universität innsbruck

26S lecture 2

Logic

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Particify
ars.uibk.ac.at
with session ID **6893 6178** for anonymous questions

Outline

Definitions

- semantic entailment

$$\varphi_1, \varphi_2, \dots, \varphi_n \vDash \psi$$

  if $\bar{v}(\psi) = \mathsf{T}$ whenever $\bar{v}(\varphi_1) = \bar{v}(\varphi_2) = \cdots = \bar{v}(\varphi_n) = \mathsf{T}$ for every valuation v
- tautology is formula φ such that $\vDash \varphi$
- formula φ is
  - valid if $\bar{v}(\varphi) = \mathsf{T}$ for every valuation v
  - satisfiable if $\bar{v}(\varphi) = \mathsf{T}$ for some valuation v
- formulas φ and ψ are semantically equivalent ($\varphi \equiv \psi$) if both $\varphi \vDash \psi$ and $\psi \vDash \varphi$

Theorem

formula φ is valid $\iff$ $\neg\varphi$ is unsatisfiable $\iff$ φ is tautology

Definitions

- literal is atom p or negation $\neg p$ of atom
- clause is disjunction $\ell_1 \vee \cdots \vee \ell_n$ of literals
- conjunctive normal form (CNF) is conjunction $C_1 \wedge \cdots \wedge C_n$ of clauses
- literals ℓ_1 and ℓ_2 are complementary if $\ell_1 = \neg\ell_2$ or $\neg\ell_1 = \ell_2$

Theorem

- for every formula φ there exists CNF ψ such that $\varphi \equiv \psi$
- validity of CNFs is efficiently decidable:

$$\text{CNF } \varphi \text{ is valid} \iff \text{every clause of } \varphi \text{ contains complementary literals}$$

Part I: Propositional Logic

algebraic normal forms, binary decision diagrams, conjunctive normal forms, DPLL, Horn formulas, natural deduction, Post's adequacy theorem, resolution, SAT, semantics, sorting networks, soundness and completeness, syntax, Tseitin's transformation

Part II: Predicate Logic

natural deduction, quantifier equivalences, resolution, semantics, Skolemization, syntax, undecidability, unification

Part III: Model Checking

adequacy, branching–time temporal logic, CTL*, fairness, linear–time temporal logic, model checking algorithms, symbolic model checking

Outline

Definitions

- Horn clause is propositional formula

$$P_1 \wedge P_2 \wedge \cdots \wedge P_n \to Q$$

  with $n \geqslant 1$ and where $P_1, \ldots, P_n, Q$ are atoms, $\bot$ or $\top$
- Horn formula is conjunction of Horn clauses

Backus–Naur Form (H)

$$P \ ::= \ p \mid \bot \mid \top \qquad C \ ::= \ A \to P$$
$$A \ ::= \ P \mid P \wedge A \qquad H \ ::= \ C \mid C \wedge H$$

Theorem

satisfiability of Horn formulas is efficiently decidable

Procedure

① maintain list of atoms, $\bot$, $\top$ occurring in φ

② mark $\top$ if it appears in list

③ **while** Horn clause $P_1 \wedge \cdots \wedge P_n \to Q$ exists in φ such that all $P_1, \dots, P_n$ are marked and Q is unmarked

mark Q

④ **if** $\bot$ is marked **then**

return unsatisfiable

else

return satisfiable satisfying assignment: $v(P) = \begin{cases} \mathsf{T} & \text{if } P \text{ is marked} \\ \mathsf{F} & \text{if } P \text{ is unmarked} \end{cases}$

Examples

1. Horn formula

$$(p \wedge q \wedge w \to \bot) \wedge (t \to \bot) \wedge (r \to \underset{\textcircled{4}}{p}) \wedge (\top \to \underset{\textcircled{2}}{r}) \wedge (\top \to \underset{\textcircled{3}}{q}) \wedge (\top \to \underset{\textcircled{5}}{u}) \wedge (u \to \underset{\textcircled{6}}{s})$$

list $p \quad q \quad r \quad s \quad t \quad u \quad w \quad \bot \quad \underset{\textcircled{1}}{\top}$

satisfiable $v(p) = v(q) = v(r) = v(s) = v(u) = \mathsf{T} \qquad v(t) = v(w) = \mathsf{F}$

2. Horn formula

$$(p \wedge q \wedge w \to \underset{\textcircled{7}}{\bot}) \wedge (t \to \bot) \wedge (r \to \underset{\textcircled{4}}{p}) \wedge (\top \to \underset{\textcircled{2}}{r}) \wedge (\top \to \underset{\textcircled{3}}{q}) \wedge (\top \to \underset{\textcircled{5}}{u}) \wedge (u \to \underset{\textcircled{6}}{w})$$

list $p \quad q \quad r \quad t \quad u \quad w \quad \bot \quad \underset{\textcircled{1}}{\top}$

unsatisfiable

Outline

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Question

Consider the formula $\varphi = (p \wedge \neg q \to \bot) \wedge (q \wedge p \to \neg q)$.

Which of the following statements hold for φ ?

- **A** φ is a CNF
- **B** φ is a Horn formula
- **C** $\varphi \equiv p \to \neg q$
- ✅ φ is satisfiable
- **E** φ is valid

Outline

Satisfiability (SAT)

instance: propositional formula φ

question: is φ satisfiable ?

Theorem

SAT is NP–complete

Links

- SAT competition
- Millennium Problems – P vs NP

SAT Applications

- bounded model checking
- combinatorial design theory
- haplotyping in bioinformatics
- hardware verification
- logic puzzles
- package management in software distributions
- planning and scheduling
- software verification
- sorting networks
- statistical physics
- term rewriting
- · · · · · · · · ·

Popular SAT Solvers

MiniSat PicoSAT Z3

Example (数独 Sudoku)

	6		1		4		5	
		8	3		5	6		
2								1
8			4		7			6
		6				3		
7			9		1			4
5								2
		7	2		6	9		
	4		5		8		7	

11	12	13	14	15	16	17	18	19
21	22	23	24	25	26	27	28	29
31	32	33	34	35	36	37	38	39
41	42	43	44	45	46	47	48	49
51	52	53	54	55	56	57	58	59
61	62	63	64	65	66	67	68	69
71	72	73	74	75	76	77	78	79
81	82	83	84	85	86	87	88	89
91	92	93	94	95	96	97	98	99

Variables

- propositional atoms x_{ijd} for $i, j, d \in \{1, \dots, 9\}$
- $v(x_{ijd}) = \mathsf{T} \iff$ cell ij contains digit d

11	12	13	14	15	16	17	18	19
21	22	23	24	25	26	27	28	29
31	32	33	34	35	36	37	38	39
41	42	43	44	45	46	47	48	49
51	52	53	54	55	56	57	58	59
61	62	63	64	65	66	67	68	69
71	72	73	74	75	76	77	78	79
81	82	83	84	85	86	87	88	89
91	92	93	94	95	96	97	98	99

Constraints

- every cell contains at least one digit
- every cell contains at most one digit
- in every row / column / 3×3 block every digit appears at most once

Cardinality Constraints

for non–empty set A of propositional atoms:

$$\text{at-least-one}(A) = \bigvee_{x \in A} x \qquad \text{at-most-one}(A) = \bigwedge_{\substack{x,y \in A \\ x \neq y}} (\neg x \vee \neg y)$$

Example

$$\text{at-least-one}(\{p,q,r\}) = p \vee q \vee r$$

$$\text{at-most-one}(\{p,q,r\}) = (\neg p \vee \neg q) \wedge (\neg p \vee \neg r) \wedge (\neg q \vee \neg r)$$

Useful Abbreviations

$D = \{1,2,3,4,5,6,7,8,9\}$

$\mathcal{G} = \{\{1,2,3\},\{4,5,6\},\{7,8,9\}\}$

$\mathcal{C} = \{\{x_{ijd} \mid j \in D\} \mid i, d \in D\} \cup \{\{x_{ijd} \mid i \in D\} \mid j, d \in D\} \cup \{\{x_{ijd} \mid (i,j) \in I \times J\} \mid I, J \in \mathcal{G}, d \in D\}$

11	12	13	14	15	16	17	18	19
21	22	23	24	25	26	27	28	29
31	32	33	34	35	36	37	38	39
41	42	43	44	45	46	47	48	49
51	52	53	54	55	56	57	58	59
61	62	63	64	65	66	67	68	69
71	72	73	74	75	76	77	78	79
81	82	83	84	85	86	87	88	89
91	92	93	94	95	96	97	98	99

	6		1		4		5	
		8	3		5	6		
2								1
8			4		7			6
		6				3		
7			9		1			4
5								2
		7	2		6	9		
	4		5		8		7	

SAT Encoding

$$\varphi: \quad \bigwedge \{\text{at-least-one}(\{x_{ijd} \mid d \in D\}) \mid i,j \in D\} \wedge \bigwedge \{\text{at-most-one}(A) \mid A \in \mathcal{C}\} \wedge$$
$$\bigwedge \{\text{at-most-one}(\{x_{ijd} \mid d \in D\}) \mid i,j \in D\} \wedge x_{126} \wedge x_{141} \wedge x_{164} \wedge \cdots \wedge x_{987}$$

- φ is satisfiable $\iff$ Sudoku puzzle has solution
- satisfying assignment gives rise to Sudoku solution

$D = \{1,2,3,4\}$

$\mathcal{G} = \{\{1,2\},\{3,4\}\}$

$\mathcal{C} = \{\{x_{ijd} \mid j \in D\} \mid i, d \in D\} \cup \{\{x_{ijd} \mid i \in D\} \mid j, d \in D\} \cup \{\{x_{ijd} \mid (i,j) \in I \times J\} \mid I, J \in \mathcal{G}, d \in D\}$

Example (2×2 数独 Sudoku)

		1	
3			
	4		

11	12	13	14
21	22	23	24
31	32	33	34
41	42	43	44

$$\begin{aligned}\mathcal{C} = {} & \{\{x_{111}, x_{121}, x_{131}, x_{141}\}, \{x_{112}, x_{122}, x_{132}, x_{142}\}, \ldots, \{x_{414}, x_{424}, x_{434}, x_{444}\}\} \\ & \cup \{\{x_{111}, x_{211}, x_{311}, x_{411}\}, \{x_{121}, x_{221}, x_{321}, x_{421}\}, \ldots, \{x_{144}, x_{244}, x_{344}, x_{444}\}\} \\ & \cup \{\{x_{111}, x_{121}, x_{211}, x_{221}\}, \{x_{112}, x_{122}, x_{212}, x_{222}\}, \ldots, \{x_{334}, x_{344}, x_{434}, x_{444}\}\}\end{aligned}$$

Pythagorean Triples Problem

can one color all natural numbers with two colors such that whenever $x^2 + y^2 = z^2$ not all of x, y, z have same color ?

Example

$$3^2 + 4^2 = 5^2 \qquad 5^2 + 12^2 = 13^2 \qquad \dots$$

1 2 3 4 5 6 7 8 9 10 11 12 13 . . . ☹

SAT Encoding

- propositional atoms x_i for $1 \leqslant i \leqslant n$
- $v(x_i) = \mathsf{T} \iff$ number i is colored red
- encoding contains clauses $(x_a \vee x_b \vee x_c)$ and $(\neg x_a \vee \neg x_b \vee \neg x_c)$ for all $a^2 + b^2 = c^2$

Solution

- NO if (and only if) $n \geqslant 7825$
- 2 days (in May 2016) on University of Texas' Stampede supercomputer with 800 processors
- 200 terabyte proof of unsatisfiability
- extensive media coverage (Nature, der Spiegel)

Example (Sports League Scheduling)

- round robin tournament scheduling for n teams and p periods consisting of $n - 1$ rounds, satisfying several other constraints like venue restrictions
- Austrian Football Bundesliga

  12 teams play 2 periods (of 11 rounds), periods 1 and 2 are mirrored
- SAT encoding
  - variables x_{ijpr} with $v(x_{ijpr}) = \mathsf{T}$ if team i plays team j at home in round r of period p
  - constraints (fragment):

$$\bigwedge_{i,p,r} \bigvee_{j \neq i} (x_{ijpr} \vee x_{jipr}) \qquad \bigwedge_{i,p,r} \bigwedge_{j \neq i} \bigwedge_{\substack{k \neq i \\ k \neq j}} (x_{ijpr} \rightarrow \neg(x_{ikpr} \vee x_{kipr})) \qquad \bigwedge_{i,j,r} (x_{ij1r} \rightarrow x_{ji2r})$$

- further details

Outline

Remark

most SAT solvers require CNF as input

Theorem

deciding satisfiability of CNF formulas is NP–complete

DIMACS Input Format

```
c
c comments
c
p cnf 4 3        4 atoms and 3 clauses
1 -2 4 0         x1 ∨ ¬x2 ∨ x4
-1 2 -3 -4 0     ¬x1 ∨ x2 ∨ ¬x3 ∨ ¬x4
3 -2 0           x3 ∨ ¬x2
```

p cnf 4 3	4 atoms and 3 clauses
1 -2 4 0	$x_1 \vee \neg x_2 \vee x_4$
-1 2 -3 -4 0	$\neg x_1 \vee x_2 \vee \neg x_3 \vee \neg x_4$
3 -2 0	$x_3 \vee \neg x_2$

Remarks

- translation from arbitrary formula to equivalent CNF is expensive
- translation to equisatisfiable CNF is possible in linear time

Definition

formulas φ and ψ are equisatisfiable ($\varphi \approx \psi$) if

$$\varphi \text{ is satisfiable} \iff \psi \text{ is satisfiable}$$

Examples

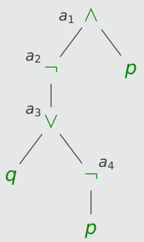
$$(p \vee q) \wedge \neg p \approx \top \qquad (p \vee q) \wedge \neg p \not\approx q \wedge \neg q$$

Example (Tseitin's Transformation)

- $\varphi = \neg(q \vee \neg p) \wedge p$
- introduce new variable for each propositional connective:

a_1	$\neg(q \vee \neg p) \wedge p$	a_3	$q \vee \neg p$
a_2	$\neg(q \vee \neg p)$	a_4	$\neg p$

- $\varphi \approx a_1 \wedge (a_1 \leftrightarrow a_2 \wedge p) \wedge (a_2 \leftrightarrow \neg a_3) \wedge (a_3 \leftrightarrow q \vee a_4) \wedge (a_4 \leftrightarrow \neg p)$



Definition

new propositional connective

- equivalence $\leftrightarrow$ $\quad p \leftrightarrow q \quad$ "p is equivalent to q"
- $\bar{v}(\varphi \leftrightarrow \psi) = \begin{cases} \mathsf{T} & \text{if } \bar{v}(\varphi) = \bar{v}(\psi) \\ \mathsf{F} & \text{otherwise} \end{cases}$

Notational Convention

binding precedence $\quad \neg \;>\; \wedge, \vee \;>\; \rightarrow, \leftrightarrow$

Lemma

$\varphi \leftrightarrow \psi \equiv (\varphi \rightarrow \psi) \wedge (\psi \rightarrow \varphi)$

Proof

φ	ψ	$\varphi \leftrightarrow \psi$	$(\varphi \rightarrow \psi) \wedge (\psi \rightarrow \varphi)$
T	T	T	T
T	F	F	F
F	T	F	F
F	F	T	T

Lemma

1. $(\varphi \leftrightarrow \neg\psi) \equiv (\varphi \vee \psi) \wedge (\neg\varphi \vee \neg\psi)$
2. $(\varphi \leftrightarrow \psi \wedge \chi) \equiv (\neg\varphi \vee \psi) \wedge (\neg\varphi \vee \chi) \wedge (\varphi \vee \neg\psi \vee \neg\chi)$
3. $(\varphi \leftrightarrow \psi \vee \chi) \equiv (\varphi \vee \neg\psi) \wedge (\varphi \vee \neg\chi) \wedge (\neg\varphi \vee \psi \vee \chi)$

Example (cont'd)

$$\varphi \approx a_1 \wedge (a_1 \leftrightarrow a_2 \wedge p) \wedge (a_2 \leftrightarrow \neg a_3) \wedge (a_3 \leftrightarrow q \vee a_4) \wedge (a_4 \leftrightarrow \neg p)$$

$$\equiv a_1 \wedge (\neg a_1 \vee a_2) \wedge (\neg a_1 \vee p) \wedge (a_1 \vee \neg a_2 \vee \neg p) \wedge (a_2 \vee a_3) \wedge (\neg a_2 \vee \neg a_3)$$
$$\wedge (a_3 \vee \neg q) \wedge (a_3 \vee \neg a_4) \wedge (\neg a_3 \vee q \vee a_4) \wedge (a_4 \vee p) \wedge (\neg a_4 \vee \neg p)$$

Definition (Tseitin's Transformation)

for propositional formula φ

- atom a_φ is defined as $a_\varphi = \begin{cases} \varphi & \text{if } \varphi \text{ is atom} \\ \text{fresh atom} & \text{otherwise} \end{cases}$
- formula $\mathrm{TT}(\varphi)$ is defined as

$$\mathrm{TT}(\varphi) = \begin{cases} a_\varphi & \text{if } \varphi \text{ is atom} \\ a_\varphi \wedge \mathrm{TT}'(a_\varphi, \varphi) & \text{otherwise} \end{cases}$$

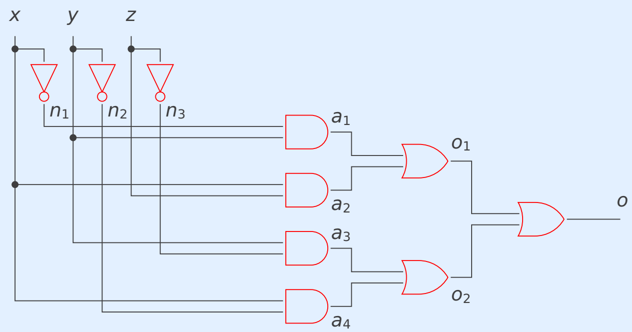
with

$$\mathrm{TT}'(a,\varphi) = \begin{cases} (a \leftrightarrow \neg a_\psi) \wedge \mathrm{TT}'(a_\psi, \psi) & \text{if } \varphi = \neg\psi \\ (a \leftrightarrow (a_{\psi_1} \wedge a_{\psi_2})) \wedge \mathrm{TT}'(a_{\psi_1}, \psi_1) \wedge \mathrm{TT}'(a_{\psi_2}, \psi_2) & \text{if } \varphi = \psi_1 \wedge \psi_2 \\ (a \leftrightarrow (a_{\psi_1} \vee a_{\psi_2})) \wedge \mathrm{TT}'(a_{\psi_1}, \psi_1) \wedge \mathrm{TT}'(a_{\psi_2}, \psi_2) & \text{if } \varphi = \psi_1 \vee \psi_2 \\ (a \leftrightarrow (a_{\psi_1} \rightarrow a_{\psi_2})) \wedge \mathrm{TT}'(a_{\psi_1}, \psi_1) \wedge \mathrm{TT}'(a_{\psi_2}, \psi_2) & \text{if } \varphi = \psi_1 \rightarrow \psi_2 \\ \top & \text{if } \varphi \text{ is atom} \end{cases}$$

Lemma

1. any satisfying valuation for φ can be (uniquely) extended to satisfying valuation for $\mathrm{TT}(\varphi)$
2. restriction of any satisfying valuation for $\mathrm{TT}(\varphi)$ to atoms in φ is satisfying valuation for φ

Logic Circuit



Equisatisfiable CNF

$$o \wedge (o \leftrightarrow o_1 \vee o_2) \wedge (o_1 \leftrightarrow a_1 \vee a_2) \wedge (o_2 \leftrightarrow a_3 \vee a_4) \wedge (a_1 \leftrightarrow n_1 \wedge y) \wedge (a_2 \leftrightarrow x \wedge z)$$
$$\wedge (a_3 \leftrightarrow y \wedge n_3) \wedge (a_4 \leftrightarrow x \wedge n_2) \wedge (n_1 \leftrightarrow \neg x) \wedge (n_2 \leftrightarrow \neg y) \wedge (n_3 \leftrightarrow \neg z)$$

Outline

Huth and Ryan

- Section 1.5

SAT and P – NP

- SAT live! [accessed March 9, 2026]
- The Science of Brute Force
  Marijn J. H. Heule and Oliver Kullmann
  Communications of the ACM 60(8), pp. 70–97, 2017
  doi: 10.1145/3107239
- Fifty Years of P vs. NP and the Possibility of the Impossible
  Lance Fortnow
  Communications of the ACM 65(1), pp. 76–85, 2022
  doi: 10.1145/3460351

Important Concepts

- DIMACS format
- equisatisfiability
- equivalence
- Horn clause
- Horn formula
- SAT
- Tseitin's transformation

homework for March 19

Outline

Tutorium

new slot: Wednesday 10:15 – 11:00 in RR 26

Mentoring

- for international master students (SKZ 648 and 921)
- Friday March 27, 10:15 – 11:00 in 3W04
- registration (group 1) required