Selected Solutions

1 **(a)**

1		$\exists x\,(P(x) \to Q)$	premise
2		$\forall x\,P(x)$	assumption
3	x_0	$P(x_0) \to Q$	assumption
4		$P(x_0)$	$\forall$e 2
5		Q	$\to$e 3, 4
6		Q	$\exists$e 1, 3–5
7		$\forall x\,P(x) \to Q$	$\to$i 2–6

(b)

1		$\forall x\,\forall y\,(P(x) \vee Q(y))$	premise
2		$\forall x\,P(x) \vee \neg\forall x\,P(x)$	LEM
3		$\forall x\,P(x)$	assumption
4		$\forall x\,P(x) \vee \forall y\,Q(y)$	$\vee$i$_1$ 3
5		$\neg\forall x\,P(x)$	assumption
6		$\neg\exists x\,\neg P(x)$	assumption
7	x_0		
8		$\neg P(x_0)$	assumption
9		$\exists x\,\neg P(x)$	$\exists$i 8
10		$\bot$	$\neg$e 9, 6
11		$P(x_0)$	PBC 8–10
12		$\forall x\,P(x)$	$\forall$i 7–11
13		$\bot$	$\neg$e 12, 5
14		$\exists x\,\neg P(x)$	PBC 6–13
15	x_0	$\neg P(x_0)$	assumption
16		$\forall y\,(P(x_0) \vee Q(y))$	$\forall$e 1
17	y_0	$P(x_0) \vee Q(y_0)$	$\forall$e 16
18		$P(x_0)$	assumption
19		$\bot$	$\neg$e 18, 15
20		$Q(y_0)$	$\bot$e 19
21		$Q(y_0)$	assumption
22		$Q(y_0)$	$\vee$e 17, 18–20, 21
23		$\forall y\,Q(y)$	$\forall$i 17–22
24		$\forall y\,Q(y)$	$\exists$e 14, 15–23
25		$\forall x\,P(x) \vee \forall y\,Q(y)$	$\vee$i$_2$ 24
26		$\forall x\,P(x) \vee \forall y\,Q(y)$	$\vee$e 2, 3–4, 5–25

(c)

1		$\forall x\, \exists y\, (P(y) \lor Q(x))$	premise
2		$\exists y\, P(y) \lor \neg \exists y\, P(y)$	LEM
3		$\exists y\, P(y)$	assumption
4	y_0	$P(y_0)$	assumption
5	x_0	$P(y_0) \lor Q(x_0)$	$\lor \mathrm{i}_1$ 4
6		$\forall x\, (P(y_0) \lor Q(x))$	$\forall$i 5
7		$\exists y\, \forall x\, (P(y) \lor Q(x))$	$\exists$i 6
8		$\exists y\, \forall x\, (P(y) \lor Q(x))$	$\exists$e 3, 4−7
9		$\neg \exists y\, P(y)$	assumption
10	x_0	$\exists y\, (P(y) \lor Q(x_0))$	$\forall$e 1
11	y_0	$P(y_0) \lor Q(x_0)$	assumption
12		$P(y_0)$	assumption
13		$\exists y\, P(y)$	$\exists$i 13
14		$\perp$	$\neg$e 13, 9
15		$Q(x_0)$	$\perp$e 15
16		$Q(x_0)$	assumption
17		$Q(x_0)$	$\lor$e 11, 12−15, 16
18		$Q(x_0)$	$\exists$e 10, 11−17
19		$P(y) \lor Q(x_0)$	$\lor \mathrm{i}_1$ 18
20		$\forall x\, (P(y) \lor Q(x))$	$\forall$i 10−19
21		$\exists y\, \forall x\, (P(y) \lor Q(x))$	$\exists$i 20
22		$\exists y\, \forall x\, (P(y) \lor Q(x))$	$\lor$e 2, 3−8, 9−21

3 (a) We first transform the given formula into an equivalent prenex normal form:

$$
\begin{aligned}
&(\exists x\, \forall y\, \neg P(x,y)) \to (\forall y\, \exists x\, \neg P(x,y)) \\
&\equiv\ (\exists x\, \forall y\, \neg P(x,y)) \to (\forall v\, \exists u\, \neg P(u,v)) \\
&\equiv\ \forall x\, \exists y\, \forall v\, \exists u\, (\neg P(x,y) \to \neg P(u,v))
\end{aligned}
$$

Next, we transform the quantifier-free part of the prenex normal form into CNF:

$$\forall x\, \exists y\, \forall v\, \exists u\, (P(x,y) \lor \neg P(u,v))$$

We obtain an equisatisfiable Skolem normal form by replacing the existentially quantified variables by fresh Skolem functions, y by $f(x)$ and u by $h(x,v)$:

$$\approx\ \forall x\, \forall v\, (P(x, f(x)) \lor \neg P(h(x,v), v))$$

(b) We first eliminate the implication:

$$
\begin{aligned}
&\forall x\, \exists y\, \forall x\, z (Q(x,y,z) \to \exists w\, R(y,z,w)) \\
&\equiv\ \forall x\, \exists y\, \forall z\, (\neg Q(x,y,z) \lor \exists w\, R(y,z,w))
\end{aligned}
$$

Next, we bring all quantifiers to the front to obtain a prenex normal form:

$$\equiv\ \forall x\, \exists y\, \forall z\, \exists w\, (\neg Q(x,y,z) \lor R(y,z,w))$$

We obtain an equisatisfiable Skolem normal form by replacing the existentially quantified variables by Skolem functions, y by $f(x)$ and w by $h(x,z)$:

$$\approx\ \forall x\, \forall z\, (\neg Q(x, f(x), z) \lor R(f(x), z, h(x,z)))$$

4 The formula is valid:

1	$\forall y\, D(y) \vee \neg\forall y\, D(y)$	LEM
2	$\forall y\, D(y)$	assumption
3	$D(a)$	assumption
4	$\forall y\, D(y)$	copy 2
5	$D(a) \rightarrow \forall y\, D(y)$	$\rightarrow$i 3 $-$ 4
6	$\exists x\,(D(x) \rightarrow \forall y\, D(y))$	$\exists$i 5
7	$\neg\forall y\, D(y)$	assumption
8	$\neg\exists y\, \neg D(y)$	assumption
9	y_0	
10	$\neg D(y_0)$	assumption
11	$\exists y\, \neg D(y)$	$\exists$i 10
12	$\perp$	$\neg$e 11, 8
13	$D(y_0)$	PBC 10 $-$ 12
14	$\forall y\, D(y)$	$\forall$i 9 $-$ 13
15	$\perp$	$\neg$e 14, 7
16	$\exists y\, \neg D(y)$	PBC 8 $-$ 15
17	y_0 $\neg D(y_0)$	assumption
18	$D(y_0)$	assumption
19	$\perp$	$\neg$e 18, 17
20	$\forall y\, D(y)$	$\perp$e 19
21	$D(y_0) \rightarrow \forall y\, D(y)$	$\rightarrow$i 18 $-$ 20
22	$\exists x\,(D(x) \rightarrow \forall y\, D(y))$	$\exists$i 21
23	$\exists x\,(D(x) \rightarrow \forall y\, D(y))$	$\exists$e 16, 17 $-$ 22
24	$\exists x\,(D(x) \rightarrow \forall y\, D(y))$	$\vee$e 1, 2 $-$ 6, 7 $-$ 23