- Prepare your solutions on paper.
- Mark the exercises in OLAT before the deadline.
- Marking an exercise means that a significant part of that exercise has been treated.

Exercise 1 *Size-Change Termination* **8 p.**

In the lecture the set of multigraphs $\mathcal{M}$ of a set of size-change graphs $\mathcal{G}$ has essentially been defined as follows:

$$\frac{G \in \mathcal{G}}{G \in \mathcal{M}} \qquad \frac{G_1 \in \mathcal{M} \quad G_2 \in \mathcal{M}}{G_1 \bullet G_2 \in \mathcal{M}}$$

Now consider the following set of multigraphs $\mathcal{N}$, defined as:

$$\frac{G \in \mathcal{G}}{G \in \mathcal{N}} \qquad \frac{G_1 \in \mathcal{G} \quad G_2 \in \mathcal{N}}{G_1 \bullet G_2 \in \mathcal{N}}$$

In this exercise we will show that both definitions are equivalent.

1. Prove $\mathcal{N} \subseteq \mathcal{M}$. (2 points)
2. Prove $\mathcal{M} \subseteq \mathcal{N}$. You can assume that $\bullet$ is associative. Most likely, you will need to prove one auxiliary property. (4 points)
3. Think about an implementation: is it faster to compute $\mathcal{M}$ or $\mathcal{N}$? Why? Provide a pseudo-code implementation for the faster algorithm. (Do NOT go into details how $\bullet$ is implemented) (2 points)

Exercise 2 *Polynomial Interpretations* **6 p.**

Consider the following functional program for computing the binary logarithm.

$$\begin{aligned}
\mathsf{half}(\mathsf{Zero}) &= \mathsf{Zero} \\
\mathsf{half}(\mathsf{Succ}(\mathsf{Zero})) &= \mathsf{Zero} \\
\mathsf{half}(\mathsf{Succ}(\mathsf{Succ}(x))) &= \mathsf{Succ}(\mathsf{half}(x)) \\
\mathsf{log2}(\mathsf{Zero}) &= \mathsf{Zero} \\
\mathsf{log2}(\mathsf{Succ}(\mathsf{Zero})) &= \mathsf{Zero} \\
\mathsf{log2}(\mathsf{Succ}(\mathsf{Succ}(x))) &= \mathsf{Succ}(\mathsf{log2}(\mathsf{Succ}(\mathsf{half}(x))))
\end{aligned}$$

1. Write down all dependency pairs that cannot be solved by the subterm criterion and determine the usable equations for these dependency pairs. (1 point)
2. Prove termination via polynomial interpretations. First setup the constraints symbolically, and then choose between manual solving and SMT-solving. For the latter you can either directly download and compile Z3 from github, or use a binary version that is distributed as part of Isabelle in the `contrib/z3...` directory. (5 points)

Exercise 3 *Usable Equations* **6 p.**

The aim of this exercise is to prove soundness of usable equations. Let $\mathcal{E}$ be the defining equations of a program and let $(\succ, \succsim)$ be a reduction pair where we further assume that $\succsim$ is reflexive.
Define $\mathcal{E}_F = \{\ell = r \in \mathcal{E} \mid \ell = F(\dots)\}$ as the equations from $\mathcal{E}$ with F as root symbol.
We say that a set of defined symbols S is *oriented* iff the following statement is satisfied: whenever $\ell = r \in \mathcal{E}_F$ with $F \in S$ then $\ell \succsim r$.
Show the following statement: whenever $\mathcal{US}(t) \subseteq \mathcal{US}(P)$ and $\mathcal{US}(P)$ is oriented and $t\sigma \overset{\mathsf{i}}{\hookrightarrow}^* s$ for some normal form substitution σ,[1] then $t\sigma \succsim s$.

[1] σ is a normal form substitution if $\sigma(x) \in NF(\overset{\mathsf{i}}{\hookrightarrow})$ for all x.