universität innsbruck

Logic	26S	LVA 703026 + 703027
Lecture 6		April 30, 2026

Selected Solutions

1 **(a)** The meaning of φ in $\mathcal{M}_1$ is "for every natural number x there exists a natural number y such that $y = x + 1$ and for all natural numbers z, if $z = y + 1$ then $x \neq z + 1$". This holds because $y = x + 1$ and $z = y + 1$ imply $z = x + 2$ and thus $x \neq z + 1$. Hence $\mathcal{M}_1$ satisfies φ.

(b) The meaning of φ in $\mathcal{M}_2$ is "for every natural number x there exists a natural number y such that $2x \neq y$ and for all natural numbers z, if $2y \neq z$ then $2z = x$". This does not hold since $2x \neq y$ and $2y \neq z$ do not imply $2z = x$. For instance, for $x = 0$ we have to choose $y \neq 0$ and can take $z = 1$. Hence $\mathcal{M}_2$ does not satisfy φ.

(c) In $\mathcal{M}_3$ the meaning of the sentence φ is "for every natural number x there exists a natural number y such that $x < y$ and for all natural numbers z, if $y < z$ then $z \geqslant x$". This holds because $x < y$ and $y < z$ imply $z > x$. So we can take $y = x + 1$. Hence $\mathcal{M}_3$ satisfies φ.

3 **(c)** The sequent $\exists x\,(P(x) \land \exists y\,(\neg(x = y) \land \neg Q(y))) \vdash \neg\forall x\,(P(x) \land Q(x))$ is valid:

1		$\exists x\,(P(x) \land \exists y\,(\neg(x = y) \land \neg Q(y)))$	premise
2		$\forall x\,(P(x) \land Q(x))$	assumption
3	x_0	$P(x_0) \land \exists y\,(\neg(x_0 = y) \land \neg Q(y))$	assumption
4		$\exists y\,(\neg(x_0 = y) \land \neg Q(y))$	$\land e_2$ 3
5	y_0	$\neg(x_0 = y_0) \land \neg Q(y_0)$	assumption
6		$\neg Q(y_0)$	$\land e_2$ 5
7		$P(y_0) \land Q(y_0)$	$\forall e$ 2
8		$Q(y_0)$	$\land e_2$ 7
9		$\bot$	$\neg e$ 8, 6
10		$\bot$	$\exists e$ 4, 5−9
11		$\bot$	$\exists e$ 1, 3−10
12		$\neg\forall x\,(P(x) \land Q(x))$	$\neg i$ 2−11

4 **(a)**

1		$\exists x\,(P \to Q(x))$	premise
2		P	assumption
3	x_0	$P \to Q(x_0)$	assumption
4		$Q(x_0)$	$\to e$ 3, 2
5		$\exists x\, Q(x)$	$\exists i$ 4
6		$\exists x\, Q(x)$	$\exists e$ 1, 3−5
7		$P \to \exists x\, Q(x)$	$\to i$ 2−6

(b)

1		$\forall x\, Q(x) \to P$	premise
2		$\exists x\, \neg Q(x) \lor \neg\exists x\, \neg Q(x)$	LEM
3		$\exists x\, \neg Q(x)$	assumption
4	x_0	$\neg Q(x_0)$	assumption
5		$Q(x_0)$	assumption
6		$\bot$	$\neg$e 5, 4
7		P	$\bot$e 6
8		$Q(x_0) \to P$	$\to$i 5–7
9		$\exists x\, (Q(x) \to P)$	$\exists$i 8
10		$\exists x\, (Q(x) \to P)$	$\exists$e 3, 4–9
11		$\neg\exists x\, \neg Q(x)$	assumption
12		$Q(z)$	assumption
13	x_0		
14		$\neg Q(x_0)$	assumption
15		$\exists x\, \neg Q(x)$	$\exists$i 14
16		$\bot$	$\neg$e 15, 11
17		$Q(x_0)$	PBC 14–16
18		$\forall x\, Q(x)$	$\forall$i 13–17
19		P	$\to$e 1, 18
20		$Q(z) \to P$	$\to$i 12–19
21		$\exists x\, (Q(x) \to P)$	$\exists$i 20
22		$\exists x\, (Q(x) \to P)$	$\lor$e 2, 3–10, 11–21

(c)

1		$P \to \exists x\, Q(x)$	premise
2		$\exists x\, Q(x) \lor \neg\exists x\, Q(x)$	LEM
3		$\exists x\, Q(x)$	assumption
4	x_0	$Q(x_0)$	assumption
5		P	assumption
6		$Q(x_0)$	copy 4
7		$P \to Q(x_0)$	$\to$i 5–6
8		$\exists x\, (P \to Q(x))$	$\exists$i 7
9		$\exists x\, (P \to Q(x))$	$\exists$e 3, 4–8
10		$\neg\exists x\, Q(x)$	assumption
11		P	assumption
12		$\exists x\, Q(x)$	$\to$e 1, 11
13		$\bot$	$\neg$e 12, 10
14		$Q(x_1)$	$\bot$e 13
15		$P \to Q(x_1)$	$\to$i 11–14
16		$\exists x\, (P \to Q(x))$	$\exists$i 15
17		$\exists x\, (P \to Q(x))$	$\lor$e 2, 3–9, 10–16