vee y) \to \neg x \wedge \neg y$$
$$\neg(x \wedge y) \to \neg x \vee \neg y$$
$$x \wedge (y \vee z) \to (x \wedge y) \vee (x \wedge z)$$
$$(y \vee z) \wedge x \to (x \wedge y) \vee (x \wedge z)$$

proving termination is

- ➜ easy because MPO (multiset path order) applies
- ➜ difficult because automatic tools like CiME and AProVE 1.0 ("Meta Combination" algorithm) fail

LEMMA

$\forall\, t \in \mathcal{T}_\infty \quad \exists\, l \to r \in \mathcal{R} \quad \exists\, \sigma \quad \exists$ non-variable subterm u of r with $u \ntrianglelefteq l$

$$t \xrightarrow{>\epsilon}{}^{*} l\sigma \xrightarrow{\epsilon} r\sigma \trianglerighteq u\sigma \in \mathcal{T}_\infty$$

(TENTATIVE) DEFINITION

$\mathcal{S} = \{l \to u \mid l \to r \in \mathcal{R},\, u \trianglelefteq r$ with $\mathrm{root}(u)$ defined, $u \ntrianglelefteq l\}$

LEMMA

$\forall\, t \in \mathcal{T}_\infty \quad \exists\, l \to u \in \mathcal{S} \quad \exists\, \sigma \qquad t \xrightarrow[\mathcal{R}]{>\epsilon}{}^{*} l\sigma \xrightarrow[\mathcal{S}]{\epsilon} u\sigma \in \mathcal{T}_\infty$

IDEA

get rid of position constraints by marking root symbols of terms in rewrite rules of $\mathcal{S}$

DEFINITION

TRS $\mathcal{R}$ over signature $\mathcal{F}$

- $\mathcal{F}^\sharp = \mathcal{F} \cup \{f^\sharp \mid f$ is defined symbol of $\mathcal{R}\}$
- if $t = f(t_1,\dots,t_n)$ with f defined then $t^\sharp = f^\sharp(t_1,\dots,t_n)$
- $\mathcal{T}_\infty^\sharp = \{t^\sharp \mid t \in \mathcal{T}_\infty\}$
- DP($\mathcal{R}$) $= \{\underbrace{l^\sharp \to u^\sharp}_{\text{dependency pair}} \mid l \to r \in \mathcal{R},\ u \trianglelefteq r$ with root(u) defined, $u \ntriangleleft l\}$

LEMMA

$\forall\, s \in \mathcal{T}_\infty \quad \exists\, t, u \in \mathcal{T}_\infty \qquad s^\sharp \xrightarrow[\mathcal{R}]{*} t^\sharp \xrightarrow[\mathsf{DP}(\mathcal{R})]{} u^\sharp$

COROLLARY

$\forall$ non-terminating TRS $\mathcal{R}$ $\;\exists$ infinite rewrite sequence

$$\mathcal{T}_\infty^\sharp \ni t_1 \xrightarrow[\mathcal{R}]{*} t_2 \xrightarrow[\mathsf{DP}(\mathcal{R})]{} t_3 \xrightarrow[\mathcal{R}]{*} t_4 \xrightarrow[\mathsf{DP}(\mathcal{R})]{} \cdots$$

LEMMA

$\forall$ non-terminating TRS $\mathcal{R}$ $\;\exists$ infinite rewrite sequence

$$\mathcal{T}_\infty^\sharp \ni t_1 \xrightarrow[\mathcal{R}]{*} t_2 \xrightarrow[\mathcal{C}]{} t_3 \xrightarrow[\mathcal{R}]{*} t_4 \xrightarrow[\mathcal{C}]{} \cdots \qquad \mathcal{C}\text{-minimal}$$

where all dependency pairs in $\mathcal{C} \subseteq$ DP($\mathcal{R}$) are used infinitely often

ASSUMPTION

all TRSs are finite

OBSERVATION

$\mathcal{C}$ is cycle in dependency graph of $\mathcal{R}$

LEADING EXAMPLE

1:	$\neg\neg x \to x$
2:	$\neg(x \vee y) \to \neg x \wedge \neg y$
3:	$\neg(x \wedge y) \to \neg x \vee \neg y$
4:	$x \wedge (y \vee z) \to (x \wedge y) \vee (x \wedge z)$
5:	$(y \vee z) \wedge x \to (x \wedge y) \vee (x \wedge z)$

dependency pairs

6:	$\neg^\sharp(x \vee y) \to \neg x \wedge^\sharp \neg y$		
7:	$\neg^\sharp(x \vee y) \to \neg^\sharp x$	11:	$x \wedge^\sharp (y \vee z) \to x \wedge^\sharp y$
8:	$\neg^\sharp(x \vee y) \to \neg^\sharp y$	12:	$x \wedge^\sharp (y \vee z) \to x \wedge^\sharp z$
9:	$\neg^\sharp(x \wedge y) \to \neg^\sharp x$	13:	$(y \vee z) \wedge^\sharp x \to x \wedge^\sharp y$
10:	$\neg^\sharp(x \wedge y) \to \neg^\sharp y$	14:	$(y \vee z) \wedge^\sharp x \to x \wedge^\sharp z$

DEFINITION

dependency graph DG($\mathcal{R}$)

nodes: $\boxed{l \to r}$ for every $l \to r \in$ DP($\mathcal{R}$)

arrows: $\boxed{l_1 \to r_1} \longrightarrow \boxed{l_2 \to r_2}$

if $\exists$ substitutions σ, τ such that $r_1\sigma \xrightarrow[\mathcal{R}]{*} l_2\tau$

THEOREM

TRS $\mathcal{R}$ is terminating if (and only if)

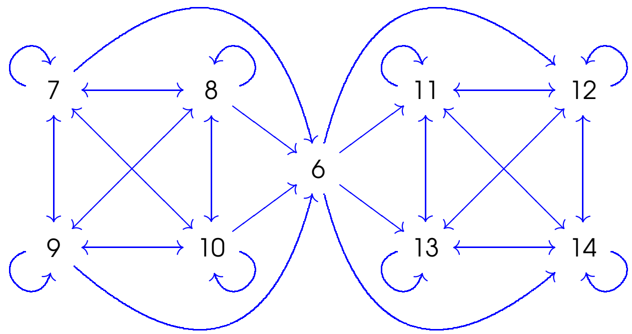
$\forall$ cycle $\mathcal{C}$ in DG($\mathcal{R}$) $\quad\neg\,\exists\, \mathcal{C}$-minimal rewrite sequence

REMARK

dependency graph is not computable in general, but good approximations exist

Leading Example

dependency graph



30 cycles: all nonempty subsets of $\{7, 8, 9, 10\}$ and $\{11, 12, 13, 14\}$

Subterm Criterion

DEFINITION

- simple projection for cycle $\mathcal{C}$ in DG($\mathcal{R}$) is mapping π that assigns to every n-ary dependency pair symbol $f^\sharp$ in $\mathcal{C}$ one of its argument positions
- extension of π to terms in $\mathcal{T}^\sharp$: $\pi(f^\sharp(t_1, \dots, t_n)) = t_{\pi(f^\sharp)}$

NEW THEOREM 1

if $\exists$ simple projection π for cycle $\mathcal{C}$ in DG($\mathcal{R}$) such that

① $\forall\, l \to r \in \mathcal{C} \quad \pi(l) = \pi(r)$ or $\pi(l) \rhd \pi(r)$

② $\exists\, l \to r \in \mathcal{C} \quad \pi(l) \rhd \pi(r)$

then $\neg\, \exists\, \mathcal{C}$-minimal rewrite sequence

THEOREM

$\forall$ non-terminating TRS $\mathcal{R}$ $\exists$ cycle $\mathcal{C}$ in DG($\mathcal{R}$)
$\exists\, \mathcal{C}$-minimal rewrite sequence $t_1 \xrightarrow[\mathcal{R}]{*} t_2 \xrightarrow[\mathcal{C}]{} t_3 \xrightarrow[\mathcal{R}]{*} t_4 \xrightarrow[\mathcal{C}]{} \cdots$

IDEA

project each dependency symbol in $\mathcal{C}$ to fixed argument position

$$\pi(t_1) \xrightarrow[\mathcal{R}]{*} \pi(t_2) \quad ? \quad \pi(t_3) \xrightarrow[\mathcal{R}]{*} \pi(t_4) \quad ? \quad \cdots$$

OBSERVATION

- $\pi(t_1)$ is terminating with respect to $\to_{\mathcal{R}}$
- hence also with respect to $\to_{\mathcal{R}} \cup \rhd$

IDEA

require: $\forall\, l \to r \in \mathcal{C} \quad \pi(l) \trianglerighteq \pi(r)$ and $\exists\, l \to r \in \mathcal{C} \quad \pi(l) \rhd \pi(r)$

Leading Example

- cycle $\mathcal{C} = \{7, 8, 9, 10\}$

7:	$\neg^\sharp(x \vee y) \to \neg^\sharp x$	9:	$\neg^\sharp(x \wedge y) \to \neg^\sharp x$
8:	$\neg^\sharp(x \vee y) \to \neg^\sharp y$	10:	$\neg^\sharp(x \wedge y) \to \neg^\sharp y$

simple projection $\pi(\neg^\sharp) = 1$

7:	$x \vee y \rhd x$	9:	$x \wedge y \rhd x$
8:	$x \vee y \rhd y$	10:	$x \wedge y \rhd y$

- cycle $\mathcal{C} = \{11, 12, 13, 14\}$

11:	$x \wedge^\sharp (y \vee z) \to x \wedge^\sharp y$	13:	$(y \vee z) \wedge^\sharp x \to x \wedge^\sharp y$
12:	$x \wedge^\sharp (y \vee z) \to x \wedge^\sharp z$	14:	$(y \vee z) \wedge^\sharp x \to x \wedge^\sharp z$

???

NEW THEOREM 1

if $\exists$ simple projection π for cycle $\mathcal{C}$ in DG($\mathcal{R}$) such that

① $\forall\, l \to r \in \mathcal{C} \quad \pi(l) = \pi(r)$ or $\pi(l) \rhd \pi(r)$

② $\exists\, l \to r \in \mathcal{C} \quad \pi(l) \rhd \pi(r)$

then $\neg\, \exists\, \mathcal{C}$-minimal rewrite sequence

REMARKS

- very easy to check
- surprisingly powerful

dependency graph

$$\begin{tikzcd} 9 \arrow[loop above] \arrow[r] & 8 \arrow[r] & 10 \arrow[r, leftrightarrow] \arrow[rr, bend left] & 11 & 12 \arrow[l] \arrow[loop above] \end{tikzcd}$$

2 strongly connected components (SCCs)

- $\{9\}$ simple projection $\pi(\mathsf{sort}^\sharp) = 1$

  9: $x : y \rhd x$

- $\{10, 11, 12\}$ simple projection $\pi(\mathsf{insert}^\sharp) = \pi(\mathsf{choose}^\sharp) = 2$

  10: $v : w \unrhd v : w$ 11: $v : w \rhd w$ 12: $v : w \unrhd v : w$

  - new SCC $\{12\}$ simple projection $\pi(\mathsf{choose}^\sharp) = 3$

    12: $\mathsf{s}(y) \rhd y$

Example

1: $\mathsf{sort}(\mathsf{nil}) \to \mathsf{nil}$

2: $\mathsf{sort}(x : y) \to \mathsf{insert}(x, \mathsf{sort}(y))$

3: $\mathsf{insert}(x, \mathsf{nil}) \to x : \mathsf{nil}$

4: $\mathsf{insert}(x, v : w) \to \mathsf{choose}(x, v : w, x, v)$

5: $\mathsf{choose}(x, v : w, y, 0) \to x : (v : w)$

6: $\mathsf{choose}(x, v : w, 0, \mathsf{s}(z)) \to v : \mathsf{insert}(x, w)$

7: $\mathsf{choose}(x, v : w, \mathsf{s}(y), \mathsf{s}(z)) \to \mathsf{choose}(x, v : w, y, z)$

dependency pairs

8: $\mathsf{sort}^\sharp(x : y) \to \mathsf{insert}^\sharp(x, \mathsf{sort}(y))$

9: $\mathsf{sort}^\sharp(x : y) \to \mathsf{sort}^\sharp(y)$

10: $\mathsf{insert}^\sharp(x, v : w) \to \mathsf{choose}^\sharp(x, v : w, x, v)$

11: $\mathsf{choose}^\sharp(x, v : w, 0, \mathsf{s}(z)) \to \mathsf{insert}^\sharp(x, w)$

12: $\mathsf{choose}^\sharp(x, v : w, \mathsf{s}(y), \mathsf{s}(z)) \to \mathsf{choose}^\sharp(x, v : w, y, z)$

Example

1: $\mathsf{intlist}(\mathsf{nil}) \to \mathsf{nil}$

2: $\mathsf{intlist}(x : y) \to \mathsf{s}(x) : \mathsf{intlist}(y)$

3: $\mathsf{int}(0, 0) \to 0 : \mathsf{nil}$

4: $\mathsf{int}(0, \mathsf{s}(y)) \to 0 : \mathsf{int}(\mathsf{s}(0), \mathsf{s}(y))$

5: $\mathsf{int}(\mathsf{s}(x), 0) \to \mathsf{nil}$

6: $\mathsf{int}(\mathsf{s}(x), \mathsf{s}(y)) \to \mathsf{intlist}(\mathsf{int}(x, y))$

7: $\mathsf{intlist}^\sharp(x : y) \to \mathsf{intlist}^\sharp(y)$

8: $\mathsf{int}^\sharp(0, \mathsf{s}(y)) \to \mathsf{int}^\sharp(\mathsf{s}(0), \mathsf{s}(y))$

9: $\mathsf{int}^\sharp(\mathsf{s}(x), \mathsf{s}(y)) \to \mathsf{intlist}^\sharp(\mathsf{int}(x, y))$

10: $\mathsf{int}^\sharp(\mathsf{s}(x), \mathsf{s}(y)) \to \mathsf{int}^\sharp(x, y)$

dependency pairs

dependency graph

$$\begin{tikzcd} 8 \arrow[r, leftrightarrow] \arrow[rr, bend left] & 10 \arrow[r] \arrow[loop below] & 9 \arrow[r] & 7 \arrow[loop below] \end{tikzcd}$$

2 SCCs:

- $\{7\}$ simple projection $\pi(\mathsf{intlist}^\sharp) = 1$
- $\{8, 10\}$ simple projection $\pi(\mathsf{int}^\sharp) = 2$

THEOREM Giesl, Arts, Ohlebusch JSC 2002

if $\exists$ argument filtering π and $\exists$ reduction pair $(\gtrsim, >)$ such that

① $\forall\, l \to r \in \mathcal{R} \quad \pi(l) \gtrsim \pi(r)$

② $\forall\, l \to r \in \mathcal{C} \quad \pi(l) \gtrsim \pi(r)$ or $\pi(l) > \pi(r)$

③ $\exists\, l \to r \in \mathcal{C} \quad \pi(l) > \pi(r)$

then $\neg\, \exists$ $\mathcal{C}$-minimal rewrite sequence

DEFINITION

➜ argument filtering is mapping π such that for every n-ary function symbol $f \in \mathcal{F}^\sharp$ one of following alternatives holds:

① $\pi(f) = i$ with $i \in \{1, \ldots, n\}$

② $\pi(f) = [i_1, \ldots, i_m]$ with $1 \leqslant i_1 \leqslant \cdots \leqslant i_m \leqslant n$

➜ $$\pi(t) = \begin{cases} t & \text{if } t \text{ is variable} \\ \pi(t_i) & \text{if } t = f(t_1, \ldots, t_n) \text{ and ①} \\ f(\pi(t_{i_1}), \ldots, \pi(t_{i_m})) & \text{if } t = f(t_1, \ldots, t_n) \text{ and ②} \end{cases}$$

THEOREM Giesl, Arts, Ohlebusch JSC 2002

if $\exists$ argument filtering π and $\exists$ reduction pair $(\gtrsim, >)$ such that

① $\forall\, l \to r \in \mathcal{R} \quad \pi(l) \gtrsim \pi(r)$

② $\forall\, l \to r \in \mathcal{C} \quad \pi(l) \gtrsim \pi(r)$ or $\pi(l) > \pi(r)$

③ $\exists\, l \to r \in \mathcal{C} \quad \pi(l) > \pi(r)$

then $\neg\, \exists$ $\mathcal{C}$-minimal rewrite sequence

REMARK

theorem provides sufficient and necessary condition for absence of $\mathcal{C}$-minimal rewrite sequences

THEOREM Giesl, Arts, Ohlebusch JSC 2002

if $\exists$ argument filtering π and $\exists$ reduction pair $(\gtrsim, >)$ such that

① $\forall\, l \to r \in \mathcal{R} \quad \pi(l) \gtrsim \pi(r)$

② $\forall\, l \to r \in \mathcal{C} \quad \pi(l) \gtrsim \pi(r)$ or $\pi(l) > \pi(r)$

③ $\exists\, l \to r \in \mathcal{C} \quad \pi(l) > \pi(r)$

then $\neg\, \exists$ $\mathcal{C}$-minimal rewrite sequence

DEFINITION

reduction pair $(\gtrsim, >)$ consists of qo $\gtrsim$ and wfo $>$ such that

① $>$ is closed under substitutions

② $\gtrsim$ is closed under contexts and substitutions

③ $\gtrsim \cdot > \subseteq >$ or $> \cdot \gtrsim \subseteq >$

➜ LPO $(>_{\mathsf{lpo}}^{=}, >_{\mathsf{lpo}})$ for strict precedence $>$

➜ polynomial interpretation $(\geqslant_{\mathbb{N}}, >_{\mathbb{N}})$

USABLE RULES

IDEA

reduce rule constraints

$$\forall\, l \to r \in \mathcal{R} \quad \pi(l) \gtrsim \pi(r)$$

by considering usable rules

DEFINITION

➜ relation $\blacktriangleright$ on signature $\mathcal{F}$ of TRS $\mathcal{R}$:

$$f \blacktriangleright g \quad \text{if } \exists\, f(l_1, \ldots, l_n) \to r \in \mathcal{R} \text{ with } g \in \mathsf{fun}(r)$$

➜ $\mathcal{R} \restriction \mathcal{G} = \{l \to r \in \mathcal{R} \mid \mathsf{root}(l) \in \mathcal{G}\}$

➜ $\mathcal{U}(t) = \mathcal{R} \restriction \{g \mid f \blacktriangleright^* g \text{ for some } f \in \mathsf{fun}(t)\}$

➜ $\mathcal{U}(\mathcal{C}) = \bigcup_{l \to r \in \mathcal{C}} \mathcal{U}(r)$

DEFINITION

TRS $\mathcal{C}_\varepsilon$ consists of rewrite rules

$$\mathsf{cons}(x,y) \to x \qquad \mathsf{cons}(x,y) \to y$$

NEW THEOREM 2

if $\exists$ argument filtering π and $\exists$ reduction pair $(\gtrsim, >)$ such that

① $\forall\, l \to r \in \mathcal{U}(\mathcal{C}) \cup \mathcal{C}_\varepsilon \quad \pi(l) \gtrsim \pi(r)$

② $\forall\, l \to r \in \mathcal{C} \quad \pi(l) \gtrsim \pi(r)$ or $\pi(l) > \pi(r)$

③ $\exists\, l \to r \in \mathcal{C} \quad \pi(l) > \pi(r)$

then $\neg\, \exists\, \mathcal{C}$-minimal rewrite sequence

REMARKS

- ➜ proof is based on subtle modification of 'interpretation' of
- ➜ result was independently obtained by Urbain IJCAR 2001
  Thiemann, Giesl, Schneider-Kamp IJCAR 2004

DEFINITION

TRS $\mathcal{C}_\varepsilon$ consists of rewrite rules

$$\mathsf{cons}(x,y) \to x \qquad \mathsf{cons}(x,y) \to y$$

NEW THEOREM 2

if $\exists$ argument filtering π and $\exists$ reduction pair $(\gtrsim, >)$ such that

① $\forall\, l \to r \in \mathcal{U}(\mathcal{C}) \cup \mathcal{C}_\varepsilon \quad \pi(l) \gtrsim \pi(r)$

② $\forall\, l \to r \in \mathcal{C} \quad \pi(l) \gtrsim \pi(r)$ or $\pi(l) > \pi(r)$

③ $\exists\, l \to r \in \mathcal{C} \quad \pi(l) > \pi(r)$

then $\neg\, \exists\, \mathcal{C}$-minimal rewrite sequence

REMARKS

- ➜ most reduction pairs can be extended such that $\pi(\mathcal{C}_\varepsilon) \subseteq \gtrsim$
- ➜ theorem provides only sufficient condition for absence of $\mathcal{C}$-minimal rewrite sequences

LEADING EXAMPLE

- ➜ cycle $\mathcal{C} = \{11, 12, 13, 14\}$

11:	$x \wedge^\sharp (y \vee z) \to x \wedge^\sharp y$	13:	$(y \vee z) \wedge^\sharp x \to x \wedge^\sharp y$
12:	$x \wedge^\sharp (y \vee z) \to x \wedge^\sharp z$	14:	$(y \vee z) \wedge^\sharp x \to x \wedge^\sharp z$

$\mathcal{U}(\mathcal{C}) = \varnothing$

polynomial interpretation $\wedge^\sharp_{\mathbb{N}}(x,y) = x + y \quad \vee_{\mathbb{N}}(x,y) = x + y + 1$

11:	$x+y+z+1 > x+y$	13:	$x+y+z+1 > x+y$
12:	$x+y+z+1 > x+z$	14:	$x+y+z+1 > x+z$

EXAMPLE

1: $\mathsf{f}(\mathsf{s}(\mathsf{a}), \mathsf{s}(\mathsf{b}), x) \to \mathsf{f}(x,x,x)$

2: $\mathsf{g}(\mathsf{f}(\mathsf{s}(x), \mathsf{s}(y), z)) \to \mathsf{g}(\mathsf{f}(x,y,z))$

dependency pairs

3: $\mathsf{f}^\sharp(\mathsf{s}(\mathsf{a}), \mathsf{s}(\mathsf{b}), x) \to \mathsf{f}^\sharp(x,x,x)$

4: $\mathsf{g}^\sharp(\mathsf{f}(\mathsf{s}(x), \mathsf{s}(y), z)) \to \mathsf{g}^\sharp(\mathsf{f}(x,y,z))$

5: $\mathsf{g}^\sharp(\mathsf{f}(\mathsf{s}(x), \mathsf{s}(y), z)) \to \mathsf{f}^\sharp(x,y,z)$

dependency graph

$$3 \longleftarrow 5 \longleftarrow 4$$

1 cycle: $\mathcal{C} = \{4\}$

$\mathcal{U}(\mathcal{C}) = \{1\}$

$\mathcal{U}(\mathcal{C}) \cup \mathcal{C}_{\mathcal{E}} \cup \mathcal{C}$ is cyclic:

$\mathsf{g}^\sharp(\mathsf{f}(\mathsf{s}(\mathsf{cons}(\mathsf{s}(\mathsf{a}),\mathsf{s}(\mathsf{b}))),\mathsf{s}(\mathsf{cons}(\mathsf{s}(\mathsf{a}),\mathsf{s}(\mathsf{b}))),\mathsf{s}(\mathsf{cons}(\mathsf{s}(\mathsf{a}),\mathsf{s}(\mathsf{b})))))$

$\to_{\mathcal{C}}$ $\mathsf{g}^\sharp(\mathsf{f}(\mathsf{cons}(\mathsf{s}(\mathsf{a}),\mathsf{s}(\mathsf{b})),\mathsf{cons}(\mathsf{s}(\mathsf{a}),\mathsf{s}(\mathsf{b})),\mathsf{s}(\mathsf{cons}(\mathsf{s}(\mathsf{a}),\mathsf{s}(\mathsf{b})))))$

$\to_{\mathcal{C}_{\mathcal{E}}}$ $\mathsf{g}^\sharp(\mathsf{f}(\mathsf{s}(\mathsf{a}),\mathsf{cons}(\mathsf{s}(\mathsf{a}),\mathsf{s}(\mathsf{b})),\mathsf{s}(\mathsf{cons}(\mathsf{s}(\mathsf{a}),\mathsf{s}(\mathsf{b})))))$

$\to_{\mathcal{C}_{\mathcal{E}}}$ $\mathsf{g}^\sharp(\mathsf{f}(\mathsf{s}(\mathsf{a}),\mathsf{s}(\mathsf{b}),\mathsf{s}(\mathsf{cons}(\mathsf{s}(\mathsf{a}),\mathsf{s}(\mathsf{b})))))$

$\to_{\mathcal{U}(\mathcal{C})}$ $\mathsf{g}^\sharp(\mathsf{f}(\mathsf{s}(\mathsf{cons}(\mathsf{s}(\mathsf{a}),\mathsf{s}(\mathsf{b}))),\mathsf{s}(\mathsf{cons}(\mathsf{s}(\mathsf{a}),\mathsf{s}(\mathsf{b}))),\mathsf{s}(\mathsf{cons}(\mathsf{s}(\mathsf{a}),\mathsf{s}(\mathsf{b})))))$

s subterm criterion
l LPO with 'some' heuristic for computing argument filterings
u usable rules criterion

	s	l	sl	ul	sul
success	128	133	149	144	152
avg time	0.01	0.07	0.01	0.01	0.01
failure	95	90	74	79	71
avg time	0.01	0.01	0.01	0.01	0.01
timeout	0	0	0	0	0
total time	1.72	10.49	2.33	2.31	2.07

EXPERIMENTS — TYROLEAN TERMINATION TOOL

CPU:	Pentium 4 2.2 GHz
memory:	512 MB
language:	OCaml (native compiled)
input:	223 TRSs from Arts & Giesl, Dershowitz, Steinbach & Kühler
timeout:	30 seconds per TRS

s subterm criterion
p linear polynomial interpretations with coefficients from $\{0,1\}$
u usable rules criterion

	s	p	sp	up	sup
success	128	139	180	179	189
avg time	0.01	0.32	0.37	0.28	0.25
failure	95	77	39	44	34
avg time	0.01	0.52	0.33	0.03	0.03
timeout	0	7	4	0	0
total time	1.72	294.13	198.83	51.30	47.79

RTA 2005

April 19–21 Nara, Japan

submission deadline: November 19, 2004