



Logic

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Outline

1. Summary of Previous Lecture
2. Natural Deduction
3. Intermezzo
4. Soundness
5. Further Reading

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ars.uibk.ac.at

with session ID **6893 6178** for anonymous questions



Definition

- ▶ **Horn clause** is propositional formula

$$P_1 \wedge P_2 \wedge \dots \wedge P_n \rightarrow Q$$

- with $n \geq 1$ and where P_1, \dots, P_n, Q are atoms, \perp or \top
- ▶ **Horn formula** is conjunction of Horn clauses

Theorem

satisfiability of Horn formulas is **efficiently** decidable

Remark

deciding satisfiability for **arbitrary** formulas is important and difficult problem (**SAT**)

Definition

formulas φ and ψ are **equisatisfiable** ($\varphi \approx \psi$) if

$$\varphi \text{ is satisfiable} \iff \psi \text{ is satisfiable}$$

Remark

Tseitin's transformation transforms arbitrary formula into equisatisfiable CNF in linear time

Lemma

- any satisfying valuation for φ can be (uniquely) extended to satisfying valuation for $TT(\varphi)$
- restriction of any satisfying valuation for $TT(\varphi)$ to atoms in φ is satisfying valuation for φ

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Part I: Propositional Logic

algebraic normal forms, binary decision diagrams, conjunctive normal forms, DPLL, Horn formulas, **natural deduction**, Post's adequacy theorem, resolution, SAT, semantics, sorting networks, **soundness** and completeness, syntax, Tseitin's transformation

Part II: Predicate Logic

natural deduction, quantifier equivalences, resolution, semantics, Skolemization, syntax, undecidability, unification

Part III: Model Checking

adequacy, branching-time temporal logic, CTL*, fairness, linear-time temporal logic, model checking algorithms, symbolic model checking

Natural Deduction

calculus for reasoning about propositions

Definitions

- sequent

$$\underbrace{\varphi_1, \varphi_2, \dots, \varphi_n}_{\text{premises}} \vdash \underbrace{\psi}_{\text{conclusion}}$$

with propositional formulas $\varphi_1, \varphi_2, \dots, \varphi_n, \psi$

- sequent $\varphi_1, \varphi_2, \dots, \varphi_n \vdash \psi$ is **valid** if ψ can be proved from premises $\varphi_1, \varphi_2, \dots, \varphi_n$ using **proof rules** of natural deduction

natural deduction consists of 17 proof rules

Definition

▶ and introduction

$$\frac{\varphi \quad \psi}{\varphi \wedge \psi} \wedge i$$

Example

$p, q, r \vdash (r \wedge q) \wedge p$ is valid:

1	p	premise
2	q	premise
3	r	premise
4	$r \wedge q$	$\wedge i$ 3,2
5	$(r \wedge q) \wedge p$	$\wedge i$ 4,1

Definition

▶ and elimination

$$\frac{\varphi \wedge \psi}{\varphi} \wedge e_1 \qquad \frac{\varphi \wedge \psi}{\psi} \wedge e_2$$

Example

$p \wedge q, r \vdash r \wedge q$ is valid:

1	$p \wedge q$	premise
2	r	premise
3	q	$\wedge e_2$ 1
4	$r \wedge q$	$\wedge i$ 2,3

Definitions

▶ double negation elimination

$$\frac{\neg\neg\varphi}{\varphi} \neg e$$

▶ double negation introduction

$$\frac{\varphi}{\neg\neg\varphi} \neg i$$

Example

$p, \neg\neg(q \wedge r) \vdash \neg\neg p \wedge r$ is valid:

1	p	premise
2	$\neg\neg(q \wedge r)$	premise
3	$\neg\neg p$	$\neg\neg i$ 1
4	$q \wedge r$	$\neg\neg e$ 2
5	r	$\wedge e_2$ 4
6	$\neg\neg p \wedge r$	$\wedge i$ 3,5

Definition

▶ implication elimination (modus ponens)

$$\frac{\varphi \rightarrow \psi \quad \varphi}{\psi} \rightarrow e$$

Example

$p, p \rightarrow q, p \rightarrow (q \rightarrow r) \vdash r$ is valid:

1	p	premise
2	$p \rightarrow q$	premise
3	$p \rightarrow (q \rightarrow r)$	premise
4	q	$\rightarrow e$ 2,1
5	$q \rightarrow r$	$\rightarrow e$ 3,1
6	r	$\rightarrow e$ 5,4

Definition

▶ modus tollens

$$\frac{\varphi \rightarrow \psi \quad \neg \psi}{\neg \varphi} \text{ MT}$$

Example

$\neg p \rightarrow q, \neg q \vdash p$ is valid:

1	$\neg p \rightarrow q$	premise
2	$\neg q$	premise
3	$\neg \neg p$	MT 1, 2
4	p	$\neg \neg$ e 3

Definition

▶ implication introduction

$$\frac{\begin{array}{|l} \varphi \\ \vdots \\ \psi \end{array}}{\varphi \rightarrow \psi} \begin{array}{l} \text{box} \\ \text{assumption} \\ \rightarrow i \end{array}$$

Example

$\neg q \rightarrow \neg p \vdash p \rightarrow q$ is valid:

1	$\neg q \rightarrow \neg p$	premise
2	p	assumption
3	$\neg \neg p$	$\neg \neg$ i 2
4	$\neg \neg q$	MT 1, 3
5	q	$\neg \neg$ e 4
6	$p \rightarrow q$	\rightarrow i 2-5

Definition

▶ or introduction

$$\frac{\varphi}{\varphi \vee \psi} \vee i_1 \quad \frac{\psi}{\varphi \vee \psi} \vee i_2$$

Example

$p \wedge q \vdash \neg q \vee p$ is valid:

1	$p \wedge q$	premise
2	p	\wedge e ₁ 1
3	$\neg q \vee p$	\vee i ₂ 2

Definition

▶ or elimination

$$\frac{\begin{array}{|l} \varphi \\ \vdots \\ \chi \end{array} \quad \begin{array}{|l} \psi \\ \vdots \\ \chi \end{array}}{\chi} \vee e$$

Example

$p \vee q \vdash q \vee p$ is valid:

1	$p \vee q$	premise
2	p	assumption
3	$q \vee p$	\vee i ₂ 2
4	q	assumption
5	$q \vee p$	\vee i ₁ 4
6	$q \vee p$	\vee e 1, 2-3, 4-5

Definition

theorem is formula φ such that sequent $\vdash \varphi$ is valid

Example

$p \vee q \rightarrow q \vee p$ is theorem:

1	$p \vee q$	assumption
2	p	assumption
3	$q \vee p$	$\vee i_2$ 2
4	q	assumption
5	$q \vee p$	$\vee i_1$ 4
6	$q \vee p$	$\vee e$ 1, 2-3, 4-5
7	$p \vee q \rightarrow q \vee p$	$\rightarrow i$ 1-6

Definitions

▶ **bottom elimination**

$$\frac{\perp}{\varphi} \perp e$$

▶ **negation elimination**

$$\frac{\varphi \quad \neg \varphi}{\perp} \neg e$$

▶ **negation introduction**

$$\frac{\boxed{\begin{array}{c} \varphi \\ \vdots \\ \perp \end{array}}}{\neg \varphi} \neg i$$

▶ **top introduction**

$$\frac{}{\top} \top i$$

Examples

$p \rightarrow q, p \rightarrow \neg q \vdash \neg p$ is valid:

1	$p \rightarrow q$	premise
2	$p \rightarrow \neg q$	premise
3	p	assumption
4	q	$\rightarrow e$ 1, 3
5	$\neg q$	$\rightarrow e$ 2, 3
6	\perp	$\neg e$ 4, 5
7	$\neg p$	$\neg i$ 3-6

$p, p \rightarrow q, p \rightarrow \neg q \vdash r$ is valid:

1	p	premise
2	$p \rightarrow q$	premise
3	$p \rightarrow \neg q$	premise
4	q	$\rightarrow e$ 2, 1
5	$\neg q$	$\rightarrow e$ 3, 1
6	\perp	$\neg e$ 4, 5
7	r	$\perp e$ 6

Example

\top is theorem: 1 \top $\top i$

Definitions

▶ **proof by contradiction**

$$\frac{\boxed{\begin{array}{c} \neg \varphi \\ \vdots \\ \perp \end{array}}}{\varphi} \text{PBC}$$

▶ **law of excluded middle**

$$\frac{}{\varphi \vee \neg \varphi} \text{LEM}$$

Example

$p \rightarrow q \vee r, q \rightarrow \neg p, \neg r \rightarrow p \vdash q \rightarrow r$ is valid:

1	$p \rightarrow q \vee r$	premise
2	$q \rightarrow \neg p$	premise
3	$\neg r \rightarrow p$	premise
4	q	assumption
5	$\neg p$	$\rightarrow e$ 2, 4
6	$\neg r$	assumption
7	p	$\rightarrow e$ 3, 6
8	\perp	$\neg e$ 7, 5
9	r	PBC 6-8
10	$q \rightarrow r$	$\rightarrow i$ 4-9

Example

$p \rightarrow q \vdash \neg p \vee q$ is valid:

1	$p \rightarrow q$	premise
2	$p \vee \neg p$	LEM
3	p	assumption
4	q	$\rightarrow e$ 1, 3
5	$\neg p \vee q$	$\vee i_2$ 4
6	$\neg p$	assumption
7	$\neg p \vee q$	$\vee i_1$ 6
8	$\neg p \vee q$	$\vee e$ 2, 3-5, 6-7

Summary of Natural Deduction 1

	introduction	elimination
\wedge	$\frac{\varphi \quad \psi}{\varphi \wedge \psi} \wedge i$	$\frac{\varphi \wedge \psi}{\varphi} \wedge e_1 \quad \frac{\varphi \wedge \psi}{\psi} \wedge e_2$
\vee	$\frac{\varphi}{\varphi \vee \psi} \vee i_1 \quad \frac{\psi}{\varphi \vee \psi} \vee i_2$	$\frac{\varphi \vee \psi \quad \boxed{\begin{smallmatrix} \varphi \\ \vdots \\ \chi \end{smallmatrix}} \quad \boxed{\begin{smallmatrix} \psi \\ \vdots \\ \chi \end{smallmatrix}}}{\chi} \vee e$
\rightarrow	$\frac{\boxed{\begin{smallmatrix} \varphi \\ \vdots \\ \psi \end{smallmatrix}}}{\varphi \rightarrow \psi} \rightarrow i$	$\frac{\varphi \rightarrow \psi \quad \varphi}{\psi} \rightarrow e$

Summary of Natural Deduction 2

	introduction	elimination
\neg	$\frac{\boxed{\begin{smallmatrix} \varphi \\ \vdots \\ \perp \end{smallmatrix}}}{\neg \varphi} \neg i$	$\frac{\varphi \quad \neg \varphi}{\perp} \neg e$
\perp		$\frac{\perp}{\varphi} \perp e$
\top	$\frac{}{\top} \top i$	
$\neg \neg$		$\frac{\neg \neg \varphi}{\varphi} \neg \neg e$

Summary of Natural Deduction 3

derived proof rules

$$\frac{\varphi \rightarrow \psi \quad \neg \psi}{\neg \varphi} \text{ MT} \qquad \frac{\varphi}{\neg \neg \varphi} \neg \text{i} \qquad \frac{\boxed{\begin{array}{c} \neg \varphi \\ \vdots \\ \perp \end{array}}}{\varphi} \text{ PBC} \qquad \frac{}{\varphi \vee \neg \varphi} \text{ LEM}$$

Theorem

proof rules MT, \neg i, PBC and LEM are **derivable** from other (**basic**) proof rules

Theorem

proof rules **MT**, \neg i, PBC and LEM are derivable from other (basic) proof rules

Proof

1	$\varphi \rightarrow \psi$	premise
2	$\neg \psi$	premise
3	φ	assumption
4	ψ	\rightarrow e 1, 3
5	\perp	\neg e 4, 2
6	$\neg \varphi$	\neg i 3-5

Theorem

proof rules MT, \neg i, PBC and LEM are derivable from other (basic) proof rules

Proof

1	φ	premise
2	$\neg \varphi$	assumption
3	\perp	\neg e 1, 2
4	$\neg \neg \varphi$	\neg i 2-3

Theorem

proof rules MT, \neg i, **PBC** and LEM are derivable from other (basic) proof rules

Proof

1	$\neg \varphi$	hypothesis
	\vdots	
n	\perp	
n+1	$\neg \neg \varphi$	\neg i 1-n
n+2	φ	$\neg \neg$ e n+1 conclusion

Theorem

proof rules MT, \neg i, PBC and **LEM** are derivable from other (basic) proof rules

Proof

1	$\neg(\varphi \vee \neg\varphi)$	assumption
2	φ	assumption
3	$\varphi \vee \neg\varphi$	$\vee i_1$ 2
4	\perp	$\neg e$ 3, 1
5	$\neg\varphi$	$\neg i$ 2-4
6	$\varphi \vee \neg\varphi$	$\vee i_2$ 5
7	\perp	$\neg e$ 6, 1
8	$\neg\neg(\varphi \vee \neg\varphi)$	$\neg i$ 1-7
9	$\varphi \vee \neg\varphi$	$\neg\neg e$ 8

Theorem

proof rules LEM, PBC and $\neg\neg e$ are **inter-derivable** (with respect to other basic proof rules)

Remark


- ▶ LEM, PBC and $\neg\neg e$ are **controversial** because they are not **constructive**
- ▶ **classical** logicians use all proof rules
- ▶ **intuitionistic** logicians do not use LEM, PBC and $\neg\neg e$

Example

- ▶ formula $((p \rightarrow q) \rightarrow p) \rightarrow p$ is valid
- ▶ sequent $\vdash ((p \rightarrow q) \rightarrow p) \rightarrow p$ is valid but proof requires LEM, PBC or $\neg\neg e$

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Question

Which of the following statements are true ?

- ✓ The sequent $p \rightarrow q \vdash \neg q \rightarrow \neg p$ is provable in natural deduction.
- ✓ The sequent $(p \rightarrow q) \rightarrow p \vdash p$ is only derivable in classical logic.
- ✓ Valid sequents can have arbitrarily long natural deduction proofs.
- ✓ Every valid sequent has a proof without \neg i.
- E** Intuitionistic logicians can prove more statements than classical logicians.
- F** Every natural deduction proof of sequent with n premises has at least n lines.



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Theorem

natural deduction is sound:

$$\varphi_1, \varphi_2, \dots, \varphi_n \vdash \psi \text{ is valid} \implies \varphi_1, \varphi_2, \dots, \varphi_n \models \psi$$

only true statements can be proved

Corollary

theorems are valid

Idea

use **induction** on length of natural deduction proof and **case analysis** of last proof step

Problem

initial part of proof need not correspond to sequent with same premises

1	$p \rightarrow q$	premise	$p \rightarrow q, p \rightarrow \neg q \vdash p \rightarrow q$
2	$p \rightarrow \neg q$	premise	$p \rightarrow q, p \rightarrow \neg q \vdash p \rightarrow \neg q$
3	p	assumption	?
4	q	$\rightarrow e$ 1, 3	?
5	$\neg q$	$\rightarrow e$ 2, 3	?
6	\perp	$\neg e$ 4, 5	?
7	$\neg p$	$\neg i$ 3-6	$p \rightarrow q, p \rightarrow \neg q \vdash \neg p$

Solution

add assumptions to sequents

1	$p \rightarrow q$	premise	$p \rightarrow q, p \rightarrow \neg q \vdash p \rightarrow q$
2	$p \rightarrow \neg q$	premise	$p \rightarrow q, p \rightarrow \neg q \vdash p \rightarrow \neg q$
3	p	assumption	$p \rightarrow q, p \rightarrow \neg q; p \vdash p$
4	q	$\rightarrow e$ 1, 3	$p \rightarrow q, p \rightarrow \neg q; p \vdash q$
5	$\neg q$	$\rightarrow e$ 2, 3	$p \rightarrow q, p \rightarrow \neg q; p \vdash \neg q$
6	\perp	$\neg e$ 4, 5	$p \rightarrow q, p \rightarrow \neg q; p \vdash \perp$
7	$\neg p$	$\neg i$ 3-6	$p \rightarrow q, p \rightarrow \neg q \vdash \neg p$

Definition

extended sequent

$$\underbrace{\varphi_1, \varphi_2, \dots, \varphi_n}_{\text{premises } \Phi_1}; \underbrace{\psi_1, \psi_2, \dots, \psi_m}_{\text{assumptions } \Phi_2} \vdash \underbrace{\chi}_{\text{conclusion}}$$

Example

1	$p \wedge q \rightarrow r$	premise	$p \wedge q \rightarrow r \vdash p \wedge q \rightarrow r$
2	p	assumption	$p \wedge q \rightarrow r; p \vdash p$
3	q	assumption	$p \wedge q \rightarrow r; p, q \vdash q$
4	$p \wedge q$	\wedge i 2, 3	$p \wedge q \rightarrow r; p, q \vdash p \wedge q$
5	r	\rightarrow e 1, 4	$p \wedge q \rightarrow r; p, q \vdash r$
6	$q \rightarrow r$	\rightarrow i 3-5	$p \wedge q \rightarrow r; p \vdash q \rightarrow r$
7	$p \rightarrow (q \rightarrow r)$	\rightarrow i 2-6	$p \wedge q \rightarrow r \vdash p \rightarrow (q \rightarrow r)$

Soundness Proof

by induction on length of proof of

$$\Phi_1; \Phi_2 \vdash \psi$$

we prove

$$\Phi_1, \Phi_2 \models \psi$$



Claim

$$\Phi_1; \Phi_2 \vdash \psi \text{ is valid} \implies \Phi_1, \Phi_2 \models \psi$$

Base Cases

- ▶ premise $\psi \in \Phi_1 \implies \Phi_1, \Phi_2 \models \psi$
- ▶ assumption $\psi \in \Phi_2 \implies \Phi_1, \Phi_2 \models \psi$
- ▶ \top i $\psi = \top \implies \Phi_1, \Phi_2 \models \psi$



Claim

$$\Phi_1; \Phi_2 \vdash \psi \text{ is valid} \implies \Phi_1, \Phi_2 \models \psi$$

Induction Step (case analysis of last proof step) \wedge i

$$\psi = \psi_1 \wedge \psi_2$$

shorter proofs $\Phi_1; \Phi_2^1 \vdash \psi_1$ and $\Phi_1; \Phi_2^2 \vdash \psi_2$ with $\Phi_2^1, \Phi_2^2 \subseteq \Phi_2$

induction hypothesis: $\Phi_1, \Phi_2^1 \models \psi_1$ and $\Phi_1, \Phi_2^2 \models \psi_2$

$$\begin{aligned} \bar{v}(\varphi) = \text{T for all } \varphi \in \Phi_1, \Phi_2 &\implies \bar{v}(\varphi) = \text{T for all } \varphi \in \Phi_1, \Phi_2^1, \Phi_2^2 \\ &\implies \bar{v}(\psi_1) = \bar{v}(\psi_2) = \text{T} \\ &\implies \bar{v}(\psi_1 \wedge \psi_2) = \text{T} \end{aligned}$$

hence $\Phi_1, \Phi_2 \models \psi$



Claim

$$\Phi_1; \Phi_2 \vdash \psi \text{ is valid} \implies \Phi_1, \Phi_2 \models \psi$$

Induction Step (case analysis of last proof step) \rightarrow i

$$\psi = \psi_1 \rightarrow \psi_2$$

shorter proof $\Phi_1; \Phi_2^1, \psi_1 \vdash \psi_2$ with $\Phi_2^1 \subseteq \Phi_2$

induction hypothesis: $\Phi_1, \Phi_2^1, \psi_1 \models \psi_2$

$\bar{v}(\varphi) = \text{T for all } \varphi \in \Phi_1, \Phi_2$

two cases

$$\bar{v}(\psi_1) = \text{F} \implies \bar{v}(\psi_1 \rightarrow \psi_2) = \text{T}$$

$$\bar{v}(\psi_1) = \text{T} \implies \bar{v}(\varphi) = \text{T for all } \varphi \in \Phi_1, \Phi_2^1, \psi_1 \implies \bar{v}(\psi_2) = \text{T} \implies \bar{v}(\psi_1 \rightarrow \psi_2) = \text{T}$$

hence $\Phi_1, \Phi_2 \models \psi$



Claim

$\Phi_1; \Phi_2 \vdash \psi$ is valid $\implies \Phi_1, \Phi_2 \models \psi$

Induction Step (case analysis of last proof step) $\neg e$

$\psi = \perp$

shorter proofs $\Phi_1; \Phi_2^1 \vdash \psi'$ and $\Phi_1; \Phi_2^2 \vdash \neg\psi'$ with $\Phi_2^1, \Phi_2^2 \subseteq \Phi_2$

induction hypothesis: $\Phi_1, \Phi_2^1 \models \psi'$ and $\Phi_1, \Phi_2^2 \models \neg\psi'$

$$\begin{aligned} \bar{v}(\varphi) = T \text{ for all } \varphi \in \Phi_1, \Phi_2 &\implies \bar{v}(\varphi) = T \text{ for all } \varphi \in \Phi_1, \Phi_2^1, \Phi_2^2 \\ &\implies \bar{v}(\psi') = T \text{ and } \bar{v}(\neg\psi') = T \quad \text{⚡} \end{aligned}$$

hence $\Phi_1, \Phi_2 \models \perp$

Claim

$\Phi_1; \Phi_2 \vdash \psi$ is valid $\implies \Phi_1, \Phi_2 \models \psi$

Induction Step (case analysis of last proof step) $\vee e$

shorter proofs $\Phi_1; \Phi_2^1 \vdash \psi_1 \vee \psi_2$ and $\Phi_1; \Phi_2^2, \psi_1 \vdash \psi$ and $\Phi_1; \Phi_2^3, \psi_2 \vdash \psi$ with $\Phi_2^1, \Phi_2^2, \Phi_2^3 \subseteq \Phi_2$

induction hypothesis: $\Phi_1, \Phi_2^1 \models \psi_1 \vee \psi_2$ and $\Phi_1; \Phi_2^2, \psi_1 \models \psi$ and $\Phi_1; \Phi_2^3, \psi_2 \models \psi$

$$\bar{v}(\varphi) = T \text{ for all } \varphi \in \Phi_1, \Phi_2 \implies \bar{v}(\psi_1 \vee \psi_2) = T$$

two cases

$$\bar{v}(\psi_1) = T \implies \bar{v}(\psi) = T$$

$$\bar{v}(\psi_2) = T \implies \bar{v}(\psi) = T$$

hence $\Phi_1, \Phi_2 \models \psi$

Claim

$\Phi_1; \Phi_2 \vdash \psi$ is valid $\implies \Phi_1, \Phi_2 \models \psi$

Induction Step (case analysis of last proof step)

remaining (basic) proof rules

$$\wedge e_1 \quad \wedge e_2 \quad \vee i_1 \quad \vee i_2 \quad \rightarrow e \quad \neg i \quad \perp e \quad \neg\neg e$$

are similar

Corollary

natural deduction is sound

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Huth and Ryan

- ▶ Section 1.2
- ▶ Sections 1.4.2 and 1.4.3

Intuitionistic Logic

- ▶ Wikipedia [accessed December 27, 2024]

Differences (slides – book)

- ▶ top introduction proof rule
- ▶ order of premises in $\rightarrow e$

Important Concepts

- ▶ and elimination
- ▶ and introduction
- ▶ assumption
- ▶ bottom elimination
- ▶ derived proof rule
- ▶ double negation introduction
- ▶ double negation elimination
- ▶ extended sequent
- ▶ implication elimination
- ▶ implication introduction
- ▶ intuitionistic logic
- ▶ law of excluded middle
- ▶ modus ponens
- ▶ modus tollens
- ▶ natural deduction
- ▶ negation elimination
- ▶ negation introduction
- ▶ or elimination
- ▶ or introduction
- ▶ premise
- ▶ proof by contradiction
- ▶ sequent
- ▶ soundness
- ▶ theorem
- ▶ top introduction
- ▶ validity of sequents

homework for March 26