universität innsbruck

Logic

Luca Campa Philipp Dablander Aaron Groß Aart Middeldorp
Alexander Montag Johannes Niederhauser Vera Schmitt

Outline

VO is streamed and recorded

Particify

ars.uibk.ac.at

with session ID **6893 6178** for anonymous questions

Important Information

- LVA 703026 (VO 3) + 703027 (PS 2)
- http://cl-informatik.uibk.ac.at/teaching/ss26/lics
- online registration for VO required until June 30
- OLAT links for VO and PS

Time and Place

VO	Monday	8:30 – 11:00	HSB 1	Aart		
TU	Tuesday	15:30 – 16:15	HS G	Aaron		
PS	Thursday	8:15 – 9:45	HS 10	Alexander	group 1	
	Thursday	12:00 – 13:30	HS 10	Vera	group 2	
	Thursday	8:30 – 10:00	HSB 4	Luca	group 3	(in English)
	Thursday	12:00 – 13:30	HSB 8	Philipp	group 4	
	Thursday	8:30 – 10:00	SR 12	Vera	group 5	
	Thursday	12:00 – 13:30	HS C	Johannes	group 6	(in English)
	Thursday	12:00 – 13:30	HS E	Alexander	group 7	

Consultation Hours

Lucas Campa	3M03	Wednesday	10:00 – 11:30
Philipp Dablander	3M03	Wednesday	10:00 – 11:30
Aart Middeldorp	3M07	Monday	12:00 – 13:30
Alexander Montag	ÖH Technik	Monday	11:30 – 12:30
Johannes Niederhauser	3M03	Wednesday	10:00 – 11:30
Vera Schmitt			by arrangement

Schedule

lecture 1	02.03 & 05.03 & 12.03	lecture 8	04.05 & 21.05
lecture 2	09.03 & 19.03	lecture 9	11.05 & 21.05
lecture 3	16.03 & 26.03	lecture 10	18.05 & 28.05
lecture 4	23.03 & 16.04	lecture 11	01.06 & 11.06
lecture 5	13.04 & 23.04	lecture 12	08.06 & 18.06
lecture 6	20.04 & 30.04	lecture 13	15.06 & 25.06
lecture 7	27.04 & 07.05	lecture 14	22.06

Announcements

- VO is streamed and recorded
- PS in presence

Grading — VO

- first exam on June 29
- registration starts 5 weeks before exam and ends 2 weeks before exam
- late registration requests will be ignored
- de−registration is possible until 23:59 on June 26
- second exam on September 15
- third exam on February 4, 2027

Grading — PS

$$\text{score} = \min\left(\tfrac{2}{3}(E+P)+B, 100\right)$$

E: points for solved exercises (at most 130)

B: points for bonus exercises (at most 20)

P: points for presentations of solutions (at most 20)

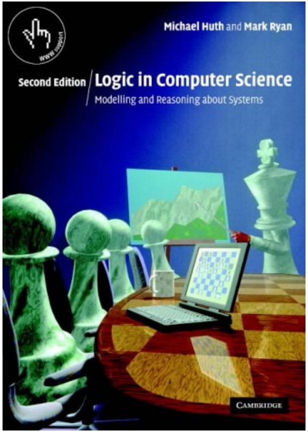
grade: $[0,50) \to \mathbf{5}$ $\quad [50,63) \to \mathbf{4}$ $\quad [63,75) \to \mathbf{3}$ $\quad [75,88) \to \mathbf{2}$ $\quad [88,100] \to \mathbf{1}$

- homework exercises are given on course web site
- solved exercises must be marked in OLAT
- solutions must be uploaded (PDF) in OLAT; deadline: 7 am on Thursday
- 10 points per PS
- two presentations of solutions are mandatory
- 20 points for two presentations; additional presentations give bonus points
- attendance is compulsory; unexcused absence is allowed twice (resulting in 0 points)

Literature

Michael Huth and Mark Ryan
Logic in Computer Science (second edition)
Cambridge University Press, 2004

in Semesterapparat digital version



Online Material

- slides are available on Thursday before lecture on Monday
- solutions to selected exercises are available after they have been discussed in PS

evaluation 25S

Evaluation 25S (selected comments)

- Fragen im ARS öfter anschauen.
- Better live streams, the board should be visible as well
- Bitte das Mikro für den Livestream einschalten
- Das Tutorium könnte länger dauern, damit man mehr Zeit hat, Inhalte zu besprechen, die unklar waren.
- Ein paar Beschreibungen in den Folien kann man falsch interpretieren. Besonders bei den Beweisen, und bei Algorithmen. Ich glaube in diesen Fällen würde ein Satz helfen, um einen Beweisschritt zu erklären. Anstatt nur ein Stichpunkt, wo man nach der Vorlesung nicht mehr weiß, wie man den Schritt interpretieren soll.
- Everything was perfect
- I am happy to learn this course and there is no suggestion because professor teach us very well
- Ich würde es bevorzugen, wenn die aktullen übungen in der gleichen woche besprochen würden
- I think the tempo is quite fast, i know there are lots of topics to talk about, but it still is very fast.
- It is one of the courses, where I actually wouldn't know what to improve.
- There were some times where I became lost on what was going on in the lecture, but that was more so a skill issue on my part being distracted and missing important parts.

Evaluation 25S (selected comments, cont'd)

- It's hard to figure out theoretical material one or two weeks after the lecture, but examples, which are given are more or less self explanatory. For each new theoretical material new example with the explanation on the slides, where explains where exactly in this example new theoretical material is used, would be very helpful.
- Really well done.
- Maybe publish the solutions of the Intermezzos after the lecture
- mehr Beispiele zu der Theorie machen
- More interactivity if possible. Might not be possible because it is a lot of content to go through
- nothing. This course is already perfect, in my opinion.
- Sometimes the slides seem a bit misleading. One example is the formula for the ANF. It was not explained how to calculate c_A but only mentioned that is is element of {0, 1}.
- Additional resource with Hut and Ryan was really helpful, including going through exercises together on each topic. Also having a live-stream makes thing a lot easier, for people who can't be there in presence for various reasons (especially when you are older and have more responsibilities). The huge amount of material to practice, including old exercises was also a really big help.
- Maybe make it more easier.

Evaluation 25S (selected comments, cont'd)

- Zu viel Stoff vielleicht ein bisschen weniger da der abgedeckte Bereich riesig ist, dürfen wir jetzt einen cheatsheet nutzen bei der Klausur?
- It really helps that in the slides are a lot of examples which ate easy to follow if one downoads the version that is presented. It is nice that there are a lot of previous exams available
- Ein wöchentliches Quiz im OLAT wäre cool.
- Dass wir die Hausaufgaben fürs PS in der VO erhalten haben und so über eine Woche Zeit dafür hatten.
- Die Beispiel in den Folien waren gut. Damit konnte man die PS Aufgaben gut bearbeiten. In anderen Fächern kann man nämlich die PS Aufgaben nicht mit dem Wissen aus der VO bearbeiten.
- Die Hausaufgaben haben mir spaß gemacht. Besonders die Bonus-Aufgaben sind cool.
- How professor tries to explain. But I don't understand
- how well organized it is, especially when taking together with the PS. If you do homework a little bit early, there are parts that are kind of difficult, but the new material in the next lecture helps understanding the previous content way better. The topics either build on each other or start something else, which feels very like it's one continuous lesson. The lecturer is sometimes funny and that makes a lot of the content easier to memorize. Quizes during the break are also a pretty good way to help comprehend the content.

Evaluation 25S (selected comments, cont'd)

- I like that during the semester we revisit past topics to refresh our knowledge on past topics and towards the end of each lecture we sometimes do a task to prepare for the exam, which is very effective. Also questions are answered with engagement. I like that for mist theorems a proof was provided (and if not there was a good reasoning not to do so) Topics (other than ANF) where explained very intuitively
- it was easy to learn the lecture material after solving homework
- Liveübertragung, Recording, Stoff wird gut rübergebracht sodass man alles wichtige versteht, tiefes Verständnis des Professors, Beispiele sind gut (mehr bitte)
- Motivated, humorous teacher that can explain very well
- The knowledge and enthusiasm of the professor. This is my third time attending this course and everytime it is basically exactly the same so how you keep your enthusiasm alive for exactly the same content is astonishing. The slides are really good.
- Although when the course started I was not too interested I grew to like the topics presented, so everything is good
- Best prepared VO I've visited sofar.

Evaluation 25S (selected comments, cont'd)

- Beste Vorlesung dieses Semester aber der Stoff ist mit am schwersten und weitaus am meisten, Rückfragen besser beantworten wäre cool, da diese meist nur ganz kurz angeschnitten werden und man nicht mitkommt
- es wäre toll wenn es Lösungen für die Altklausuren geben würde, dann könnte man seine Lösungen überprüfen
- For me this is really, really hard and boring i don't really see the value in knowing this stuff outside of working for maybe NASA or continuing with a Master in logic and working later on for study groups at the University.
- Manchmal wird, aus dem nichts, eine Interaktiver Teil geführt, generell sehr gut, aber einzelne S. reagieren schlecht auf diesen cold call.
- One of the best VOs I've visited
- Please make it easier maybe
- Thank you very much to teach us and i learned lots of things !

Outline

Logic in Innsbruck

Donnerstag, 11. Mai 2006 **WISSEN HEUTE** *LFU INNSBRUCK* **5**

ZEITTAFEL der LFU Innsbruck

1669	Gründung der Universität Innsbruck aus dem seit 100 Jahren bestehenden Jesuitengymnasiums durch Leopold I.
1669/70	Aufnahme des Lehrbetriebs durch die Jesuiten. Erster Universitätskurs wird im Fach Logik abgehalten.
1677	Durch die Bestätigung der Errichtung durch Papst Innozenz XI. erlangt die LFU ihre volle Rechtsgültigkeit. Vor dem Hintergrund wissenschaftlich aufblühender protestantischer Hochschulen sollte Innsbruck das katholische Bollwerk zwischen Deutschland und Italien werden.
1781	Kaiser Joseph I. stufte die Universität Innsbruck zu einem Lyzeum zu Gunsten der Zentraluniversitäten Wien und Prag herab.
1792	Wiedereinrichtung durch Leopold II.
1809	Studentenkompanien beteiligen sich am Tiroler Freiheitskampf.
1810	Aufhebung durch die Bayern

WISSENSWERT

Mit Maria-Theresia kam die Bibliothek an die Universität: Am 22. Mai 1745 genehmigte Maria Theresia die Errichtung einer Innsbrucker Bibliothek. Grundstein bildete die Büchersammlung der Tiroler Habsburger. Die Bibliothek war öffentlich zugänglich und die Benützerordnung war streng: Es durfte immer nur ein Buch vor Ort gelesen werden und auf Bücherentwendungen stand die Exmatrikulation.

Innsbruck zieht Studierende an: 1684 meldet die Chronik des

Formal Logic at Department of Christian Philosophy

Logic in Computer Science

- During the past 30 years, there has been an extensive and growing interaction between logic and computer science.
- Concepts and methods of logic occupy a central place in computer science, insomuch that logic has been called the calculus of computer science.
- Logic has been much more effective in computer science than it has been in mathematics.

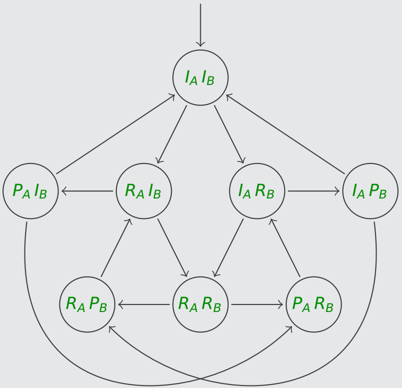
Phokion G. Kolaitis, Moshe Y. Vardi (2001)

Example（数独 Sudoku）

8								
		3	6					
	7			9		2		
	5				7			
				4	5	7		
			1				3	
		1					6	8
		8	5			4	1	
	9							

propositional logic is very useful to quickly develop efficient solver for Sudoku and all kinds of other tasks

Example (Printer Manager)



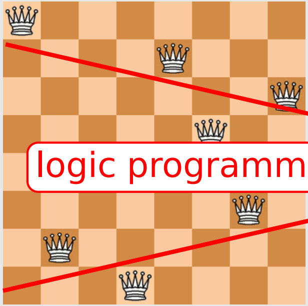
two users A and B

I_i	user i is idle
R_i	print request by user i
P_i	printing document for user i

some questions

- is every P_i preceded by R_i ?
- is every R_i eventually followed by P_i ?

Example (Eight Queens Puzzle)



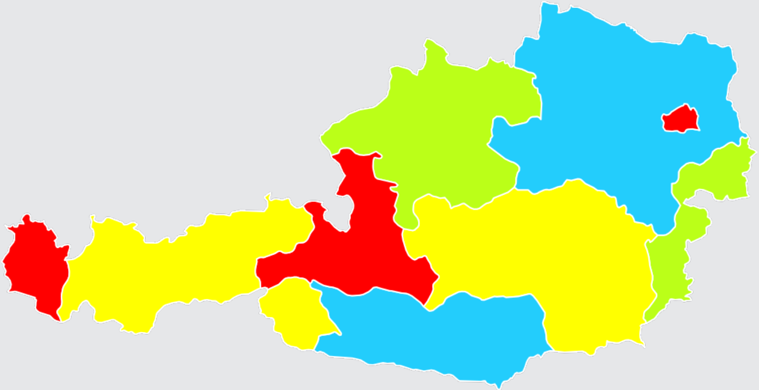
Prolog code

```
:- use_module(library(clpfd)).
nqueens(N,Qs) :-
  length(Qs,N), Qs ins 1 .. N,
  all_different(Qs),
  constraint_queens(Qs), label(Qs).
constraint_queens([])
  constraint_queens(Qs).
noattack(_,[],_).
noattack(X,[Q|Qs],N) :-
  X #\= Q+N, X #\= Q-N, M is N+1,
  noattack(X,Qs,M).
```

```
?- nqueens(8,Xs).
```

logic programming is sometimes taught in elective module

Example (Coloring Austria)



question: do three colors suffice to color Austria ?

Example (Programming Task)

function that computes $\sum_{i=1}^{n} i$ (exercise in C programming course at Nagoya University)

some answers:

```
int sum(int x) {
  int i, j, z;
  z = 0;
  for (i = 0; i <= x; i++)
    for (j = 0; j < i; j++)
      z++;
  return z;
}
```

```
int sum(int n) {
  if (n <= 0) {
    return 0;
  } else {
    return (n*(n+1)/2);
  }
}
```

question: are these programs correct ?

Greek Alphabet

alpha	α	A	eta	η	H	nu	ν	N	tau	τ	T
beta	β	B	theta	θ ϑ	Θ	xi	ξ	Ξ	upsilon	υ	Υ
gamma	γ	Γ	iota	ι	I	omicron	o	O	phi	ϕ φ	Φ
delta	δ	Δ	kappa	κ	K	pi	π	Π	chi	χ	X
epsilon	ϵ ε	E	lambda	λ	Λ	rho	ρ	P	psi	ψ	Ψ
zeta	ζ	Z	mu	μ	M	sigma	σ ς	Σ	omega	ω	Ω

Natural Numbers

$0 \in \mathbb{N}$

Outline

Part I: Propositional Logic

algebraic normal forms, binary decision diagrams, conjunctive normal forms, DPLL, Horn formulas, natural deduction, Post's adequacy theorem, resolution, SAT, semantics, sorting networks, soundness and completeness, syntax, Tseitin's transformation

Part II: Predicate Logic

natural deduction, quantifier equivalences, resolution, semantics, Skolemization, syntax, undecidability, unification

Part III: Model Checking

adequacy, branching–time temporal logic, CTL*, fairness, linear–time temporal logic, model checking algorithms, symbolic model checking

Outline

Definition

propositional formulas are built from

- atoms $p, q, r, p_1, p_2, \ldots$
- bottom $\bot$
- top $\top$
- negation $\neg$ $\neg p$ "not p"
- conjunction $\wedge$ $p \wedge q$ "p and q"
- disjunction $\vee$ $p \vee q$ "p or q"
- implication $\rightarrow$ $p \rightarrow q$ "if p then q"

according to following Backus–Naur Form:

$$\varphi \ ::= \ p \mid \bot \mid \top \mid (\neg\varphi) \mid (\varphi \wedge \varphi) \mid (\varphi \vee \varphi) \mid (\varphi \rightarrow \varphi)$$

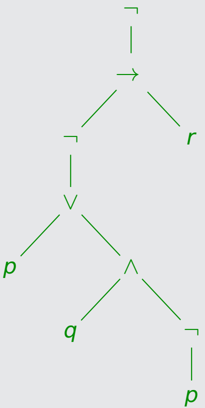
Notational Conventions

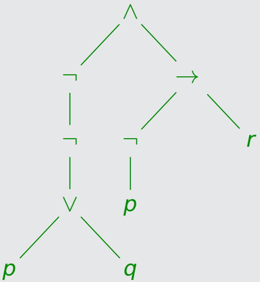
- binding precedence $\neg \quad > \quad \wedge, \vee \quad > \quad \rightarrow$
- omit outer parentheses
- $\rightarrow$, $\wedge$, $\vee$ are right–associative: $p \rightarrow q \rightarrow r$ denotes $p \rightarrow (q \rightarrow r)$

Example

formula $\neg(\neg(p \vee (q \wedge \neg p)) \to r)$ $\neg\neg(p \vee q) \wedge (\neg p \to r)$ $\neg\neg p \vee q \wedge \neg p \to r$

parse tree





?

Outline

Definitions (Boole 1854)

- valuation (truth assignment) is mapping $v \colon \{p \mid p \text{ is atom}\} \to \{\mathrm{T}, \mathrm{F}\}$ — truth values
- $\bar{v}$ is extension of v to formulas:

$$\bar{v}(p) = v(p) \qquad \bar{v}(\varphi \vee \psi) = \begin{cases} \mathrm{F} & \text{if } \bar{v}(\varphi) = \bar{v}(\psi) = \mathrm{F} \\ \mathrm{T} & \text{otherwise} \end{cases}$$

$$\bar{v}(\bot) = \mathrm{F} \qquad \bar{v}(\top) = \mathrm{T} \qquad \bar{v}(\varphi \wedge \psi) = \begin{cases} \mathrm{T} & \text{if } \bar{v}(\varphi) = \bar{v}(\psi) = \mathrm{T} \\ \mathrm{F} & \text{otherwise} \end{cases}$$

$$\bar{v}(\neg\varphi) = \begin{cases} \mathrm{T} & \text{if } \bar{v}(\varphi) = \mathrm{F} \\ \mathrm{F} & \text{otherwise} \end{cases} \qquad \bar{v}(\varphi \to \psi) = \begin{cases} \mathrm{F} & \text{if } \bar{v}(\varphi) = \mathrm{T} \text{ and } \bar{v}(\psi) = \mathrm{F} \\ \mathrm{T} & \text{otherwise} \end{cases}$$

Example

$v(p) = \mathrm{T}$ and $v(q) = \mathrm{F} \implies \bar{v}(p \wedge \neg q \to \neg p) = \mathrm{F}$

Definition

truth tables

φ	$\neg\varphi$
T	F
F	T

φ	ψ	$\varphi \wedge \psi$	$\varphi \vee \psi$	$\varphi \rightarrow \psi$
T	T	T	T	T
T	F	F	T	F
F	T	F	T	T
F	F	F	F	T

truth tables for propositional formulas are constructed bottom−up

Example ❶

p	q	$\neg p$	$\neg q$	$p \to \neg q$	$q \lor \neg p$	$(p \to \neg q) \to (q \lor \neg p)$
T	T	F	F	F	T	T
T	F	F	T	T	F	F
F	T	T	F	T	T	T
F	F	T	T	T	T	T

Example ❷

p	q	$(p$	$\to$	$\neg$	$q)$	$\to$	$(q$	$\lor$	$\neg$	$p)$
T	T					T		T		
T	F		T	T		F		F	F	
F	T		T			T		T		
F	F		T			T		T	T	

Definitions

- semantic entailment

$$\varphi_1, \varphi_2, \ldots, \varphi_n \vDash \psi$$

if $\bar{v}(\psi) = \mathrm{T}$ whenever $\bar{v}(\varphi_1) = \bar{v}(\varphi_2) = \cdots = \bar{v}(\varphi_n) = \mathrm{T}$ for every valuation v

- tautology is formula φ such that $\vDash \varphi$

Examples ①

p	q	$p \to q$	$\vDash$	$\neg p \vee q$
T	T	T		T
T	F	F		
F	T	T		T
F	F	T		T

p	q	$p \to q,$	$p \to \neg q$	$\vDash$	$\neg p$
T	T	T	F		
T	F	F			
F	T	T	T		T
F	F	T	T		T

Examples ❷

p	q	$p \to q$	$\not\models$	$q \to p$
T	T	T		T
T	F	F		
F	T	T		F
F	F	T		

p	q	$p \to q,$	$p \to \neg q$	$\not\models$	q
T	T	T	F		
T	F	F			
F	T	T	T		T
F	F	T	T		F

p	q	$p \to q,$	$p \wedge \neg q$	$\models$	$\bot$
T	T	T	F		
T	F	F			
F	T	T	F		
F	F	T	F		

Question

$$\models \begin{array}{l}(\neg p \wedge \neg q) \vee (s \wedge u) \vee (r \wedge w) \vee (\neg t \wedge \neg u) \vee (p \wedge r) \vee (q \wedge s) \\ \vee\, (p \wedge t) \vee (q \wedge u) \vee (\neg r \wedge \neg s) \vee (t \wedge v) \vee (\neg v \wedge \neg w)\end{array} \quad ?$$

... truth table has $2^8 = 256$ rows ...

	6		1		4		5	
		8	3		5	6		
2								1
8			4		7			6
		6				3		
7			9		1			4
5								2
		7	2		6	9		
	4		5		8		7	

... truth table has $2^{459} > 2^{4\times 100} = 16^{100} > 10^{100}$ rows ...

Outline

Definitions

formula φ is

- valid if $\bar{v}(\varphi) = \mathrm{T}$ for every valuation v
- satisfiable if $\bar{v}(\varphi) = \mathrm{T}$ for some valuation v

Theorem

formula φ is valid $\iff$ $\neg\varphi$ is unsatisfiable

Proof

$$
\begin{aligned}
\varphi \text{ is valid} &\iff \bar{v}(\varphi) = \mathrm{T} \quad \text{for every valuation } v \\
&\iff \bar{v}(\neg\varphi) = \mathrm{F} \quad \text{for every valuation } v \\
&\iff \bar{v}(\neg\varphi) = \mathrm{T} \quad \text{for no valuation } v \\
&\iff \neg\varphi \text{ is not satisfiable} \iff \neg\varphi \text{ is unsatisfiable}
\end{aligned}
$$

Definition

formulas φ and ψ are semantically equivalent ($\varphi \equiv \psi$) if both $\varphi \vDash \psi$ and $\psi \vDash \varphi$

Examples

$$\neg(\varphi \vee \psi) \equiv \neg\varphi \wedge \neg\psi \qquad \neg\neg\varphi \equiv \varphi \qquad \varphi \vee (\psi \wedge \chi) \equiv (\varphi \vee \psi) \wedge (\varphi \vee \chi)$$

Theorem

validity and satisfiability are decidable

Proof

construct truth table of φ and inspect last column:

- φ is valid if and only if all entries are T
- φ is satisfiable if and only if T entry exists

Outline

Question

Given that one and only one answer is correct, which of the following is true ?

- **A** All of the below.
- **B** None of the below.
- **C** One of the above.
- **D** All of the above.
- ✓ None of the above.
- **F** None of the above.

Outline

Definitions

- literal is atom p or negation $\neg p$ of atom
- clause is disjunction $\ell_1 \vee \cdots \vee \ell_n$ of literals
- conjunctive normal form (CNF) is conjunction $C_1 \wedge \cdots \wedge C_n$ of clauses
- literals ℓ_1 and ℓ_2 are complementary if $\ell_1 = \neg \ell_2$ or $\neg \ell_1 = \ell_2$

Theorem

validity of CNFs is efficiently decidable:

$$\text{CNF } \varphi \text{ is valid} \iff \text{every clause of } \varphi \text{ contains complementary literals}$$

Examples

1. CNF

$$(p \vee q \vee \neg r) \wedge (\neg p \vee \neg r \vee p) \wedge (\neg q)$$

complementary literals

not valid   witness: $v(p) = v(q) = \mathsf{F}$ and $v(r) = \mathsf{T}$

2. CNF

$$(p \vee q \vee \neg p) \wedge (\neg r \vee \neg p \vee r) \wedge (\neg q \vee q)$$

valid

Special Cases

- $\bot$ represents empty clause (no literals)
- $\top$ represents empty CNF (no clauses)

Theorem

for every formula φ there exists CNF ψ such that $\varphi \equiv \psi$

Procedure

① eliminate implications

② push negations towards atoms and remove double negations

③ distribute disjunction over conjunction

$$\varphi \to \psi \xrightarrow{①} \neg\varphi \vee \psi \qquad \neg\neg\varphi \xrightarrow{②} \varphi$$

$$\neg(\varphi \wedge \psi) \xrightarrow{②} \neg\varphi \vee \neg\psi \qquad \varphi \vee (\psi \wedge \chi) \xrightarrow{③} (\varphi \vee \psi) \wedge (\varphi \vee \chi)$$

$$\neg(\varphi \vee \psi) \xrightarrow{②} \neg\varphi \wedge \neg\psi \qquad (\varphi \wedge \psi) \vee \chi \xrightarrow{③} (\varphi \vee \chi) \wedge (\psi \vee \chi)$$

Remark

CNF ψ for formula φ might be exponentially larger

Example (CNFs are not unique)

$$\varphi = \neg(p \vee (q \wedge r))$$
$$\xrightarrow{②} \neg p \wedge \neg(q \wedge r) \xrightarrow{②} \neg p \wedge (\neg q \vee \neg r)$$

$$\varphi = \neg(p \vee (q \wedge r))$$
$$\xrightarrow{③} \neg((p \vee q) \wedge (p \vee r)) \xrightarrow{②} \neg(p \vee q) \vee \neg(p \vee r)$$
$$\xrightarrow{②} (\neg p \wedge \neg q) \vee \neg(p \vee r) \xrightarrow{②} (\neg p \wedge \neg q) \vee (\neg p \wedge \neg r)$$
$$\xrightarrow{③} ((\neg p \wedge \neg q) \vee \neg p) \wedge ((\neg p \wedge \neg q) \vee \neg r)$$
$$\xrightarrow{③} ((\neg p \vee \neg p) \wedge (\neg q \vee \neg p)) \wedge ((\neg p \wedge \neg q) \vee \neg r)$$
$$\xrightarrow{③} ((\neg p \vee \neg p) \wedge (\neg q \vee \neg p)) \wedge ((\neg p \vee \neg r) \wedge (\neg q \vee \neg r))$$

CNFs are not unique, even if rules ①, ②, ③ are applied in order

Procedure (extended)

⓪ simplify formulas with $\bot$ and $\top$

① eliminate implications

② push negations towards atoms and remove double negations

③ distribute disjunction over conjunction

$\neg\bot \xrightarrow{⓪} \top$	$\bot \wedge \varphi \xrightarrow{⓪} \bot$	$\bot \vee \varphi \xrightarrow{⓪} \varphi$	$\bot \rightarrow \varphi \xrightarrow{⓪} \top$
$\neg\top \xrightarrow{⓪} \bot$	$\top \wedge \varphi \xrightarrow{⓪} \varphi$	$\top \vee \varphi \xrightarrow{⓪} \top$	$\top \rightarrow \varphi \xrightarrow{⓪} \varphi$
	$\varphi \wedge \bot \xrightarrow{⓪} \bot$	$\varphi \vee \bot \xrightarrow{⓪} \varphi$	$\varphi \rightarrow \bot \xrightarrow{⓪} \neg\varphi$
	$\varphi \wedge \top \xrightarrow{⓪} \varphi$	$\varphi \vee \top \xrightarrow{⓪} \top$	$\varphi \rightarrow \top \xrightarrow{⓪} \top$

Example

$$p \vee (q \wedge (\top \to (\neg p \vee \bot)) \to (\top \wedge \neg q))$$

$$\overset{⓪}{\longrightarrow} \quad p \vee (q \wedge (\top \to \neg p) \to (\top \wedge \neg q)) \quad \overset{⓪}{\longrightarrow} \quad p \vee (q \wedge (\top \to \neg p) \to \neg q)$$

$$\overset{⓪}{\longrightarrow} \quad p \vee (q \wedge \neg p \to \neg q) \quad \overset{①}{\longrightarrow} \quad p \vee (\neg (q \wedge \neg p) \vee \neg q)$$

$$\overset{②}{\longrightarrow} \quad p \vee ((\neg q \vee \neg\neg p) \vee \neg q) \quad \overset{②}{\longrightarrow} \quad p \vee ((\neg q \vee p) \vee \neg q)$$

Example (CNF from truth table)

$\varphi = \neg(p \vee (q \wedge r))$

p	q	r	φ
T	T	T	F
T	T	F	F
T	F	T	F
T	F	F	F

p	q	r	φ
F	T	T	F
F	T	F	T
F	F	T	T
F	F	F	T

$$\varphi \equiv \neg\big((p \wedge q \wedge r) \vee (p \wedge q \wedge \neg r) \vee (p \wedge \neg q \wedge r) \vee (p \wedge \neg q \wedge \neg r) \vee (\neg p \wedge q \wedge r)\big)$$

$$\equiv (\neg p \vee \neg q \vee \neg r) \wedge (\neg p \vee \neg q \vee r) \wedge (\neg p \vee q \vee \neg r) \wedge (\neg p \vee q \vee r) \wedge (p \vee \neg q \vee \neg r)$$

Theorem

for every formula φ there exists CNF ψ such that $\varphi \equiv \psi$

Deterministic Procedure

① eliminate implications

function IMPL_FREE(φ):

begin function

case	φ is atom:	**return**	φ
	φ is $\neg\varphi_1$:	**return**	$\neg$ IMPL_FREE(φ_1)
	φ is $\varphi_1 \wedge \varphi_2$:	**return**	IMPL_FREE(φ_1) $\wedge$ IMPL_FREE(φ_2)
	φ is $\varphi_1 \vee \varphi_2$:	**return**	IMPL_FREE(φ_1) $\vee$ IMPL_FREE(φ_2)
	φ is $\varphi_1 \to \varphi_2$:	**return**	$\neg$ IMPL_FREE(φ_1) $\vee$ IMPL_FREE(φ_2)

end case

end function

Theorem

for every formula φ there exists CNF ψ such that $\varphi \equiv \psi$

Deterministic Procedure

② push negations towards atoms and remove double negations

function NNF(φ):

begin function

case	φ is literal:	**return**	φ
	φ is $\neg\neg\varphi_1$:	**return**	$\text{NNF}(\varphi_1)$
	φ is $\varphi_1 \wedge \varphi_2$:	**return**	$\text{NNF}(\varphi_1) \wedge \text{NNF}(\varphi_2)$
	φ is $\varphi_1 \vee \varphi_2$:	**return**	$\text{NNF}(\varphi_1) \vee \text{NNF}(\varphi_2)$
	φ is $\neg(\varphi_1 \wedge \varphi_2)$:	**return**	$\text{NNF}(\neg\varphi_1) \vee \text{NNF}(\neg\varphi_2)$
	φ is $\neg(\varphi_1 \vee \varphi_2)$:	**return**	$\text{NNF}(\neg\varphi_1) \wedge \text{NNF}(\neg\varphi_2)$

end case

end function

Theorem

for every formula φ there exists CNF ψ such that $\varphi \equiv \psi$

Deterministic Procedure

③ distribute disjunction over conjunction

function CNF(φ):

begin function

case	φ is literal:	**return**	φ
	φ is $\varphi_1 \wedge \varphi_2$:	**return**	$\mathrm{CNF}(\varphi_1) \wedge \mathrm{CNF}(\varphi_2)$
	φ is $\varphi_1 \vee \varphi_2$:	**return**	DISTR(CNF(φ_1), CNF(φ_2))

end case

end function

Theorem

for every formula φ there exists CNF ψ such that $\varphi \equiv \psi$

Deterministic Procedure

③ distribute disjunction over conjunction

function DISTR(η_1, η_2):

begin function

case	η_1 is $\eta_{11} \wedge \eta_{12}$:	**return**	DISTR$(\eta_{11}, \eta_2) \wedge$ DISTR(η_{12}, η_2)
	η_2 is $\eta_{21} \wedge \eta_{22}$:	**return**	DISTR$(\eta_1, \eta_{21}) \wedge$ DISTR(η_1, η_{22})
	otherwise:	**return**	$\eta_1 \vee \eta_2$

end case

end function

Theorem

for every formula φ there exists CNF ψ such that $\varphi \equiv \psi$

Deterministic Procedure

① eliminate implications

② push negations towards atoms and remove double negations

③ distribute disjunction over conjunction

Theorem

1. $\mathsf{CNF}(\mathsf{NNF}(\mathsf{IMPL\_FREE}(\varphi)))$ is CNF
2. $\mathsf{CNF}(\mathsf{NNF}(\mathsf{IMPL\_FREE}(\varphi))) \equiv \varphi$
3. executing $\mathsf{CNF}(\mathsf{NNF}(\mathsf{IMPL\_FREE}(\varphi)))$ terminates

Outline

Huth and Ryan

- Section 1.1
- Section 1.3
- Sections 1.4.1 and 1.4.2
- Sections 1.5.1 and 1.5.2

Differences (slides – book)

- role of $\bot$ and $\top$
- terminology concerning CNFs

Important Concepts

- atom
- bottom
- clause
- complementary literals
- conjunction
- conjunctive normal form
- disjunction
- disjunctive normal form
- implication
- literal
- negation
- top
- right−associativity
- satisfiability
- semantic entailment
- semantic equivalence
- tautology
- truth table
- truth values
- validity
- valuation

homework for March 5 and 12