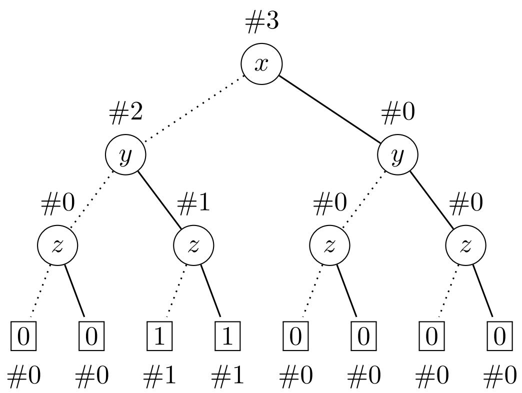
Selected Solutions

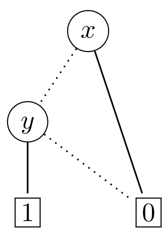
1 (a) From the table

x	y	z	$x \oplus y$	$\overline{x}$	$\mathrm{HWB}_4(x, x \oplus y, \overline{x}, z)$	$x\overline{z}$	$f(x,y,z)$
0	0	0	0	1	0	0	0
0	0	1	0	1	0	0	0
0	1	0	1	1	1	0	1
0	1	1	1	1	1	0	1
1	0	0	1	0	1	1	0
1	0	1	1	0	0	0	0
1	1	0	0	0	1	1	0
1	1	1	0	0	0	0	0

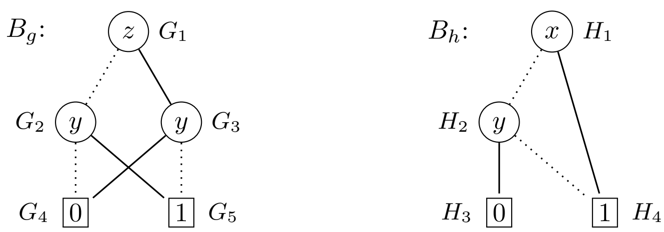
we obtain the binary decision tree



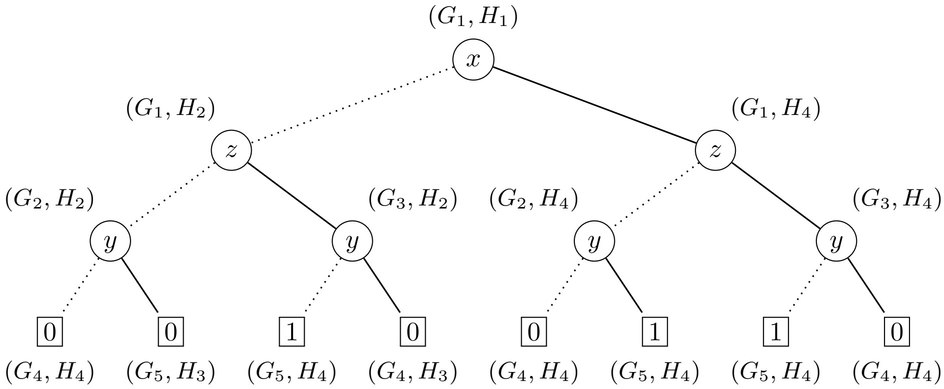
Applying the `reduce` algorithm produces the desired reduced OBDD



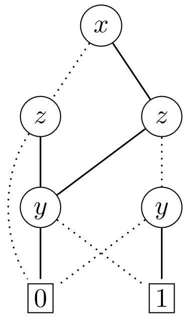
(b) For easy reference, we label the nodes of B_g and B_h:



When computing $\mathsf{apply}(\cdot, B_g, B_h)$, we can choose between the two variable orderings $[z, x, y]$ and $[x, z, y]$, which are compatible with both OBDDs. Taking the latter we obtain the intermediate OBDD

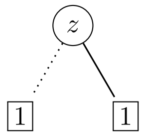


Applying the reduce algorithm produces the desired reduced OBDD $B_{g \cdot h}$:



(c) Since $\exists y.g = g[0/y] + g[1/y]$, we start by computing OBDDs for $B_{g[0/y]}$ and $B_{g[1/y]}$:

Next we compute apply$(+, B_{g[0/y]}, B_{g[0/y]})$, obtaining the intermediate OBDD



Applying the reduce algorithm produces the desired reduced OBDD $B_{\exists y.g}$:

2 (a) We have $\exists x.\varphi = \varphi[0/x] + \varphi[1/x]$.

- $\Leftarrow$ If $\exists x.\varphi$ is satisfiable then there is an assignment v such that $\bar{v}(\varphi[0/x] + \varphi[1/x]) = 1$. Hence $\bar{v}(\varphi[0/x]) = 1$ or $\bar{v}(\varphi[1/x]) = 1$ (or both). In the former case we can update v by the binding $v(x) = 0$ and in the latter case by $v(x) = 1$ in order obtain an assignment that satisfies φ.
- $\Rightarrow$ If φ is satisfiable then there is an assignment v such that $\bar{v}(\varphi) = 1$. Note that in this assignment, either $v(x) = 0$ or $v(x) = 1$. In both cases v also satisfies $\varphi[0/x] + \varphi[1/x]$.

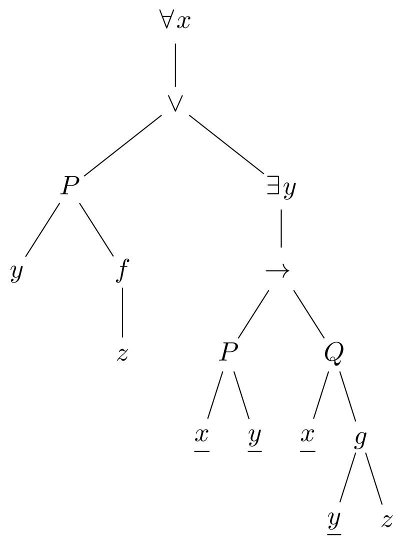
(b) We have $\forall x.\varphi = \varphi[0/x] \cdot \varphi[1/x]$.

- $\Leftarrow$ If $\forall x.\varphi$ is valid then for all assignments v we have $\bar{v}(\varphi[0/x] \cdot \varphi[1/x]) = 1$. Hence both $\bar{v}(\varphi[0/x]) = 1$ and $\bar{v}(\varphi[1/x]) = 1$. Shannon's expansion yields $\varphi = \bar{x} \cdot \varphi[0/x] + x \cdot \varphi[1/x]$ and thus

$$\bar{v}(\varphi) = \bar{v}(\bar{x} \cdot \varphi[0/x] + x \cdot \varphi[1/x]) = \bar{v}(\bar{x}) \cdot 1 + \bar{v}(x) \cdot 1 = \bar{v}(\bar{x}) + \bar{v}(x) = 1$$

- $\Rightarrow$ If φ is valid then for all assignments v we have $\bar{v}(\varphi) = 1$, so in particular both $\bar{v}(\varphi[0/x]) = 1$ and $\bar{v}(\varphi[1/x]) = 1$.

4 (a)



There are 7 subformulas: $Q(x, g(y, z))$, $P(x, y)$, $P(x, y) \to Q(x, g(y, z))$, $\exists y\,(P(x, y) \to Q(x, g(y, z)))$, $P(y, f(z))$, $P(y, f(z)) \vee \exists y\,(P(x, y) \to Q(x, g(y, z)))$ and $\forall x\,(P(y, f(z)) \vee \exists y\,(P(x, y) \to Q(x, g(y, z))))$.

(b) The underlined variable occurrences in the parse tree are bound, the others are free.

(c) i. We have

$$\begin{aligned}
\varphi[f(y)/x] &= \varphi \\
\varphi[f(y)/y] &= \forall x\,(P(\underline{f(y)}, f(z)) \vee \exists y\,(P(x, y) \to Q(y, g(y, z)))) \\
\varphi[f(y)/z] &= \forall x\,(P(y, f(\underline{f(y)})) \vee \exists y\,(P(x, y) \to Q(y, g(y, \underline{f(y)}))))
\end{aligned}$$

The term $f(y)$ is free for x and y but not for z.

ii. We have

$$\begin{aligned}
\varphi[g(x, z)/x] &= \varphi \\
\varphi[g(x, z)/y] &= \forall x\,(P(\underline{g(x, z)}, f(z)) \vee \exists y\,(P(x, y) \to Q(y, g(y, z)))) \\
\varphi[g(x, z)/z] &= \forall x\,(P(y, f(\underline{g(x, z)})) \vee \exists y\,(P(x, y) \to Q(y, g(y, \underline{g(x, z)}))))
\end{aligned}$$

The term $g(x, z)$ is free for x but not for y and z.

iii. We have

$$\begin{aligned}
\varphi[g(f(z), z)/x] &= \varphi \\
\varphi[g(f(z), z)/y] &= \forall x\,(P(\underline{g(f(z), z)}, f(z)) \vee \exists y\,(P(x, y) \to Q(y, g(y, z)))) \\
\varphi[g(f(z), z)/z] &= \forall x\,(P(y, f(\underline{g(f(z), z)})) \vee \exists y\,(P(x, y) \to Q(y, g(y, \underline{g(f(z), z)}))))
\end{aligned}$$

The term $g(f(z), z)$ is free for x, y and z.