Automatic Complexity Analysis for Rewrite Systems

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Joint Spectral Radius Theory for Automated Complexity Analysis of Rewrite Systems

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joint work with

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Johannes Waldmann Harald Zankl

Outline

- Introduction
- History
- Matrix Interpretations
- Algebraic Methods
- Automata-Based Methods
- Unifying Algebraic and Automata-Based Methods
- Concluding Remarks

Example

```
let rec reverse =
function
| [] -> []
| x :: xs -> (reverse xs) @ (x :: []) ;;


let rec shuffle =
function
| [] -> []
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`shuffle([0,1,2,3,4])` evaluates to `[0,4,1,3,2]`

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$$\begin{aligned} \mathsf{reverse}(\mathsf{nil}) &= \mathsf{nil} \\ \mathsf{reverse}(x :: xs) &= \mathsf{append}(\mathsf{reverse}(xs), x :: \mathsf{nil}) \end{aligned}$$

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Example

signature $\quad \mathsf{nil}\ \mathsf{0}$ (constants) $\quad \mathsf{reverse}\ \mathsf{shuffle}\ \mathsf{s}$ (unary) $\quad \mathsf{append}\ ::$ (binary)

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Example

signature nil 0 (constants) reverse shuffle s (unary) append :: (binary)

terms s(s(0))

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rewriting $\quad \mathsf{shuffle}(\mathsf{0} :: \mathsf{s}(\mathsf{0}) :: \mathsf{s}(\mathsf{s}(\mathsf{0})) :: \mathsf{s}(\mathsf{s}(\mathsf{s}(\mathsf{0}))) :: \mathsf{s}(\mathsf{s}(\mathsf{s}(\mathsf{s}(\mathsf{0})))) :: \mathsf{nil})$

Example

signature: nil 0 (constants) reverse shuffle s (unary) append :: (binary)

terms: $\mathsf{s}(\mathsf{s}(0))$ $\mathsf{shuffle}(0 :: \mathsf{nil})$ $\mathsf{reverse}(x :: xs)$ $\mathsf{s}(\mathsf{append}(\mathsf{s}(x), 0))$

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rewriting: $\mathsf{shuffle}(0 :: 1 :: 2 :: 3 :: 4 :: \mathsf{nil})$

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\to\ & \cdots \to 0 :: 4 :: 1 :: 3 :: 2 :: \mathsf{nil}
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Definitions

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Definitions

- derivation height $\quad \mathrm{dh}_{\mathcal{R}}(t) = \max\{\, n \mid t \to_{\mathcal{R}}^{n} u \text{ for some term } u \,\}$
- derivational complexity $\quad \mathrm{dc}_{\mathcal{R}}(k) = \max\{\, \mathrm{dh}(t) \mid |t| \leqslant k \,\}$

Example

TRS $\mathcal{R}$

$$0 + y \to y \qquad\qquad \mathsf{s}(x) + y \to \mathsf{s}(x + y)$$

Example

TRS $\mathcal{R}$ is terminating

$$0 + y \to y \qquad\qquad \mathsf{s}(x) + y \to \mathsf{s}(x + y)$$

polynomial interpretation

$$0_{\mathbb{N}} = 0 \qquad \mathsf{s}_{\mathbb{N}}(x) = x + 1 \qquad +_{\mathbb{N}}(x, y) = 2x + y + 1$$

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derivational complexity

$$t_i = \underbrace{\mathsf{s}(0) + \cdots + \mathsf{s}(0)}_{i+1} \qquad t_i = \begin{cases} \mathsf{s}(0) & \text{if } i = 0 \\ t_{i-1} + \mathsf{s}(0) & \text{if } i > 0 \end{cases}$$

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$$|t_i| = 3i + 2 \qquad \mathsf{dh}_{\mathcal{R}}(t_i) = \begin{cases} 0 & \text{if } i = 0 \\ \mathsf{dh}_{\mathcal{R}}(t_{i-1}) + i + 1 & \text{if } i > 0 \end{cases}$$

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polynomial interpretation

$$0_{\mathbb{N}} = 0 \qquad \mathsf{s}_{\mathbb{N}}(x) = x + 1 \qquad +_{\mathbb{N}}(x, y) = 2x + y + 1$$

derivation height

$$\mathsf{dh}_{\mathcal{R}}(\mathsf{s}^m(0) + \mathsf{s}^n(0)) = m + 1$$

derivational complexity

$$t_i = \underbrace{\mathsf{s}(0) + \cdots + \mathsf{s}(0)}_{i+1} \qquad t_i = \begin{cases} \mathsf{s}(0) & \text{if } i = 0 \\ t_{i-1} + \mathsf{s}(0) & \text{if } i > 0 \end{cases} \qquad \mathsf{dc}_{\mathcal{R}}(k) \in \Theta(k^2)$$

$$|t_i| = 3i + 2 \qquad \mathsf{dh}_{\mathcal{R}}(t_i) = \begin{cases} 0 & \text{if } i = 0 \\ \mathsf{dh}_{\mathcal{R}}(t_{i-1}) + i + 1 & \text{if } i > 0 \end{cases}$$

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	termination	**derivational complexity** 1967
1967	Knuth-Bendix order	

	termination	**derivational complexity**	1968
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	termination	**derivational complexity**	1969
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termination	derivational complexity	1970
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	termination	**derivational complexity**	1971
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	termination	**derivational complexity**	1972
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	termination	**derivational complexity**	1973
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	termination	**derivational complexity**	1974
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	termination	**derivational complexity**	1975
1967	Knuth-Bendix order		
1975	polynomial interpretations		

	termination	derivational complexity	1976
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1975	polynomial interpretations		

	termination	**derivational complexity**	1979
1967	Knuth-Bendix order		
1975	polynomial interpretations		
1979	simple path order		

	termination	**derivational complexity**	1980
1967	Knuth-Bendix order		
1975	polynomial interpretations		
1979	simple path order		
1980	lexicographic path order semantic path order		

	termination	**derivational complexity**	1981
1967	Knuth-Bendix order		
1975	polynomial interpretations		
1979	simple path order		
1980	lexicographic path order semantic path order		
1981	recursive decomposition order		

	termination	**derivational complexity**	1982
1967	Knuth-Bendix order		
1975	polynomial interpretations		
1979	simple path order		
1980	lexicographic path order semantic path order		
1981	recursive decomposition order		
1982	multiset path order		

	termination	**derivational complexity**	1983
1967	Knuth-Bendix order		
1975	polynomial interpretations		
1979	simple path order		
1980	lexicographic path order semantic path order		
1981	recursive decomposition order		
1982	multiset path order		
1983	recursive path order		

	termination	**derivational complexity**	1984
1967	Knuth-Bendix order		
1975	polynomial interpretations		
1979	simple path order		
1980	lexicographic path order semantic path order		
1981	recursive decomposition order		
1982	multiset path order		
1983	recursive path order		

	termination	**derivational complexity**	1985
1967	Knuth-Bendix order		
1975	polynomial interpretations		
1979	simple path order		
1980	lexicographic path order semantic path order		
1981	recursive decomposition order		
1982	multiset path order		
1983	recursive path order		

	termination	derivational complexity	1986
1967	Knuth-Bendix order		
1975	polynomial interpretations		
1979	simple path order		
1980	lexicographic path order semantic path order		
1981	recursive decomposition order		
1982	multiset path order		
1983	recursive path order		

	termination	derivational complexity	1987
1967	Knuth-Bendix order		
1975	polynomial interpretations		
1979	simple path order		
1980	lexicographic path order semantic path order		
1981	recursive decomposition order		
1982	multiset path order		
1983	recursive path order		

	termination	**derivational complexity**	1988
1967	Knuth-Bendix order		
1975	polynomial interpretations		
1979	simple path order		
1980	lexicographic path order semantic path order		
1981	recursive decomposition order		
1982	multiset path order		
1983	recursive path order		

termination		derivational complexity 1989	
1967	Knuth-Bendix order		
1975	polynomial interpretations	1989	Hofbauer and Lautemann
1979	simple path order		
1980	lexicographic path order semantic path order		
1981	recursive decomposition order		
1982	multiset path order		
1983	recursive path order		

Theorem (Hofbauer and Lautemann 1989)

interpretation in $\mathbb{N}$	bound on derivational complexity
polynomial	double-exponential

Example

rewrite system

$$
\begin{array}{ccc}
x + 0 \to x & \mathsf{d}(0) \to 0 & \mathsf{q}(0) \to 0 \\
x + \mathsf{s}(y) \to \mathsf{s}(x + y) & \mathsf{d}(\mathsf{s}(x)) \to \mathsf{s}(\mathsf{s}(\mathsf{d}(x))) & \mathsf{q}(\mathsf{s}(x)) \to \mathsf{q}(x) + \mathsf{s}(\mathsf{d}(x))
\end{array}
$$

interpretations

$$
0_{\mathbb{N}} = 2 \quad \mathsf{s}_{\mathbb{N}}(x) = x + 1 \quad +_{\mathbb{N}}(x, y) = x + 2y \quad \mathsf{d}_{\mathbb{N}}(x) = 3x \quad \mathsf{q}_{\mathbb{N}}(x) = x^3
$$

Theorem (Hofbauer and Lautemann 1989)

	interpretation in $\mathbb{N}$	bound on derivational complexity
	polynomial	double-exponential
$a_1 x_1 + \cdots + a_n x_n + b$	linear	exponential
$x_1 + \cdots + x_n + b$	strongly linear	linear

Example

rewrite system

$$x + 0 \to x \qquad \mathsf{d}(0) \to 0 \qquad \mathsf{q}(0) \to 0$$

$$x + \mathsf{s}(y) \to \mathsf{s}(x + y) \qquad \mathsf{d}(\mathsf{s}(x)) \to \mathsf{s}(\mathsf{s}(\mathsf{d}(x))) \qquad \mathsf{q}(\mathsf{s}(x)) \to \mathsf{q}(x) + \mathsf{s}(\mathsf{d}(x))$$

interpretations

$$0_{\mathbb{N}} = 2 \quad \mathsf{s}_{\mathbb{N}}(x) = x + 1 \quad +_{\mathbb{N}}(x, y) = x + 2y \quad \mathsf{d}_{\mathbb{N}}(x) = 3x \quad \mathsf{q}_{\mathbb{N}}(x) = x^3$$

	termination		derivational complexity 1990
1967	Knuth-Bendix order		
1975	polynomial interpretations	1989	Hofbauer and Lautemann
1979	simple path order		
1980	lexicographic path order semantic path order		
1981	recursive decomposition order		
1982	multiset path order	1990	Hofbauer
1983	recursive path order		
1990	transformation order		

Theorem (Hofbauer 1990)

termination proof by multiset path order implies primitive recursive upper bound on derivational complexity

	termination		derivational complexity 1991
1967	Knuth-Bendix order		
1975	polynomial interpretations	1989	Hofbauer and Lautemann
1979	simple path order		
1980	lexicographic path order semantic path order		
1981	recursive decomposition order		
1982	multiset path order	1990	Hofbauer
1983	recursive path order		
1990	transformation order		

	termination		derivational complexity	1992
1967	Knuth-Bendix order			
1975	polynomial interpretations	1989	Hofbauer and Lautemann	
1979	simple path order			
1980	lexicographic path order semantic path order			
1981	recursive decomposition order			
1982	multiset path order	1990	Hofbauer	
1983	recursive path order			
1990	transformation order			
1992	elementary interpretations type introduction			

	termination		derivational complexity 1993
1967	Knuth-Bendix order		
1975	polynomial interpretations	1989	Hofbauer and Lautemann
1979	simple path order		
1980	lexicographic path order semantic path order		
1981	recursive decomposition order		
1982	multiset path order	1990	Hofbauer
1983	recursive path order		
1990	transformation order		
1992	elementary interpretations type introduction		

	termination		**derivational complexity** 1994
1967	Knuth-Bendix order		
1975	polynomial interpretations	1989	Hofbauer and Lautemann
1979	simple path order		
1980	lexicographic path order semantic path order		
1981	recursive decomposition order		
1982	multiset path order	1990	Hofbauer
1983	recursive path order		
1990	transformation order		
1992	elementary interpretations type introduction		

	termination		derivational complexity 1995
1967	Knuth-Bendix order		
1975	polynomial interpretations	1989	Hofbauer and Lautemann
1979	simple path order		
1980	lexicographic path order semantic path order	1995	Weiermann
1981	recursive decomposition order		
1982	multiset path order	1990	Hofbauer
1983	recursive path order		
1990	transformation order		
1992	elementary interpretations type introduction		
1995	general path order semantic labeling dummy elimination		

Theorem (Hofbauer 1990)

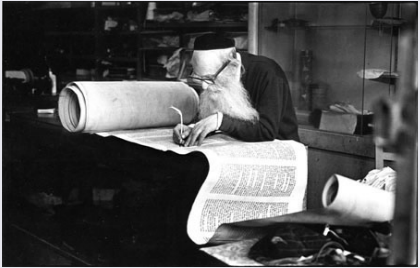
termination proof by multiset path order implies primitive recursive upper bound on derivational complexity

Theorem (Weiermann 1995)

termination proof by lexicographic path order implies multiple recursive upper bound on derivational complexity

	termination		**derivational complexity** 1996
1967	Knuth-Bendix order		
1975	polynomial interpretations	1989	Hofbauer and Lautemann
1979	simple path order		
1980	lexicographic path order semantic path order	1995	Weiermann
1981	recursive decomposition order		
1982	multiset path order	1990	Hofbauer
1983	recursive path order		
1990	transformation order		
1992	elementary interpretations type introduction		
1995	general path order semantic labeling dummy elimination		

Termination and Complexity Research



	termination	**derivational complexity**	1997
1997	dependency pairs		

	termination	**derivational complexity**	1998
1997	dependency pairs		

	termination	**derivational complexity**	1999
1997	dependency pairs		

	termination	**derivational complexity**	2000
1997	dependency pairs		
2000	monotonic semantic path order		

	termination		derivational complexity 2001
1967	Knuth-Bendix order	2001	Lepper
1975	polynomial interpretations	1989	Hofbauer and Lautemann
1979	simple path order		
1980	lexicographic path order semantic path order	1995	Weiermann
1981	recursive decomposition order		
1982	multiset path order	1990	Hofbauer
1983	recursive path order		
1990	transformation order		
1992	elementary interpretations type introduction		
1995	general path order semantic labeling dummy elimination		

Theorem (Hofbauer 1990)

termination proof by multiset path order implies primitive recursive upper bound on derivational complexity

Theorem (Weiermann 1995)

termination proof by lexicographic path order implies multiple recursive upper bound on derivational complexity

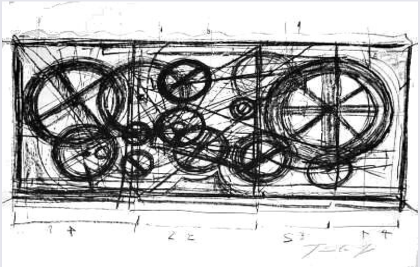
Theorem (Lepper 2001)

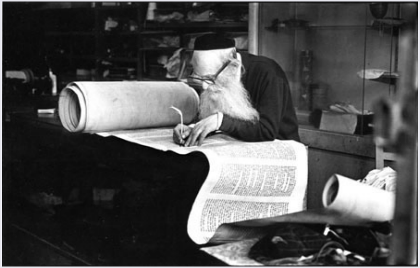
termination proof by Knuth-Bendix order implies multiple recursive upper bound on derivational complexity

	termination	**derivational complexity**	2001
1997	dependency pairs		
2000	monotonic semantic path order		
2001	context-dependent interpretations 2001		

	termination	**derivational complexity**	2002
1997	dependency pairs		
2000	monotonic semantic path order		
2001	context-dependent interpretations 2001		

Termination Research | Complexity Research





Termination Tools

CiME, $\mathsf{T_TT_2}$, AProVE, Termptation, Cariboo, Torpa, TPA, Matchbox, Jambox, MuTerm, NTI, VMTL, ...

	termination	derivational complexity	2003
1997	dependency pairs		
2000	monotonic semantic path order		
2001	context-dependent interpretations 2001		
2003	match-bounds size-change principle		
2003	**termination competition**		

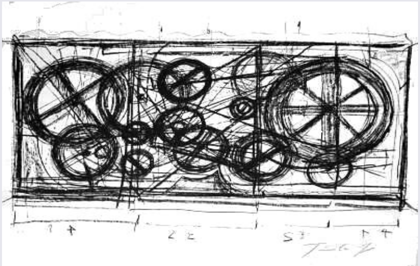
	termination		derivational complexity	2004
1997	dependency pairs			
2000	monotonic semantic path order			
2001	context-dependent interpretations	2001		
2003	match-bounds size-change principle	2004		
2003	**termination competition**			

	termination		derivational complexity	2005
1997	dependency pairs			
2000	monotonic semantic path order			
2001	context-dependent interpretations	2001		
2003	match-bounds size-change principle	2004		
2003	**termination competition**			

	termination		derivational complexity	2006
1997	dependency pairs			
2000	monotonic semantic path order			
2001	context-dependent interpretations	2001		
2003	match-bounds size-change principle	2004		
2003	**termination competition**			
2006	matrix interpretations predictive labeling uncurrying			

	termination		**derivational complexity**	2007
1997	dependency pairs			
2000	monotonic semantic path order			
2001	context-dependent interpretations	2001		
2003	match-bounds size-change principle	2004		
2003	**termination competition**			
2006	matrix interpretations predictive labeling uncurrying			
2007	bounded increase quasi-periodic interpretations			

Termination and Complexity Research



Termination Tools

CiME, $\mathrm{T_TT_2}$, AProVE, Matchbox, Jambox, MuTerm, VMTL, ...

Complexity Tools

$\mathrm{T_CT}$, Matchbox, CaT

	termination	derivational complexity	2008
1997	dependency pairs		
2000	monotonic semantic path order		
2001	context-dependent interpretations	2001, 2008	
2003	match-bounds size-change principle	2004	
2003	**termination competition**		
2006	matrix interpretations predictive labeling uncurrying	2008	
2007	bounded increase quasi-periodic interpretations		
2008	arctic interpretations root-labeling	2008	**complexity competition**

	termination		derivational complexity	2009
1997	dependency pairs	2009		
2000	monotonic semantic path order			
2001	context-dependent interpretations	2001, 2008		
2003	match-bounds size-change principle	2004		
2003	**termination competition**			
2006	matrix interpretations predictive labeling uncurrying	2008		
2007	bounded increase quasi-periodic interpretations			
2008	arctic interpretations root-labeling ...	2008	**complexity competition**	

	termination		**derivational complexity** 2010
1997	dependency pairs	2009	
2000	monotonic semantic path order		
2001	context-dependent interpretations	2001, 2008	
2003	match-bounds size-change principle	2004	
2003	**termination competition**		
2006	matrix interpretations predictive labeling uncurrying	2008, 2010	
2007	bounded increase quasi-periodic interpretations		
2008	arctic interpretations root-labeling ...	2008	**complexity competition**

	termination	derivational complexity	2011
1997	dependency pairs	2009, 2011	
2000	monotonic semantic path order		
2001	context-dependent interpretations	2001, 2008	
2003	match-bounds size-change principle	2004	
2003	**termination competition**		
2006	matrix interpretations predictive labeling uncurrying	2008, 2010, 2011	
2007	bounded increase quasi-periodic interpretations		
2008	arctic interpretations root-labeling ...	2008 **complexity competition**	

Outline

- Introduction
- History
- Matrix Interpretations
- Algebraic Methods
- Automata-Based Methods
- Unifying Algebraic and Automata-Based Methods
- Concluding Remarks

Definition

algebra $\mathcal{M}$ with well-founded order $>$

- carrier of $\mathcal{M}$ is $\mathbb{N}^d$ with $d > 0$
- $(x_1, \ldots, x_d)^\mathsf{T} > (y_1, \ldots, y_d)^\mathsf{T} \iff x_1 > y_1 \ \wedge \ \bigwedge_{i=2}^{d} x_i \geqslant y_i$
- interpretations (for every n-ary f)

$$f_\mathcal{M}(\vec{x}_1, \ldots, \vec{x}_n) = F_1\, \vec{x}_1 + \cdots + F_n\, \vec{x}_n + f$$

  with

  - matrices $F_1, \ldots, F_n \in \mathbb{N}^{d \times d}$ with $(F_i)_{1,1} \geqslant 1$ for all $1 \leqslant i \leqslant n$
  - vector $f \in \mathbb{N}^d$

Lemma

$(\mathcal{M}, >)$ *is well-founded monotone algebra*

Theorem

termination proof by matrix interpretation implies exponential upper bound on derivational complexity

Example

rewrite rule

$$\mathsf{a}(\mathsf{b}(x)) \to \mathsf{b}(\mathsf{b}(\mathsf{a}(x)))$$

matrix interpretation (linear polynomial interpretation)

$$\mathsf{a}_{\mathcal{M}}(x) = 3x \qquad \mathsf{b}_{\mathcal{M}}(x) = x + 1$$

derivational complexity is exponential

$$\mathsf{a}^2\mathsf{b} \to^3 \mathsf{b}^4\mathsf{a}^2 \qquad \mathsf{a}^3\mathsf{b} \to^7 \mathsf{b}^8\mathsf{a}^3 \qquad \mathsf{a}^4\mathsf{b} \to^{15} \mathsf{b}^{16}\mathsf{a}^4 \qquad \cdots$$

Theorem

termination proof by matrix interpretation implies exponential upper bound on derivational complexity

Example

rewrite rule

$$\mathsf{a}(\mathsf{b}(x)) \to \mathsf{b}(\mathsf{b}(\mathsf{a}(x)))$$

matrix interpretation (linear polynomial interpretation)

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Aim

restrict matrix interpretations to obtain polynomial derivational complexity

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restrict matrix interpretations to obtain polynomial derivational complexity

Original Approach (Moser, Schnabl, Waldmann 2008)

allow only special upper triangular matrices in interpretations

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restrict matrix interpretations to obtain polynomial derivational complexity

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allow only special upper triangular matrices in interpretations

Two Extensions

1. using weighted automata techniques (Waldmann 2010)

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restrict matrix interpretations to obtain polynomial derivational complexity

Original Approach (Moser, Schnabl, Waldmann 2008)

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Two Extensions

1. using weighted automata techniques (Waldmann 2010)
2. using linear algebra techniques (Neurauter, Zankl, Middeldorp 2010)

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restrict matrix interpretations to obtain polynomial derivational complexity

Original Approach (Moser, Schnabl, Waldmann 2008)

allow only special upper triangular matrices in interpretations

Two Extensions

1. using weighted automata techniques (Waldmann 2010)
2. using linear algebra techniques (Neurauter, Zankl, Middeldorp 2010)

joint spectral radius theory to unify and strengthen two extensions

Aim

restrict matrix interpretations to obtain polynomial derivational complexity

Original Approach (Moser, Schnabl, Waldmann 2008)

allow only special upper triangular matrices in interpretations

Two Extensions

1. using weighted automata techniques (Waldmann 2010)
2. using linear algebra techniques (Neurauter, Zankl, Middeldorp 2010)

joint spectral radius theory to unify and strengthen two extensions

are matrix interpretations complete for polynomial derivational complexity ?

are matrix interpretations complete for polynomial derivational complexity ?

Question

given TRS $\mathcal{R}$ which has polynomial derivational complexity

given compatible matrix interpretation $\mathcal{M}$

$\exists$ compatible matrix interpretation $\mathcal{N}$ that is polynomially bounded ?

Outline

- Introduction
- History
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  - Spectral Radius
  - Joint Spectral Radius
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Definition

$S_{\mathcal{M}}$ is set of matrices occurring in matrix interpretation $\mathcal{M}$:

$$S_{\mathcal{M}} \;=\; \bigcup_{n\text{-ary } f} \{\, F_1, \ldots, F_n \mid f_{\mathcal{M}}(\vec{x}_1, \ldots, \vec{x}_n) = F_1\,\vec{x}_1 + \cdots + F_n\,\vec{x}_n + f \,\}$$

Definition

$S_{\mathcal{M}}$ is set of matrices occurring in matrix interpretation $\mathcal{M}$:

$$S_{\mathcal{M}} = \bigcup_{n\text{-ary } f} \{ F_1, \dots, F_n \mid f_{\mathcal{M}}(\vec{x}_1, \dots, \vec{x}_n) = F_1 \vec{x}_1 + \cdots + F_n \vec{x}_n + f \}$$

Definition

matrix interpretation $\mathcal{M}$ is polynomially bounded[①] (with degree d) if growth of entries of matrix products

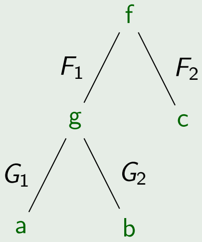
$$A_1 \cdots A_k$$

with $A_1, \dots, A_k \in S_{\mathcal{M}}$ is polynomial (with degree d) in k

Example

term t

$$\mathsf{f}(\mathsf{g}(\mathsf{a},\mathsf{b}),\mathsf{c})$$



matrix interpretation $\mathcal{M}$

$$\mathsf{a}_{\mathcal{M}} = a \qquad \mathsf{b}_{\mathcal{M}} = b \qquad \mathsf{c}_{\mathcal{M}} = c$$

$$\mathsf{f}_{\mathcal{M}}(\vec{x},\vec{y}) = F_1\,\vec{x} + F_2\,\vec{y} + f$$

$$\mathsf{g}_{\mathcal{M}}(\vec{x},\vec{y}) = G_1\,\vec{x} + G_2\,\vec{y} + g$$

interpretation of t

$$[t]_{\mathcal{M}} = F_1\,G_1\,a + F_1\,G_2\,b + F_1\,g + F_2\,c + f$$

Remark

term t of size at most k

- *. . . has at most k subterms*
- *. . . each subterm corresponds to product of at most k matrices*

Remark

term t of size at most k

- *... has at most k subterms*
- *... each subterm corresponds to product of at most k matrices*

Lemma

*if $\mathcal{R}$ has compatible matrix interpretation $\mathcal{M}$ that is polynomially bounded*① *with degree d then $dc_{\mathcal{R}}(k) \in \mathcal{O}(k^{d+1})$*

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Approximation

over-approximate growth of entries of matrix products

$$A_1 \cdots A_k \in S^*_{\mathcal{M}} \quad \text{by} \quad M^k$$

where $M_{ij} = \max \{ A_{ij} \mid A \in S_{\mathcal{M}} \}$

Definitions

square matrix $A \in R^{n \times n}$ over ring R ($\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$)

- spectral radius $\rho(A)$ of A is maximum of absolute values of its eigenvalues
- minimal polynomial $m_A(x)$ of A is unique monic polynomial of minimum degree that annihilates A

Theorem

if $\mathcal{R}$ *has compatible matrix interpretation* $\mathcal{M}$ *such that* $\rho(M) \leqslant 1$ *then*

$$dc_{\mathcal{R}}(k) \in \mathcal{O}(k^{d+1})$$

where $d = \max_{\lambda}(0, \# m_M(\lambda) - 1)$ *and*

- λ *ranges over eigenvalues of* A *with absolute value exactly one*
- $\# m_M(\lambda)$ *denotes multiplicity of* λ

Example

rewrite system $\mathcal{R}$

$$\mathsf{f}(\mathsf{f}(x)) \to \mathsf{f}(\mathsf{g}(\mathsf{f}(x))) \qquad \mathsf{g}(\mathsf{g}(x)) \to x \qquad \mathsf{b}(x) \to x$$

compatible matrix interpretation $\mathcal{M}$

$$\mathsf{f}_{\mathcal{M}}(\vec{x}) = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \vec{x} + \begin{pmatrix} 0 \\ 4 \\ 0 \end{pmatrix} \qquad \mathsf{g}_{\mathcal{M}}(\vec{x}) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \vec{x} + \begin{pmatrix} 1 \\ 0 \\ 3 \end{pmatrix}$$

$$\mathsf{b}_{\mathcal{M}}(\vec{x}) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \vec{x} + \begin{pmatrix} 4 \\ 4 \\ 0 \end{pmatrix}$$

Example

rewrite system $\mathcal{R}$

$$\mathsf{f}(\mathsf{f}(x)) \to \mathsf{f}(\mathsf{g}(\mathsf{f}(x))) \qquad \mathsf{g}(\mathsf{g}(x)) \to x \qquad \mathsf{b}(x) \to x$$

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$$\mathsf{b}_{\mathcal{M}}(\vec{x}) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \vec{x} + \begin{pmatrix} 4 \\ 4 \\ 0 \end{pmatrix} \qquad M = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{pmatrix}$$

Example

rewrite system $\mathcal{R}$

$$\mathsf{f}(\mathsf{f}(x)) \to \mathsf{f}(\mathsf{g}(\mathsf{f}(x))) \qquad \mathsf{g}(\mathsf{g}(x)) \to x \qquad \mathsf{b}(x) \to x$$

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$$\mathsf{b}_{\mathcal{M}}(\vec{x}) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \vec{x} + \begin{pmatrix} 4 \\ 4 \\ 0 \end{pmatrix} \qquad M = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{pmatrix} \quad \rho(M) = 2$$

derivational complexity is linear but

$$M^k = \begin{pmatrix} 1 & 2^{k-1} & 2^{k-1} - 1 \\ 0 & 2^{k-1} & 2^{k-1} \\ 0 & 2^{k-1} & 2^{k-1} \end{pmatrix}$$

Example

rewrite system $\mathcal{R}$

$$\mathsf{f}(\mathsf{f}(x)) \to \mathsf{f}(\mathsf{g}(\mathsf{f}(x))) \qquad \mathsf{g}(\mathsf{g}(x)) \to x \qquad \mathsf{b}(x) \to x$$

no component-wise maximum matrix of compatible matrix interpretation is polynomially bounded①

Example

rewrite system $\mathcal{R}$

$$\mathsf{f}(\mathsf{f}(x)) \to \mathsf{f}(\mathsf{g}(\mathsf{f}(x))) \qquad \mathsf{g}(\mathsf{g}(x)) \to x \qquad \mathsf{b}(x) \to x$$

no component-wise maximum matrix of compatible matrix interpretation is polynomially bounded①:

- compatible matrix interpretation $\mathcal{M}$ of dimension n:

$$\mathsf{b}_{\mathcal{M}}(\vec{x}) = B\vec{x} + b \qquad \mathsf{f}_{\mathcal{M}}(\vec{x}) = F\vec{x} + f \qquad \mathsf{g}_{\mathcal{M}}(\vec{x}) = G\vec{x} + g$$

Example

rewrite system $\mathcal{R}$

$$\mathsf{f}(\mathsf{f}(x)) \to \mathsf{f}(\mathsf{g}(\mathsf{f}(x))) \qquad \mathsf{g}(\mathsf{g}(x)) \to x \qquad \mathsf{b}(x) \to x$$

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- $Ff > FGf + Fg$

Example

rewrite system $\mathcal{R}$

$$\mathsf{f}(\mathsf{f}(x)) \to \mathsf{f}(\mathsf{g}(\mathsf{f}(x))) \qquad \mathsf{g}(\mathsf{g}(x)) \to x \qquad \mathsf{b}(x) \to x$$

no component-wise maximum matrix of compatible matrix interpretation is polynomially bounded[1]:

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- $Ff > FGf$

Example

rewrite system $\mathcal{R}$

$$\mathsf{f}(\mathsf{f}(x)) \to \mathsf{f}(\mathsf{g}(\mathsf{f}(x))) \qquad \mathsf{g}(\mathsf{g}(x)) \to x \qquad \mathsf{b}(x) \to x$$

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  hence $(M^k)_{ii}$ grows exponentially

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compatible matrix interpretation $\mathcal{M}$

$$\mathsf{f}_{\mathcal{M}}(\vec{x}) = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \vec{x} + \begin{pmatrix} 0 \\ 4 \\ 0 \end{pmatrix} \qquad \mathsf{g}_{\mathcal{M}}(\vec{x}) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \vec{x} + \begin{pmatrix} 1 \\ 0 \\ 3 \end{pmatrix}$$

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derivational complexity is linear: joint spectral radius

$$\rho\left(\left\{\begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\right\}\right) = 1$$

Outline

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- History
- Matrix Interpretations
- Algebraic Methods
  - Spectral Radius
  - Joint Spectral Radius
- Automata-Based Methods
- Unifying Algebraic and Automata-Based Methods
- Concluding Remarks

Definitions

finite set $S \subseteq \mathbb{R}^{n \times n}$ of real square matrices

- growth function

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Theorem

$\mathit{growth}_S(k) \in \mathcal{O}(k^d)$ for some $d \in \mathbb{N}$ if and only if $\rho(S) \leqslant 1$

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problem

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question:	$\rho(S) \leqslant 1$ *?*

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Theorem (based on Jungers, Protasov, Blondel 2008)

if $\rho(S) \leqslant 1$ *for finite set* $S \subseteq \mathbb{N}^{n \times n}$ *then*

$$growth_S(k) \in \Theta(k^{d})$$

where d *is largest integer such that* $\exists$ d *different pairs of indices* $(i_1, j_1), \ldots, (i_d, j_d)$

- $\forall\ 1 \leqslant n \leqslant d \quad i_n \neq j_n$ *and* $\exists$ *product* $A \in S^*$ *such that* $A_{i_n\, i_n}, A_{i_n\, j_n}, A_{j_n\, j_n} \geqslant 1$
- $\forall\ 1 \leqslant n < d \quad \exists$ *product* $B \in S^*$ *such that* $B_{j_n\, i_{n+1}} \geqslant 1$

Corollary

if $\mathcal{R}$ has compatible matrix interpretation $\mathcal{M}$ such that

$$\rho(S_{\mathcal{M}}) \leqslant 1$$

then $dc_{\mathcal{R}}(k) \in \mathcal{O}(k^{d+1})$ where d is largest integer such that ...

Remark

degree $d + 1$ can be computed in polynomial time

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matrix interpretation $\mathcal{M}$

- growth function of $\mathcal{M}$

$$\mathsf{growth}_{\mathcal{M}}(k) \;=\; \max\{\, [t]_1 \mid |t| \leqslant k \,\}$$

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- $\mathcal{M}$ is polynomially bounded[②] with degree d if $\mathrm{growth}_{\mathcal{M}}(k) \in \mathcal{O}(k^d)$

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matrix interpretation $\mathcal{M}$ of dimension 3

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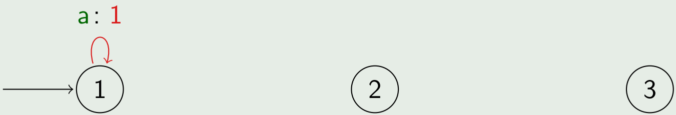

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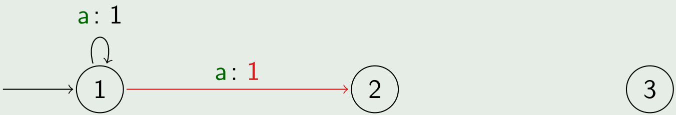

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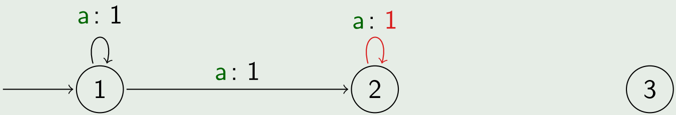

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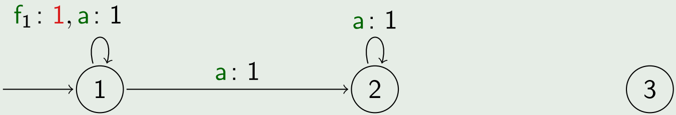

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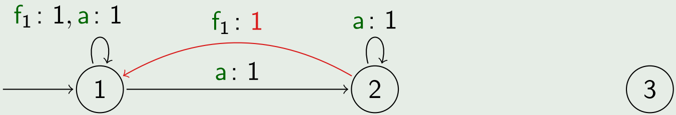

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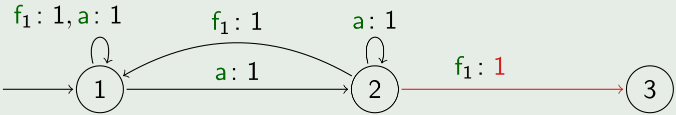

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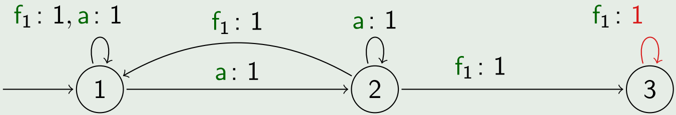

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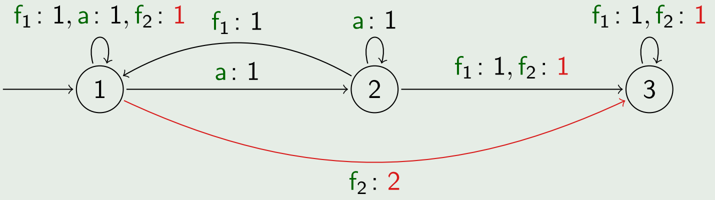

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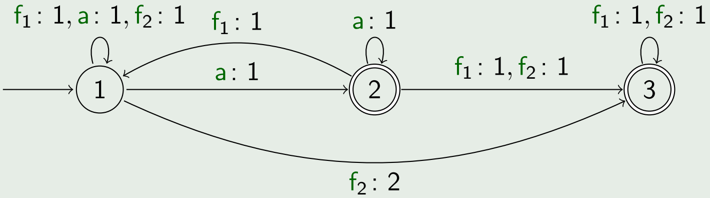

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1. Q: finite set of states
2. Σ: finite alphabet
3. $\lambda \in Q$ initial state
4. $\mu \colon \Sigma \to \mathbb{N}^{|Q| \times |Q|}$ transition matrix
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weight of string $x \in \Sigma^*$

$$\mathrm{weight}_{\mathcal{A}}(x) = \sum_{q \in \gamma} \mu(x)_{\lambda q}$$

Definition

growth function of weighted automaton $\mathcal{A} = (Q, \Sigma, \lambda, \mu, \gamma)$

$$\mathsf{growth}_{\mathcal{A}}(k) = \max \{ \mathsf{weight}_{\mathcal{A}}(x) \mid x \in \Sigma^k \}$$

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Definition

given matrix interpretation $\mathcal{M}$ of dimension n for signature $\mathcal{F}$
define weighted automaton $\mathcal{A} = (Q, \Sigma, \lambda, \mu, \gamma)$ as follows:

- $Q = \{1, \ldots, n\}$
- $\Sigma = \{ f_i \mid f \in \mathcal{F}$ has arity m and $1 \leqslant i \leqslant m \}$
- $\lambda = 1$
- $\mu(f_i) = F_i$ where F_i denotes i-th matrix of $f_{\mathcal{M}}$
- $\gamma = \{i \mid c_i > 0$ for some vector c in $\mathcal{M}\}$

Definition

weighted automaton $\mathcal{A} = (Q, \Sigma, \lambda, \mu, \gamma)$

- state q is useful if $\mathcal{A}$ contains path from initial to final state containing q

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$\forall$ *weighted automaton* $\mathcal{A}$ $\quad\exists$ *trim automaton* $\mathcal{B}$ $\quad$ *such that*

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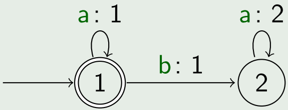
Example

weighted automaton $\mathcal{A}$ is not trim: state 2 is not useful



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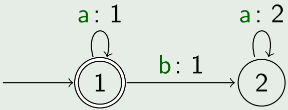
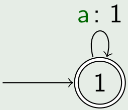


weighted automaton $\mathcal{B}$



Theorem

matrix interpretation $\mathcal{M}$ and corresponding weighted automaton $\mathcal{A}$

$$\mathit{growth}_{\mathcal{A}}(k) \in \mathcal{O}(k^d) \quad \iff \quad \mathit{growth}_{\mathcal{M}}(k) \in \mathcal{O}(k^{d+1})$$

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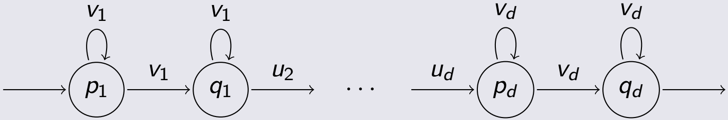
Definitions (based on Weber and Seidl 1991)

weighted automaton $\mathcal{A} = (Q, \Sigma, \lambda, \mu, \gamma)$

EDA $\quad \exists\, q \in Q\ \exists\, x \in \Sigma^*$ such that q is useful and $\mu(x)_{qq} \geqslant 2$

IDA$_d$ $\quad \exists\, p_1, q_1, \ldots, p_d, q_d \in Q\ \exists\, v_1, u_2, v_2, \ldots, u_d, v_d \in \Sigma^*$ such that
$\forall\, i \geqslant 1$ p_i and q_i are useful, $p_i \neq q_i$ and $p_i \xrightarrow{v_i} p_i \xrightarrow{v_i} q_i \xrightarrow{v_i} q_i$
$\forall\, i \geqslant 2$ $q_{i-1} \xrightarrow{u_i} p_i$



Theorem

matrix interpretation $\mathcal{M}$ *and corresponding weighted automaton* $\mathcal{A}$

$\mathit{growth}_{\mathcal{M}}(k) \in \mathcal{O}(k^{d+1}) \quad \iff \quad \mathcal{A} \not\models \mathsf{EDA},\ \mathcal{A} \not\models \mathsf{IDA}_{d+1}$

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Remark

conditions are decidable in time $\mathcal{O}(|Q|^6 \cdot |\Sigma|)$ *for* $\mathcal{A} = (Q, \Sigma, \lambda, \mu, \gamma)$

Corollary

if $\mathcal{R}$ *has compatible matrix interpretation* $\mathcal{M}$ *such that corresponding weighted automaton does not comply with* EDA *nor with* IDA_{d+1} *then* $\mathit{dc}_{\mathcal{R}}(k) \in \mathcal{O}(k^{d+1})$

Outline

- Introduction
- History
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- Algebraic Methods
- Automata-Based Methods
- Unifying Algebraic and Automata-Based Methods
- Concluding Remarks

Example

rewrite rule

$$\mathsf{f}(x) \to x$$

compatible matrix interpretation $\mathcal{M}$

$$\mathsf{f}_{\mathcal{M}}(\vec{x}) = \begin{pmatrix} 1 & 1 \\ 0 & 2 \end{pmatrix} \vec{x} + \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

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Lemma

for every TRS $\mathcal{R}$

$\forall$ compatible matrix interpretation $\mathcal{M}$ $\exists$ compatible matrix interpretation $\mathcal{N}$

such that corresponding automaton is trim and $\mathit{growth}_{\mathcal{M}}(k) = \mathit{growth}_{\mathcal{N}}(k)$

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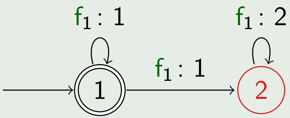

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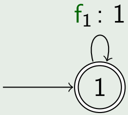

is trim and $\mathcal{M}$ is polynomially bounded[1]

Corollary

for every TRS $\mathcal{R}$

$$\exists \text{ compatible matrix interpretation } \mathcal{M} \text{ that is polynomially bounded}^{①}$$

$$\iff$$

$$\exists \text{ compatible matrix interpretation } \mathcal{N} \text{ that is polynomially bounded}^{②}$$

$\mathsf{growth}_{\mathcal{N}}(k) \in \mathcal{O}(k^{d+1})$ if and only if growth of entries of products $A_1 \cdots A_k$ of matrices in $\mathcal{M}$ is polynomial with degree d in k

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Theorem

for every TRS $\mathcal{R}$

$\mathrm{dc}_{\mathcal{R}}(k) \in \mathcal{O}(k^d)$ can be shown using automata-based approach

$$\iff$$

$\mathrm{dc}_{\mathcal{R}}(k) \in \mathcal{O}(k^d)$ can be shown using algebraic approach

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Lemma

matrix interpretations are incomplete for polynomial derivational complexity

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Example

rewrite system $\mathcal{R}$ with linear derivational complexity

$$\mathsf{f}(\mathsf{f}(x)) \to \mathsf{f}(\mathsf{g}(\mathsf{f}(x))) \qquad \mathsf{g}(\mathsf{g}(x)) \to x \qquad \mathsf{b}(x) \to x$$

compatible matrix interpretation $\mathcal{M}$

$$\mathsf{f}_{\mathcal{M}}(\vec{x}) = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \vec{x} + \begin{pmatrix} 0 \\ 4 \\ 0 \end{pmatrix} \qquad \mathsf{g}_{\mathcal{M}}(\vec{x}) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \vec{x} + \begin{pmatrix} 1 \\ 0 \\ 3 \end{pmatrix}$$

$$\mathsf{b}_{\mathcal{M}}(\vec{x}) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \vec{x} + \begin{pmatrix} 4 \\ 4 \\ 3 \end{pmatrix} \qquad \rho(\{\dots\}) = 1$$

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no polynomially bounded compatible matrix interpretation

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- compatible matrix interpretation $\mathcal{M}$ of dimension n:

$$\mathsf{b}_{\mathcal{M}}(\vec{x}) = B\vec{x} + b \qquad \mathsf{f}_{\mathcal{M}}(\vec{x}) = F\vec{x} + f \qquad \mathsf{g}_{\mathcal{M}}(\vec{x}) = G\vec{x} + g$$

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- $B \geqslant \max(I_n, G)$

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- $B \geqslant \max(I_n, G)$ ... hence entries in B^k grows exponentially

Remarks

- automation by mapping to finite-domain constraint systems (...)

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Definitions

- runtime complexity $\mathsf{rc}_{\mathcal{R}}(k) = \max\{\, \mathsf{dh}(t) \mid t \text{ is basic term and } |t| \leqslant k \,\}$

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Definitions

- runtime complexity $\mathsf{rc}_{\mathcal{R}}(k) = \max\{\,\mathsf{dh}(t) \mid t$ is basic term and $|t| \leqslant k\,\}$
- term $f(t_1, \dots, t_n)$ is basic if
  1. f is defined symbol
  2. $t_1, \dots, t_n$ are constructor terms

Example

rewrite system $\mathcal{R}$

$$
\begin{aligned}
\mathsf{reverse}(\mathsf{nil}) &\to \mathsf{nil} \\
\mathsf{reverse}(x :: xs) &\to \mathsf{append}(\mathsf{reverse}(xs), x :: \mathsf{nil}) \\
\mathsf{shuffle}(\mathsf{nil}) &\to \mathsf{nil} \\
\mathsf{shuffle}(x :: xs) &\to x :: \mathsf{shuffle}(\mathsf{reverse}(xs)) \\
\mathsf{append}(\mathsf{nil}, ys) &\to ys \\
\mathsf{append}(x :: xs, ys) &\to x :: \mathsf{append}(xs, ys)
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... beyond reach of complexity tools