universität innsbruck

Term Rewriting

Philipp Dablander and **Aart Middeldorp**

Outline

Definitions

- $\mathcal{T}(\mathcal{F},\mathcal{V})$ is set of terms built from signature (function symbols with arities) $\mathcal{F}$ and infinitely many variables $\mathcal{V}$ with $\mathcal{F} \cap \mathcal{V} \neq \varnothing$
- context is term with one hole: element of $\mathcal{T}(\mathcal{F} \cup \{\square\},\mathcal{V})$ that contains exactly one occurrence of special constant $\square$
- $C[t]$ denotes result of replacing hole in context C by term t
- binary relation R on terms is closed under contexts if for all terms s, t

$$s \mathrel{R} t \quad\Longrightarrow\quad C[s] \mathrel{R} C[t] \quad \text{for all contexts } C$$

- substitution is mapping $\sigma \colon \mathcal{V} \to \mathcal{T}(\mathcal{F},\mathcal{V})$ such that its domain

$$\mathcal{D}\mathrm{om}(\sigma) = \{x \in \mathcal{V} \mid \sigma(x) \neq x\}$$

is finite

Definitions

- application of substitution σ to term t

$$t\sigma = \begin{cases} \sigma(t) & \text{if } t \in \mathcal{V} \\ f(t_1\sigma, \ldots, t_n\sigma) & \text{if } t = f(t_1, \ldots, t_n) \end{cases}$$

- binary relation R on terms is closed under substitutions if for all terms s, t

$$s \mathrel{R} t \implies s\sigma \mathrel{R} t\sigma \quad \text{for all substitutions } \sigma$$

Theorem

matching problem

instance: terms s, t

question: $\exists$ substitution σ: $s\sigma = t$? (t is instance of s, $t \geqslant s$)

is decidable (in linear time)

Outline

1. **Summary of Lecture 1**
2. **Abstract Rewrite Systems**
   Definitions Properties Relationships
3. **Newman's Lemma**
4. **Exercises**
5. **Further Reading**

Motivation

concrete rewrite formalisms

- string rewriting

Motivation

concrete rewrite formalisms

- string rewriting
- term rewriting

Motivation

concrete rewrite formalisms

- string rewriting
- term rewriting
- graph rewriting

Motivation

concrete rewrite formalisms

- string rewriting
- term rewriting
- graph rewriting
- λ–calculus

Motivation

concrete rewrite formalisms

- string rewriting
- term rewriting
- graph rewriting
- λ–calculus
- interaction nets

Motivation

concrete rewrite formalisms

- string rewriting
- term rewriting
- graph rewriting
- λ–calculus
- interaction nets
- $\cdots$

Motivation

concrete rewrite formalisms

- string rewriting
- term rewriting
- graph rewriting
- λ–calculus
- interaction nets
- $\cdots$

abstract rewriting

- no structure on objects that are rewritten

Motivation

concrete rewrite formalisms

- string rewriting
- term rewriting
- graph rewriting
- λ–calculus
- interaction nets
- $\cdots$

abstract rewriting

- no structure on objects that are rewritten
- uniform presentation of properties and proofs

Outline

Definitions (Abstract Rewrite System)

- abstract rewrite system (ARS) is set A equipped with binary relation $\rightarrow$

Definitions (Abstract Rewrite System)

- abstract rewrite system (ARS) is set A equipped with binary relation $\to$

Example

- ARS $\mathcal{A} = \langle A, \to \rangle$ with $A = \{a, b, c, d, e, f, g\}$ and

$$\to \; = \left\{ \begin{matrix} (a,e), (b,a), (b,c), (c,d), (c,f) \\ (e,b), (e,g), (f,e), (f,g) \end{matrix} \right\}$$

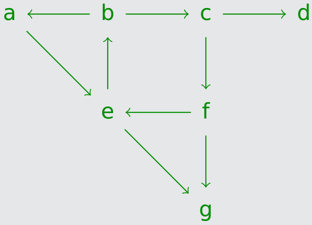
Definitions (Abstract Rewrite System)

- abstract rewrite system (ARS) is set A equipped with binary relation $\to$

Example

- ARS $\mathcal{A} = \langle A, \to \rangle$ with $A = \{\mathsf{a}, \mathsf{b}, \mathsf{c}, \mathsf{d}, \mathsf{e}, \mathsf{f}, \mathsf{g}\}$ and

$$\to \; = \left\{ \begin{array}{l} (\mathsf{a},\mathsf{e}),\ (\mathsf{b},\mathsf{a}),\ (\mathsf{b},\mathsf{c}),\ (\mathsf{c},\mathsf{d}),\ (\mathsf{c},\mathsf{f}) \\ (\mathsf{e},\mathsf{b}),\ (\mathsf{e},\mathsf{g}),\ (\mathsf{f},\mathsf{e}),\ (\mathsf{f},\mathsf{g}) \end{array} \right\}$$



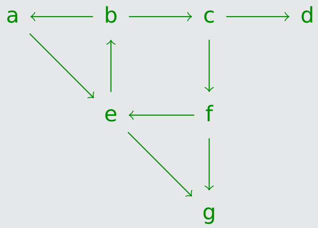
Definitions (Abstract Rewrite System)

- abstract rewrite system (ARS) is set A equipped with binary relation $\to$
- elements of $\to$ are called rewrite steps

Example

- ARS $\mathcal{A} = \langle A, \to \rangle$ with $A = \{a, b, c, d, e, f, g\}$ and

$$\to \; = \left\{ \begin{matrix} (a,e), (b,a), (b,c), (c,d), (c,f) \\ (e,b), (e,g), (f,e), (f,g) \end{matrix} \right\}$$



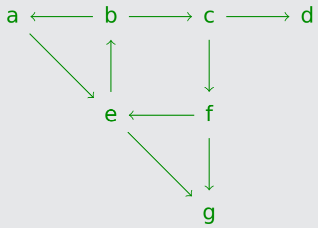
Definitions (Abstract Rewrite System)

- abstract rewrite system (ARS) is set A equipped with binary relation $\to$
- elements of $\to$ are called rewrite steps
- (finite) rewrite sequence is sequence of $(a_0, \ldots, a_n)$ such that $a_i \to a_{i+1}$ for all $0 \leqslant i < n$

Example

- ARS $\mathcal{A} = \langle A, \to \rangle$ with $A = \{\mathsf{a}, \mathsf{b}, \mathsf{c}, \mathsf{d}, \mathsf{e}, \mathsf{f}, \mathsf{g}\}$ and

$$\to \; = \begin{Bmatrix} (\mathsf{a},\mathsf{e}),\ (\mathsf{b},\mathsf{a}),\ (\mathsf{b},\mathsf{c}),\ (\mathsf{c},\mathsf{d}),\ (\mathsf{c},\mathsf{f}) \\ (\mathsf{e},\mathsf{b}),\ (\mathsf{e},\mathsf{g}),\ (\mathsf{f},\mathsf{e}),\ (\mathsf{f},\mathsf{g}) \end{Bmatrix}$$



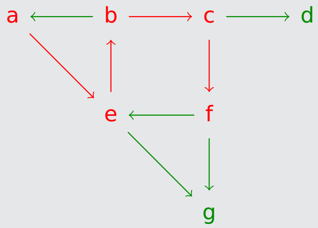
Definitions (Abstract Rewrite System)

- abstract rewrite system (ARS) is set A equipped with binary relation $\to$
- elements of $\to$ are called rewrite steps
- (finite) rewrite sequence is sequence of $(a_0, \ldots, a_n)$ such that $a_i \to a_{i+1}$ for all $0 \leqslant i < n$

Example

- ARS $\mathcal{A} = \langle A, \to \rangle$ with $A = \{a, b, c, d, e, f, g\}$ and

$$\to \; = \left\{ \begin{matrix} (a, e), (b, a), (b, c), (c, d), (c, f) \\ (e, b), (e, g), (f, e), (f, g) \end{matrix} \right\}$$



- rewrite sequences
  - finite $\quad a \to e \to b \to c \to f$

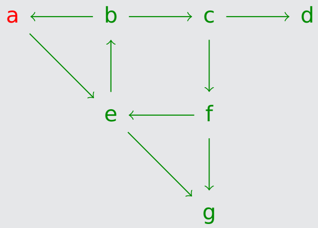
Definitions (Abstract Rewrite System)

- abstract rewrite system (ARS) is set A equipped with binary relation $\to$
- elements of $\to$ are called rewrite steps
- (finite) rewrite sequence is sequence of $(a_0, \ldots, a_n)$ such that $a_i \to a_{i+1}$ for all $0 \leqslant i < n$

Example

- ARS $\mathcal{A} = \langle A, \to \rangle$ with $A = \{\mathsf{a}, \mathsf{b}, \mathsf{c}, \mathsf{d}, \mathsf{e}, \mathsf{f}, \mathsf{g}\}$ and

$$\to \; = \left\{ \begin{matrix} (\mathsf{a},\mathsf{e}), (\mathsf{b},\mathsf{a}), (\mathsf{b},\mathsf{c}), (\mathsf{c},\mathsf{d}), (\mathsf{c},\mathsf{f}) \\ (\mathsf{e},\mathsf{b}), (\mathsf{e},\mathsf{g}), (\mathsf{f},\mathsf{e}), (\mathsf{f},\mathsf{g}) \end{matrix} \right\}$$



- rewrite sequences
  - finite $\mathsf{a} \to \mathsf{e} \to \mathsf{b} \to \mathsf{c} \to \mathsf{f}$
  - empty a

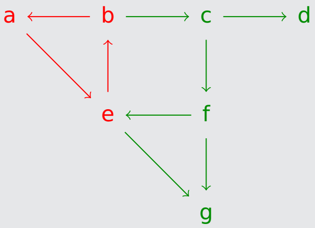
Definitions (Abstract Rewrite System)

- abstract rewrite system (ARS) is set A equipped with binary relation $\to$
- elements of $\to$ are called rewrite steps
- (finite) rewrite sequence is sequence of $(a_0, \ldots, a_n)$ such that $a_i \to a_{i+1}$ for all $0 \leqslant i < n$

Example

- ARS $\mathcal{A} = \langle A, \to \rangle$ with $A = \{\mathsf{a}, \mathsf{b}, \mathsf{c}, \mathsf{d}, \mathsf{e}, \mathsf{f}, \mathsf{g}\}$ and

$$\to \; = \begin{cases} (\mathsf{a},\mathsf{e}),\ (\mathsf{b},\mathsf{a}),\ (\mathsf{b},\mathsf{c}),\ (\mathsf{c},\mathsf{d}),\ (\mathsf{c},\mathsf{f}) \\ (\mathsf{e},\mathsf{b}),\ (\mathsf{e},\mathsf{g}),\ (\mathsf{f},\mathsf{e}),\ (\mathsf{f},\mathsf{g}) \end{cases}$$



- rewrite sequences
  - finite $\mathsf{a} \to \mathsf{e} \to \mathsf{b} \to \mathsf{c} \to \mathsf{f}$
  - empty a
  - infinite $\mathsf{a} \to \mathsf{e} \to \mathsf{b} \to \mathsf{a} \to \mathsf{e} \to \mathsf{b} \to \cdots$

Definitions (Derived Relations)

- $\leftarrow$ inverse of $\rightarrow$

Definitions (Derived Relations)

- $\leftarrow$ inverse of $\rightarrow$
- $\rightarrow^*$ transitive and reflexive closure of $\rightarrow$

Definitions (Derived Relations)

- $\leftarrow$ inverse of $\rightarrow$
- $\rightarrow^*$ transitive and reflexive closure of $\rightarrow$
- ${}^*\!\leftarrow$ inverse of $\rightarrow^*$

Definitions (Derived Relations)

- $\leftarrow$ inverse of $\rightarrow$
- $\rightarrow^*$ transitive and reflexive closure of $\rightarrow$
- ${}^*\!\leftarrow$ inverse of $\rightarrow^*$
- $\downarrow$ joinability $\downarrow \, = \, \rightarrow^* \cdot {}^*\!\leftarrow$

Definitions (Derived Relations)

- $\leftarrow$ inverse of $\rightarrow$
- $\rightarrow^*$ transitive and reflexive closure of $\rightarrow$
- ${}^*\!\leftarrow$ inverse of $\rightarrow^*$
- $\downarrow$ joinability $\downarrow \; = \; \rightarrow^* \cdot {}^*\!\leftarrow$

$\cdot$ denotes relation composition: $R \cdot S = \{(a, c) \mid a \; R \; b \text{ and } b \; S \; c\}$

Definitions (Derived Relations)

- $\leftarrow$ inverse of $\rightarrow$
- $\rightarrow^*$ transitive and reflexive closure of $\rightarrow$
- ${}^*\!\leftarrow$ inverse of $\rightarrow^*$
- $\downarrow$ joinability $\downarrow \,=\, \rightarrow^* \cdot {}^*\!\leftarrow$
- $\leftrightarrow$ symmetric closure of $\rightarrow$

$\cdot$ denotes relation composition: $R \cdot S = \{(a,c) \mid a\ R\ b \text{ and } b\ S\ c\}$

Definitions (Derived Relations)

- $\leftarrow$ inverse of $\rightarrow$
- $\rightarrow^*$ transitive and reflexive closure of $\rightarrow$
- ${}^*\!\leftarrow$ inverse of $\rightarrow^*$
- $\downarrow$ joinability $\downarrow \; = \; \rightarrow^* \cdot {}^*\!\leftarrow$
- $\leftrightarrow$ symmetric closure of $\rightarrow$
- $\leftrightarrow^*$ conversion (equivalence relation generated by $\rightarrow$)

$\cdot$ denotes relation composition: $R \cdot S = \{(a,c) \mid a \; R \; b \text{ and } b \; S \; c\}$

Definitions (Derived Relations)

- $\leftarrow$ inverse of $\rightarrow$
- $\rightarrow^*$ transitive and reflexive closure of $\rightarrow$
- ${}^*\!\leftarrow$ inverse of $\rightarrow^*$
- $\downarrow$ joinability $\quad \downarrow \,=\, \rightarrow^* \cdot {}^*\!\leftarrow$
- $\leftrightarrow$ symmetric closure of $\rightarrow$
- $\leftrightarrow^*$ conversion (equivalence relation generated by $\rightarrow$)
- $\rightarrow^+$ transitive closure of $\rightarrow$

$\cdot$ denotes relation composition: $R \cdot S = \{(a,c) \mid a\ R\ b \text{ and } b\ S\ c\}$

Definitions (Derived Relations)

- $\leftarrow$ inverse of $\rightarrow$
- $\rightarrow^*$ transitive and reflexive closure of $\rightarrow$
- ${}^*\!\leftarrow$ inverse of $\rightarrow^*$
- $\downarrow$ joinability $\downarrow \, = \, \rightarrow^* \cdot {}^*\!\leftarrow$
- $\leftrightarrow$ symmetric closure of $\rightarrow$
- $\leftrightarrow^*$ conversion (equivalence relation generated by $\rightarrow$)
- $\rightarrow^+$ transitive closure of $\rightarrow$
- $\rightarrow^=$ reflexive closure of $\rightarrow$

$\cdot$ denotes relation composition: $R \cdot S = \{(a,c) \mid a \; R \; b \text{ and } b \; S \; c\}$

Definitions (Derived Relations)

- $\leftarrow$ inverse of $\rightarrow$
- $\rightarrow^*$ transitive and reflexive closure of $\rightarrow$
- ${}^*\!\leftarrow$ inverse of $\rightarrow^*$
- $\downarrow$ joinability $\downarrow \, = \, \rightarrow^* \cdot {}^*\!\leftarrow$
- $\leftrightarrow$ symmetric closure of $\rightarrow$
- $\leftrightarrow^*$ conversion (equivalence relation generated by $\rightarrow$)
- $\rightarrow^+$ transitive closure of $\rightarrow$
- $\rightarrow^=$ reflexive closure of $\rightarrow$
- $\uparrow$ meetability $\uparrow \, = \, {}^*\!\leftarrow \cdot \rightarrow^*$

$\cdot$ denotes relation composition: $R \cdot S = \{(a,c) \mid a \mathrel{R} b \text{ and } b \mathrel{S} c\}$

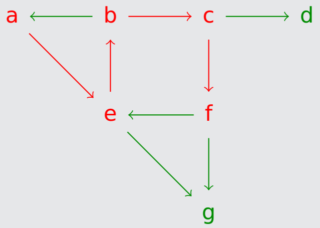
Terminology

- if $x \to^* y$ then x rewrites to y and y is reduct of x

Terminology

- if $x \to^* y$ then x rewrites to y and y is reduct of x

Example

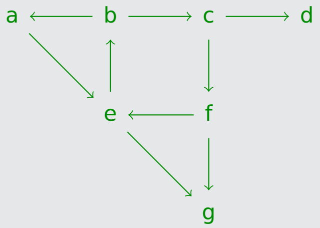


- $a \to^* f$

Terminology

- if $x \to^* y$ then x rewrites to y and y is reduct of x
- if $x \to^* z \; {}^*\!\!\leftarrow y$ then z is common reduct of x and y

Example

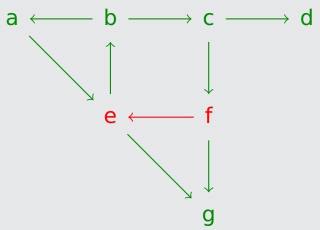


- $a \to^* f$

Terminology

- if $x \to^* y$ then x rewrites to y and y is reduct of x
- if $x \to^* z \; {}^*\!\!\leftarrow y$ then z is common reduct of x and y

Example

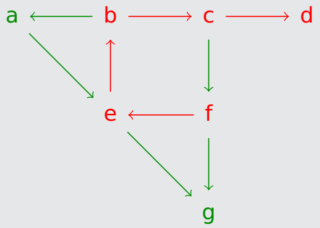


- $a \to^* f$
- $e \downarrow f$

Terminology

- if $x \to^* y$ then x rewrites to y and y is reduct of x
- if $x \to^* z \; {}^*\!\!\leftarrow y$ then z is common reduct of x and y

Example

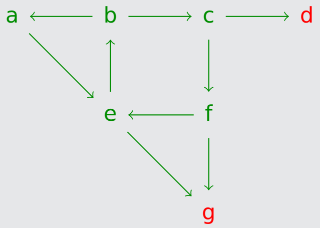


- $a \to^* f$
- $e \downarrow f$ $\quad$ $f \downarrow d$

Terminology

- if $x \rightarrow^* y$ then x rewrites to y and y is reduct of x
- if $x \rightarrow^* z \ {}^*\!\!\leftarrow y$ then z is common reduct of x and y

Example

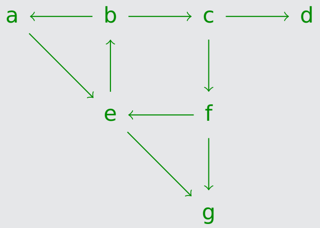


- $\mathsf{a} \rightarrow^* \mathsf{f}$
- $\mathsf{e} \downarrow \mathsf{f}$ $\quad$ $\mathsf{f} \downarrow \mathsf{d}$ $\quad$ $\mathsf{g} \not\downarrow \mathsf{d}$

Terminology

- if $x \to^* y$ then x rewrites to y and y is reduct of x
- if $x \to^* z \;{}^*\!\!\leftarrow y$ then z is common reduct of x and y
- if $x \leftrightarrow^* y$ then x and y are convertible

Example

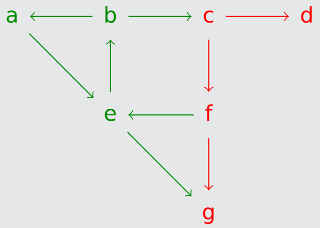


- $\mathsf{a} \to^* \mathsf{f}$
- $\mathsf{e} \downarrow \mathsf{f}$ $\quad \mathsf{f} \downarrow \mathsf{d}$ $\quad \mathsf{g} \not\downarrow \mathsf{d}$

Terminology

- if $x \to^* y$ then x rewrites to y and y is reduct of x
- if $x \to^* z \;{}^*\!\!\leftarrow y$ then z is common reduct of x and y
- if $x \leftrightarrow^* y$ then x and y are convertible

Example



- $a \to^* f$
- $e \downarrow f$ $\quad f \downarrow d$ $\quad g \not\downarrow d$
- $g \leftrightarrow^* d$

Definitions (Normal Forms)

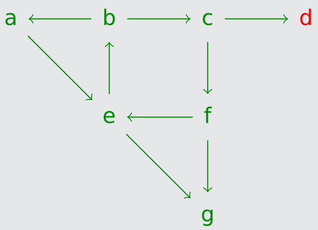
- normal form is element x such that $x \not\to y$ for all y

Definitions (Normal Forms)

- normal form is element x such that $x \not\to y$ for all y

Example

- ARS $\mathcal{A} = \langle A, \to \rangle$
- d is normal form

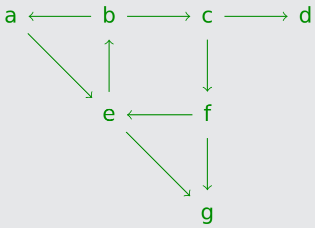


Definitions (Normal Forms)

- normal form is element x such that $x \not\to y$ for all y
- NF$(\mathcal{A})$ denotes set of normal forms of ARS $\mathcal{A}$

Example

- ARS $\mathcal{A} = \langle A, \to \rangle$
- d is normal form

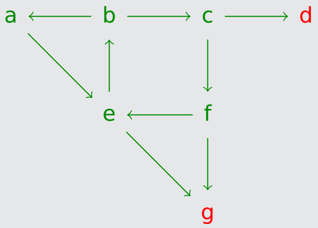


Definitions (Normal Forms)

- normal form is element x such that $x \not\to y$ for all y
- $\mathsf{NF}(\mathcal{A})$ denotes set of normal forms of ARS $\mathcal{A}$

Example

- ARS $\mathcal{A} = \langle A, \to \rangle$
- d is normal form
- $\mathsf{NF}(\mathcal{A}) = \{\mathsf{d}, \mathsf{g}\}$

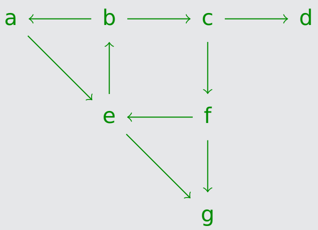


Definitions (Normal Forms)

- normal form is element x such that $x \not\to y$ for all y
- $\mathrm{NF}(\mathcal{A})$ denotes set of normal forms of ARS $\mathcal{A}$
- $x \to^! y$ if $x \to^* y$ for normal form y (x has normal form y)

Example

- ARS $\mathcal{A} = \langle A, \to \rangle$
- d is normal form
- $\mathrm{NF}(\mathcal{A}) = \{\mathrm{d}, \mathrm{g}\}$

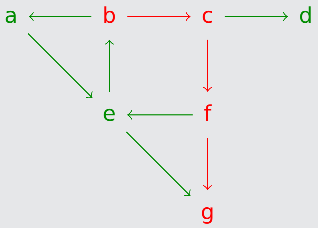


Definitions (Normal Forms)

- normal form is element x such that $x \not\to y$ for all y
- $\mathsf{NF}(\mathcal{A})$ denotes set of normal forms of ARS $\mathcal{A}$
- $x \to^! y$ if $x \to^* y$ for normal form y (x has normal form y)

Example

- ARS $\mathcal{A} = \langle A, \to \rangle$
- d is normal form
- $\mathsf{NF}(\mathcal{A}) = \{\mathsf{d}, \mathsf{g}\}$
- $\mathsf{b} \to^! \mathsf{g}$



Outline

1. **Summary of Lecture 1**
2. **Abstract Rewrite Systems**

   Definitions Properties Relationships
3. **Newman's Lemma**
4. **Exercises**
5. **Further Reading**

Definitions (Properties)

- SN strong normalization
  - no infinite rewrite sequences

Definitions (Properties)

- SN strong normalization termination
  - no infinite rewrite sequences

Definitions (Properties)

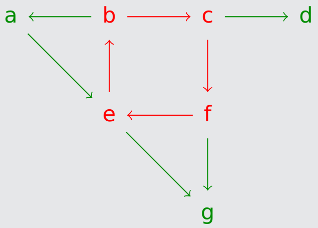
- SN strong normalization termination
  - no infinite rewrite sequences
- WN (weak) normalization
  - every element has at least one normal form

Definitions (Properties)

- SN strong normalization termination
  - no infinite rewrite sequences
- WN (weak) normalization
  - every element has at least one normal form
  - $\forall\, a\; \exists\, b \quad a \to^{!} b$

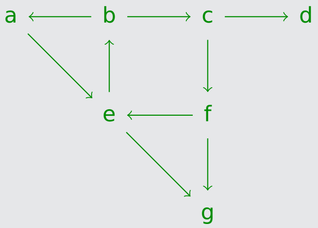
Example

- ARS $\mathcal{A} = \langle A, \to \rangle$
- not SN: $\mathsf{b} \to^+ \mathsf{b}$



Example

- ARS $\mathcal{A} = \langle A, \to \rangle$
- not SN: b $\to^+$ b
- WN



Outline

Lemmata

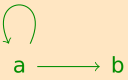
- $\mathsf{SN} \implies \mathsf{WN}$

Lemmata

- $\mathsf{SN} \implies \mathsf{WN}$
- $\mathsf{WN} \;\not\Longrightarrow\; \mathsf{SN}$

Lemmata

- $\mathsf{SN} \implies \mathsf{WN}$
- $\mathsf{WN} \not\implies \mathsf{SN}$



Definitions (Properties)

- SN strong normalization termination
  - no infinite rewrite sequences
- WN (weak) normalization
  - every element has at least one normal form
  - $\forall\, a\, \exists\, b \quad a \to^{!} b$
- UN unique normal forms
  - no element has more than one normal form

Definitions (Properties)

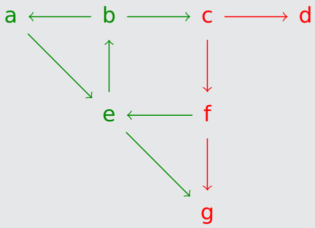
- SN strong normalization termination
  - no infinite rewrite sequences
- WN (weak) normalization
  - every element has at least one normal form
  - $\forall\, a\, \exists\, b \quad a \to^! b$
- UN unique normal forms
  - no element has more than one normal form
  - $\forall\, a, b, c$ if $a \to^! b$ and $a \to^! c$ then $b = c$

Definitions (Properties)

- SN strong normalization termination
  - no infinite rewrite sequences
- WN (weak) normalization
  - every element has at least one normal form
  - $\forall a \, \exists b \;\; a \to^! b$
- UN unique normal forms
  - no element has more than one normal form
  - $\forall a, b, c$ if $a \to^! b$ and $a \to^! c$ then $b = c$
  - ${}^!\!\leftarrow \cdot \to^! \; \subseteq \; =$

Example

- ARS $\mathcal{A} = \langle A, \to \rangle$
- not SN: $\mathsf{b} \to^+ \mathsf{b}$
- WN
- not UN: $\mathsf{c} \to^! \mathsf{d}$ and $\mathsf{c} \to^! \mathsf{g}$

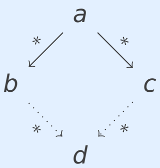


Definitions (Properties)

- CR confluence
  - $\uparrow \subseteq \downarrow$

Definitions (Properties)

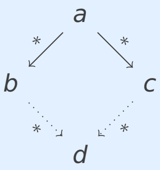
- CR confluence
  - $\uparrow \subseteq \downarrow$
  - $\forall\, a, b, c$



$\exists\, d$

Definitions (Properties)

- CR confluence
  - $\uparrow \subseteq \downarrow$
  - $\forall\, a, b, c$

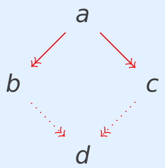


$\exists\, d$

Wikimedia

Definitions (Properties)

- CR confluence
  - $\uparrow \subseteq \downarrow$
  - $\forall\, a, b, c$



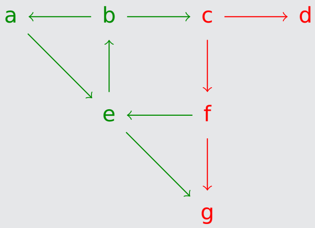
in diagrams: $\twoheadrightarrow$ for $\to^*$

$\exists\, d$

Wikimedia

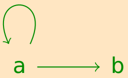
Example

- ARS $\mathcal{A} = \langle A, \rightarrow \rangle$
- not SN: b $\rightarrow^{+}$ b
- WN
- not UN: c $\rightarrow^{!}$ d and c $\rightarrow^{!}$ g
- not CR: d $\uparrow$ g and d $\not\downarrow$ g



Lemmata

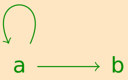
- $\mathsf{SN} \implies \mathsf{WN}$
- $\mathsf{WN} \not\implies \mathsf{SN}$



- $\mathsf{CR} \iff \leftrightarrow^* \subseteq \downarrow$ — Church–Rosser property

Lemmata

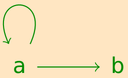
- $\mathrm{SN} \implies \mathrm{WN}$
- $\mathrm{WN} \not\Longrightarrow \mathrm{SN}$



- $\mathrm{CR} \iff\ \leftrightarrow^* \subseteq\ \downarrow\ \iff\ \leftrightarrow^* =\ \downarrow$ Church–Rosser property

Lemmata

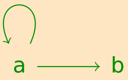
- $\mathrm{SN} \implies \mathrm{WN}$
- $\mathrm{WN} \not\Longrightarrow \mathrm{SN}$



- $\mathrm{CR} \iff\ \leftrightarrow^* \subseteq\ \downarrow\ \iff\ \leftrightarrow^* =\ \downarrow$ Church–Rosser property
- $\mathrm{CR} \implies \mathrm{UN}$

Lemmata

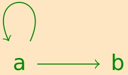
- SN $\Longrightarrow$ WN
- WN $\not\Longrightarrow$ SN



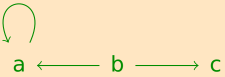
- CR $\Longleftrightarrow$ $\leftrightarrow^* \subseteq \downarrow$ $\Longleftrightarrow$ $\leftrightarrow^* = \downarrow$ Church–Rosser property
- CR $\Longrightarrow$ UN
- UN $\not\Longrightarrow$ CR

Lemmata

- $\mathrm{SN} \implies \mathrm{WN}$



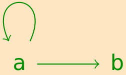
- $\mathrm{WN} \not\implies \mathrm{SN}$
- $\mathrm{CR} \iff\ \leftrightarrow^* \subseteq \downarrow\ \iff\ \leftrightarrow^* = \downarrow$ Church–Rosser property
- $\mathrm{CR} \implies \mathrm{UN}$



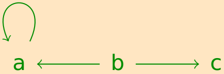
- $\mathrm{UN} \not\implies \mathrm{CR}$

Lemmata

- $\mathsf{SN} \implies \mathsf{WN}$



- $\mathsf{WN} \not\Longrightarrow \mathsf{SN}$
- $\mathsf{CR} \iff {\leftrightarrow^*} \subseteq {\downarrow} \iff {\leftrightarrow^*} = {\downarrow}$ Church–Rosser property
- $\mathsf{CR} \implies \mathsf{UN}$

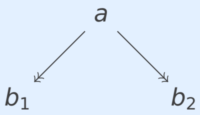


- $\mathsf{UN} \not\Longrightarrow \mathsf{CR}$
- $\mathsf{WN}\ \&\ \mathsf{UN} \implies \mathsf{CR}$

Lemma

WN & UN $\Longrightarrow$ CR

Proof

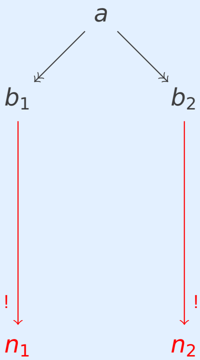


Lemma

WN & UN $\Longrightarrow$ CR

Proof

- WN $\Longrightarrow$ $\exists\, n_1, n_2$ such that $b_1 \to^! n_1$ and $b_2 \to^! n_2$

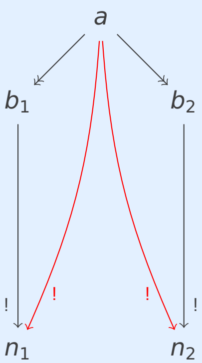


Lemma

WN & UN $\Longrightarrow$ CR

Proof

- WN $\Longrightarrow$ $\exists\, n_1, n_2$ such that $b_1 \to^! n_1$ and $b_2 \to^! n_2$
- $\to^* \cdot \to^! \subseteq \to^!$

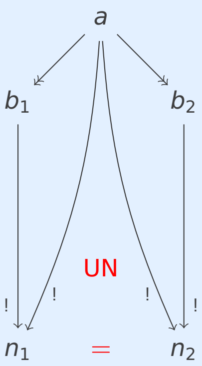


Lemma

WN & UN $\Longrightarrow$ CR

Proof

- WN $\Longrightarrow$ $\exists\ n_1, n_2$ such that $b_1 \to^! n_1$ and $b_2 \to^! n_2$
- $\to^* \cdot \to^! \subseteq \to^!$
- UN $\Longrightarrow$ $n_1 = n_2$

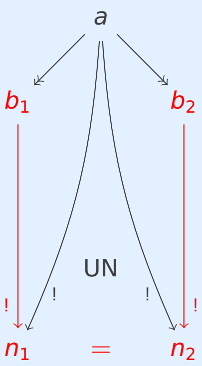


Lemma

WN & UN $\implies$ CR

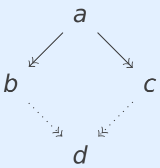
Proof

- WN $\implies$ $\exists\ n_1, n_2$ such that $b_1 \to^! n_1$ and $b_2 \to^! n_2$
- $\to^* \cdot \to^! \subseteq \to^!$
- UN $\implies$ $n_1 = n_2$ $\implies$ $b_1 \downarrow b_2$



Definitions (Properties)

- CR confluence
  - $\uparrow \subseteq \downarrow$
  - $\forall\, a, b, c$ $\exists\, d$



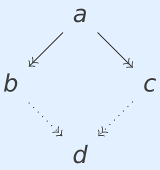
in diagrams: $\twoheadrightarrow$ for $\rightarrow^*$

Wikimedia

- WCR local confluence
  - $\leftarrow \cdot \rightarrow \subseteq \downarrow$

Definitions (Properties)

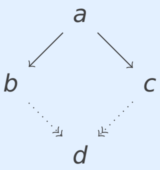
- CR confluence
  - $\uparrow \subseteq \downarrow$
  - $\forall\, a, b, c$ $\exists\, d$



in diagrams: $\twoheadrightarrow$ for $\to^*$

Wikimedia

- WCR local confluence
  - $\leftarrow \cdot \rightarrow \subseteq \downarrow$
  - $\forall\, a, b, c$ $\exists\, d$



Definitions (Properties)

- CR confluence
  - $\uparrow \subseteq \downarrow$
  - $\forall\, a, b, c$ $\exists\, d$

$$\begin{array}{ccc} & a & \\ \swarrow & & \searrow \\ b & & c \\ \searrow\!\!\!\twoheadrightarrow & & \swarrow\!\!\!\twoheadleftarrow \\ & d & \end{array}$$

in diagrams: $\twoheadrightarrow$ for $\to^*$

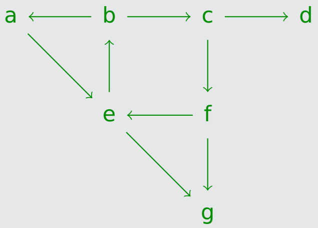
Wikimedia

- WCR local confluence weak Church–Rosser property
  - $\leftarrow \cdot \rightarrow \subseteq \downarrow$
  - $\forall\, a, b, c$ $\exists\, d$

$$\begin{array}{ccc} & a & \\ \swarrow & & \searrow \\ b & & c \\ \searrow\!\!\!\twoheadrightarrow & & \swarrow\!\!\!\twoheadleftarrow \\ & d & \end{array}$$

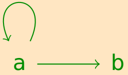
Example

- ARS $\mathcal{A} = \langle A, \rightarrow \rangle$
- not SN: $\mathsf{b} \rightarrow^{+} \mathsf{b}$
- WN
- not UN: $\mathsf{c} \rightarrow^{!} \mathsf{d}$ and $\mathsf{c} \rightarrow^{!} \mathsf{g}$
- not CR: $\mathsf{d} \uparrow \mathsf{g}$ and $\mathsf{d} \not\downarrow \mathsf{g}$
- WCR



Lemmata

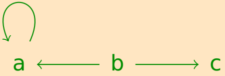
- $\mathrm{SN} \implies \mathrm{WN}$
- $\mathrm{WN} \not\Longrightarrow \mathrm{SN}$



- $\mathrm{CR} \iff\ \leftrightarrow^* \subseteq\ \downarrow\ \iff\ \leftrightarrow^* =\ \downarrow$

Church–Rosser property

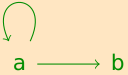
- $\mathrm{CR} \implies \mathrm{UN}$
- $\mathrm{UN} \not\Longrightarrow \mathrm{CR}$



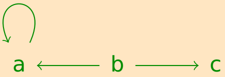
- $\mathrm{WN}\ \&\ \mathrm{UN} \implies \mathrm{CR}$
- $\mathrm{CR} \implies \mathrm{WCR}$

Lemmata

- $\text{SN} \implies \text{WN}$
- $\text{WN} \not\Longrightarrow \text{SN}$



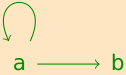
- $\text{CR} \iff {\leftrightarrow^*} \subseteq {\downarrow} \iff {\leftrightarrow^*} = {\downarrow}$ Church–Rosser property
- $\text{CR} \implies \text{UN}$
- $\text{UN} \not\Longrightarrow \text{CR}$



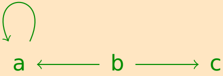
- $\text{WN} \;\&\; \text{UN} \implies \text{CR}$
- $\text{CR} \implies \text{WCR}$
- $\text{WCR} \not\Longrightarrow \text{CR}$

Lemmata

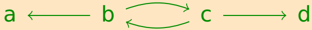
- SN $\Longrightarrow$ WN
- WN $\not\Longrightarrow$ SN



- CR $\Longleftrightarrow$ $\leftrightarrow^* \subseteq \downarrow$ $\Longleftrightarrow$ $\leftrightarrow^* = \downarrow$ Church–Rosser property
- CR $\Longrightarrow$ UN
- UN $\not\Longrightarrow$ CR

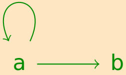


- WN & UN $\Longrightarrow$ CR
- CR $\Longrightarrow$ WCR
- WCR $\not\Longrightarrow$ CR

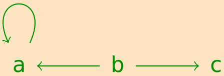


Lemmata

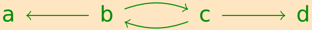
- $\mathsf{SN} \implies \mathsf{WN}$



- $\mathsf{WN} \not\Longrightarrow \mathsf{SN}$
- $\mathsf{CR} \iff \leftrightarrow^* \subseteq \downarrow \iff \leftrightarrow^* = \downarrow$ Church–Rosser property
- $\mathsf{CR} \implies \mathsf{UN}$



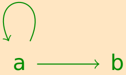
- $\mathsf{UN} \not\Longrightarrow \mathsf{CR}$
- $\mathsf{WN}\ \&\ \mathsf{UN} \implies \mathsf{CR}$
- $\mathsf{CR} \implies \mathsf{WCR}$
- $\mathsf{WCR} \not\Longrightarrow \mathsf{CR}$



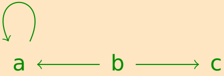
- $\mathsf{SN}\ \&\ \mathsf{WCR} \implies \mathsf{CR}$

Lemmata

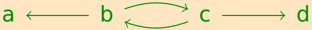
- $\mathsf{SN} \implies \mathsf{WN}$



- $\mathsf{WN} \centernot\implies \mathsf{SN}$
- $\mathsf{CR} \iff {\leftrightarrow^*} \subseteq {\downarrow} \iff {\leftrightarrow^*} = {\downarrow}$ Church–Rosser property
- $\mathsf{CR} \implies \mathsf{UN}$

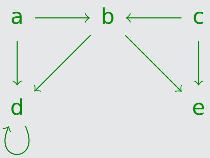


- $\mathsf{UN} \centernot\implies \mathsf{CR}$
- $\mathsf{WN}\ \&\ \mathsf{UN} \implies \mathsf{CR}$
- $\mathsf{CR} \implies \mathsf{WCR}$
- $\mathsf{WCR} \centernot\implies \mathsf{CR}$

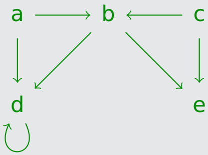


- $\mathsf{SN}\ \&\ \mathsf{WCR} \implies \mathsf{CR}$ Newman's Lemma

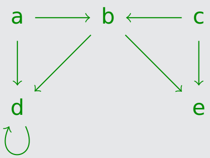
Examples

ARS	SN	WN	CR	WCR	UN
	?				

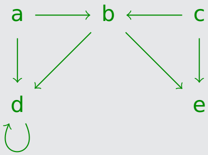
Examples

ARS	SN	WN	CR	WCR	UN
	🙁	?			

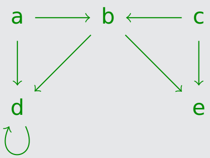
Examples

ARS	SN	WN	CR	WCR	UN
	🙁	🙁	?		

Examples

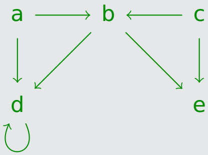
ARS	SN	WN	CR	WCR	UN
	🙁	🙁	🙁	?	

Examples

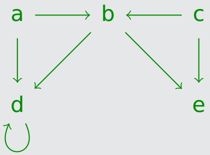
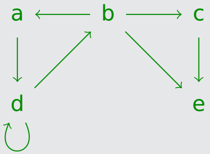


ARS	SN	WN	CR	WCR	UN
	🙁	🙁	🙁	🙁	?

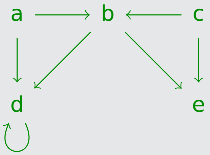
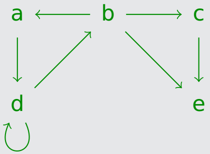
Examples

ARS	SN	WN	CR	WCR	UN
	🙁	🙁	🙁	🙁	🙂

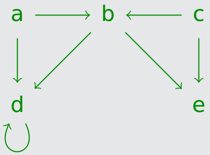
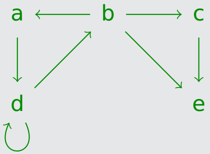
Examples

ARS	SN	WN	CR	WCR	UN
	🙁	🙁	🙁	🙁	🙂
	?				

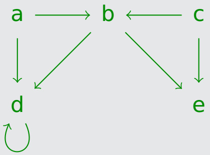
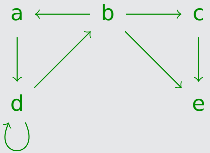
Examples

ARS	SN	WN	CR	WCR	UN
	🙁	🙁	🙁	🙁	🙂
	🙁	?			

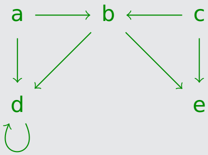
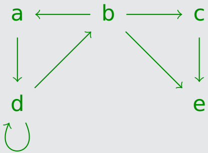
Examples

ARS	SN	WN	CR	WCR	UN
	🙁	🙁	🙁	🙁	🙂
	🙁	🙂	?		

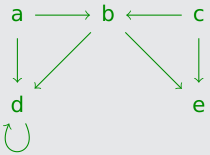
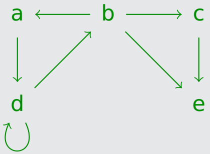
Examples

ARS	SN	WN	CR	WCR	UN
	🙁	🙁	🙁	🙁	🙂
	🙁	🙂	🙂	?	

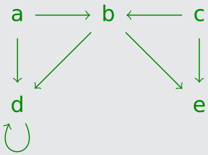
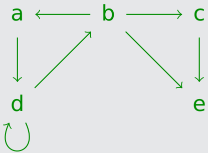
Examples

ARS	SN	WN	CR	WCR	UN
	🙁	🙁	🙁	🙁	🙂
	🙁	🙂	🙂	🙂	?

Examples

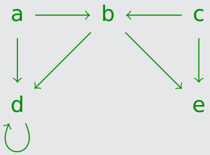
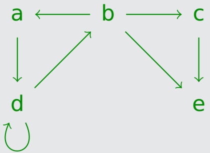
ARS	SN	WN	CR	WCR	UN
	🙁	🙁	🙁	🙁	🙂
	🙁	🙂	🙂	🙂	🙂

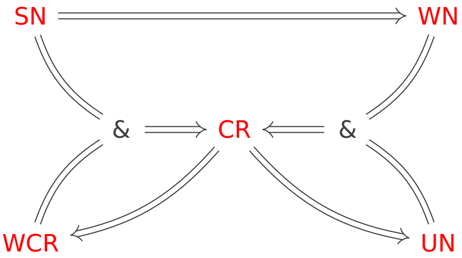
Examples

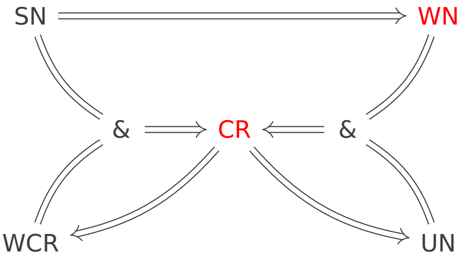
ARS	SN	WN	CR	WCR	UN
	🙁	🙁	🙁	🙁	🙂
	🙁	🙂	🙂	🙂	🙂

$\langle \mathbb{N}, \to \rangle$ with $\to\ = \{(0,2),\ (1,3)\} \cup \{(x^2, x),\ (x+3, x) \mid x \in \mathbb{N}\}$

Examples

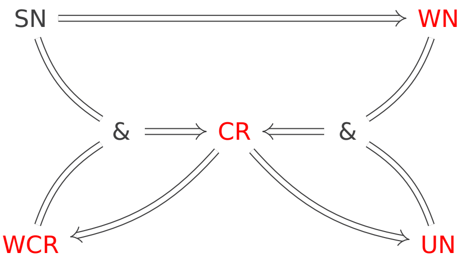
ARS	SN	WN	CR	WCR	UN
	🙁	🙁	🙁	🙁	🙂
	🙁	🙂	🙂	🙂	🙂
$\langle \mathbb{N}, \rightarrow \rangle$ with $\rightarrow \; = \{(0,2),\, (1,3)\} \cup \{(x^2, x),\, (x+3, x) \mid x \in \mathbb{N}\}$	?	?	?	?	?


SN
WN
&
CR
&
WCR
UN



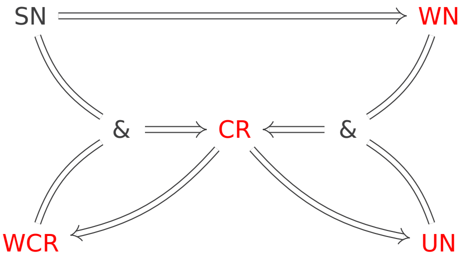
Definitions (Properties)

- semi–completeness
  - CR & WN



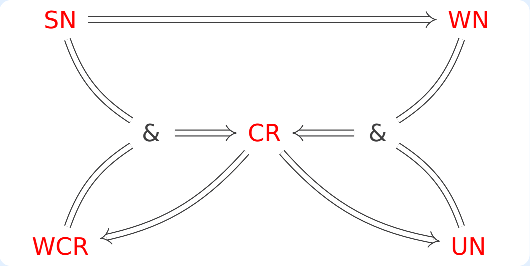
Definitions (Properties)

- semi–completeness
  - CR & WN



Definitions (Properties)

- semi–completeness
  - CR & WN
  - every element has unique normal form



Definitions (Properties)

- semi–completeness
  - CR & WN
  - every element has unique normal form
- completeness
  - CR & SN

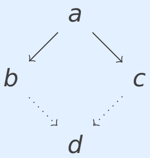
Definition (Diamond Property)

- diamond property
  - $\leftarrow \cdot \rightarrow \ \subseteq \ \rightarrow \cdot \leftarrow$

Definition (Diamond Property)

- diamond property
  - $\leftarrow \cdot \rightarrow \;\subseteq\; \rightarrow \cdot \leftarrow$
  - $\forall\, a, b, c$

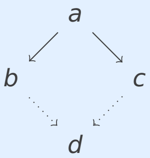
    $\exists\, d$



Definition (Diamond Property)

- diamond property $\diamond$
  - $\leftarrow \cdot \rightarrow \;\subseteq\; \rightarrow \cdot \leftarrow$
  - $\forall\, a, b, c$

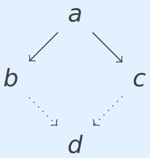
    $\exists\, d$



Definition (Diamond Property)

- diamond property $\diamond$
  - $\leftarrow \cdot \rightarrow \subseteq \rightarrow \cdot \leftarrow$
  - $\forall\, a, b, c$

    $\exists\, d$

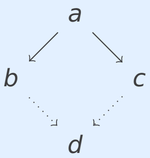


Lemmata

- every ARS with diamond property is confluent

Definition (Diamond Property)

- diamond property $\diamond$
  - $\leftarrow \cdot \rightarrow \;\subseteq\; \rightarrow \cdot \leftarrow$
  - $\forall\, a, b, c$

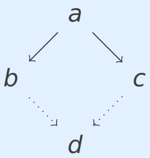
    $\exists\, d$

Lemmata

- every ARS with diamond property is confluent
- ARS $\langle A, \rightarrow \rangle$ is confluent $\iff$ ARS $\langle A, \rightarrowtail \rangle$ with $\rightarrowtail^* = \rightarrow^*$ is confluent

Definition (Diamond Property)

- diamond property $\diamond$
  - $\leftarrow \cdot \rightarrow \;\subseteq\; \rightarrow \cdot \leftarrow$
  - $\forall\, a, b, c$ $\exists\, d$



Lemmata

- every ARS with diamond property is confluent
- ARS $\langle A, \rightarrow \rangle$ is confluent $\iff$ ARS $\langle A, \rightarrowtail \rangle$ with $\rightarrowtail^* = \rightarrow^*$ is confluent

Corollary

ARS $\langle A, \rightarrow \rangle$ is confluent if $\rightarrow \;\subseteq\; \rightarrowtail \;\subseteq\; \rightarrow^*$ for some relation $\rightarrowtail$ with diamond property

Outline

Well–Founded Induction

given

- property P of ARSs that satisfies $\mathsf{P}(\mathcal{A}) \iff \forall\, a\ \ \mathsf{P}(a)$

Definitions (Properties of Elements)

ARS $\mathcal{A} = \langle A, \rightarrow \rangle$ and $a \in A$

- $\mathrm{CR}(a) \iff b \downarrow c$ whenever $b \; {}^*\!\!\leftarrow a \rightarrow^* c$

Definitions (Properties of Elements)

ARS $\mathcal{A} = \langle A, \to \rangle$ and $a \in A$

- $\mathrm{CR}(a) \iff b \downarrow c$ whenever $b \mathrel{{}^{*}{\leftarrow}} a \to^{*} c$
- $\mathrm{WCR}(a) \iff b \downarrow c$ whenever $b \leftarrow a \to c$

Definitions (Properties of Elements)

ARS $\mathcal{A} = \langle A, \to \rangle$ and $a \in A$

- $\mathrm{CR}(a) \quad \iff \quad b \downarrow c$ whenever $b \mathrel{{}^*{\leftarrow}} a \to^* c$
- $\mathrm{WCR}(a) \quad \iff \quad b \downarrow c$ whenever $b \leftarrow a \to c$
- $\mathrm{SN}(a) \quad \iff$ there are no infinite rewrite sequences starting from a

Definitions (Properties of Elements)

ARS $\mathcal{A} = \langle A, \to \rangle$ and $a \in A$

- $\mathrm{CR}(a) \iff b \downarrow c$ whenever $b \, {}^*\!\!\leftarrow a \to^* c$
- $\mathrm{WCR}(a) \iff b \downarrow c$ whenever $b \leftarrow a \to c$
- $\mathrm{SN}(a) \iff$ there are no infinite rewrite sequences starting from a
- $\mathrm{WN}(a) \iff a \to^! b$ for some b

Definitions (Properties of Elements)

ARS $\mathcal{A} = \langle A, \to \rangle$ and $a \in A$

- $\mathsf{CR}(a) \iff b \downarrow c$ whenever $b \;{}^*{\leftarrow}\; a \to^* c$
- $\mathsf{WCR}(a) \iff b \downarrow c$ whenever $b \leftarrow a \to c$
- $\mathsf{SN}(a) \iff$ there are no infinite rewrite sequences starting from a
- $\mathsf{WN}(a) \iff a \to^! b$ for some b
- $\mathsf{UN}(a) \iff b = c$ whenever $b \;{}^!{\leftarrow}\; a \to^! c$

Definitions (Properties of Elements)

ARS $\mathcal{A} = \langle A, \to \rangle$ and $a \in A$

- $\mathsf{CR}(a) \iff b \downarrow c$ whenever $b \; {}^{*}\!\!\leftarrow a \to^{*} c$
- $\mathsf{WCR}(a) \iff b \downarrow c$ whenever $b \leftarrow a \to c$
- $\mathsf{SN}(a) \iff$ there are no infinite rewrite sequences starting from a
- $\mathsf{WN}(a) \iff a \to^{!} b$ for some b
- $\mathsf{UN}(a) \iff b = c$ whenever $b \; {}^{!}\!\!\leftarrow a \to^{!} c$

Lemma

ARS $\mathcal{A} = \langle A, \to \rangle$

- $\mathsf{CR}(\mathcal{A}) \iff \forall a \in A \quad \mathsf{CR}(a)$
- $\cdots$

Well–Founded Induction

given

- property P of ARSs that satisfies $\mathsf{P}(\mathcal{A}) \iff \forall\, a \;\; \mathsf{P}(a)$
- terminating ARS $\mathcal{A} = \langle A, \to \rangle$

Well–Founded Induction

given

- property P of ARSs that satisfies $\mathsf{P}(\mathcal{A}) \iff \forall\, a\ \mathsf{P}(a)$
- terminating ARS $\mathcal{A} = \langle A, \to \rangle$

to conclude

- $\mathsf{P}(\mathcal{A})$

Well–Founded Induction

given

- property P of ARSs that satisfies $\mathsf{P}(\mathcal{A}) \iff \forall\, a\ \mathsf{P}(a)$
- terminating ARS $\mathcal{A} = \langle A, \to \rangle$

to conclude

- $\mathsf{P}(\mathcal{A})$

it is sufficient to prove

- if $\underbrace{\mathsf{P}(b) \text{ for every } b \text{ with } a \to b}_{\text{induction hypothesis}}$ then $\mathsf{P}(a)$

for arbitrary element $a \in A$

Well–Founded Induction

given

- property P of ARSs that satisfies $\mathsf{P}(\mathcal{A}) \iff \forall\, a\ \mathsf{P}(a)$
- terminating ARS $\mathcal{A} = \langle A, \to \rangle$

to conclude

- $\mathsf{P}(\mathcal{A})$

it is sufficient to prove

- if $\underbrace{\mathsf{P}(b) \text{ for every } b \text{ with } a \to b}_{\text{induction hypothesis}}$ then $\mathsf{P}(a)$

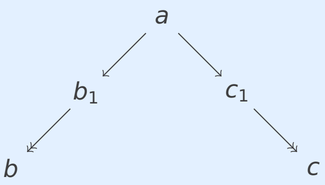
for arbitrary element $a \in A$

$$\Big(\forall\, a\ \big(\forall\, b\ (a \to b \implies \mathsf{P}(b))\big) \implies \mathsf{P}(a)\Big) \implies \forall\, a\ \mathsf{P}(a)$$

Newman's Lemma

SN & WCR $\implies$ CR

First Proof



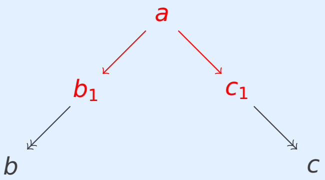
induction hypothesis

$\forall\, a' \quad \text{if } a \to a' \text{ then } \mathsf{CR}(a')$

Newman's Lemma

SN & WCR $\Longrightarrow$ CR

First Proof



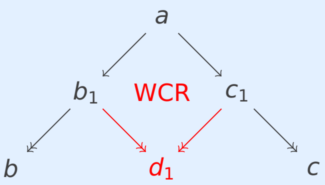
induction hypothesis

$\forall\, a' \quad \text{if } a \to a' \text{ then } \mathrm{CR}(a')$

Newman's Lemma

SN & WCR $\Longrightarrow$ CR

First Proof



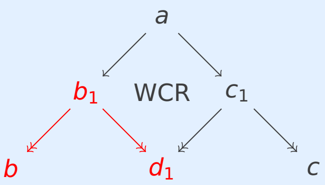
induction hypothesis

$\forall\, a' \quad \text{if } a \to a' \text{ then } \mathsf{CR}(a')$

Newman's Lemma

SN & WCR $\Longrightarrow$ CR

First Proof



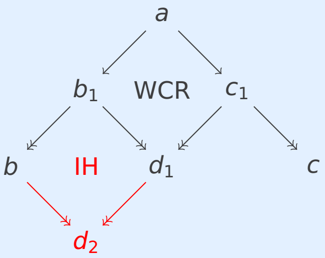
induction hypothesis

$\forall\, a' \quad \text{if } a \to a' \text{ then } \mathrm{CR}(a')$

Newman's Lemma

SN & WCR $\Longrightarrow$ CR

First Proof



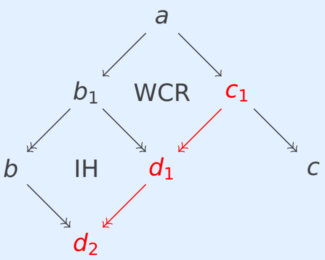
induction hypothesis CR(b_1)

$\forall\, a'$ if $a \to a'$ then CR(a')

Newman's Lemma

SN & WCR $\Longrightarrow$ CR

First Proof



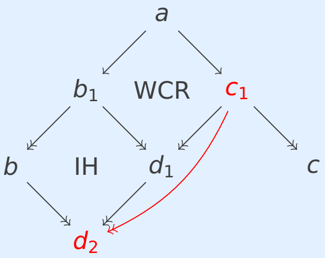
induction hypothesis

$\forall\, a' \quad \text{if } a \to a' \text{ then } \mathsf{CR}(a')$

Newman's Lemma

SN & WCR $\Longrightarrow$ CR

First Proof



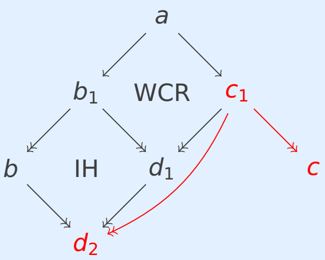
induction hypothesis

$\forall\, a' \quad \text{if } a \to a' \text{ then } \mathrm{CR}(a')$

Newman's Lemma

SN & WCR $\Longrightarrow$ CR

First Proof



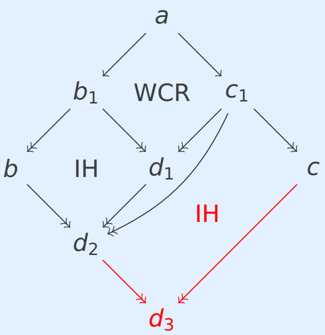
induction hypothesis

$\forall\, a' \quad \text{if } a \to a' \text{ then } \mathrm{CR}(a')$

Newman's Lemma

SN & WCR $\Longrightarrow$ CR

First Proof



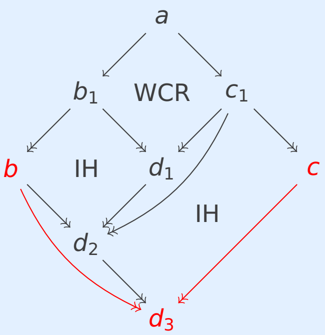
induction hypothesis $\mathsf{CR}(c_1)$

$\forall\, a'$ if $a \to a'$ then $\mathsf{CR}(a')$

Newman's Lemma

SN & WCR $\implies$ CR

First Proof



induction hypothesis

$\forall\, a' \quad \text{if } a \to a' \text{ then } \mathrm{CR}(a')$

Newman's Lemma

SN & WCR $\Longrightarrow$ CR

Second Proof

- it suffices to show UN because SN $\Longrightarrow$ WN and WN & UN $\Longrightarrow$ CR

Newman's Lemma

SN & WCR $\Longrightarrow$ CR

Second Proof

- it suffices to show UN
- suppose $B = \{a \in A \mid \neg \mathrm{UN}(a)\} \neq \varnothing$

Newman's Lemma

SN & WCR $\Longrightarrow$ CR

Second Proof

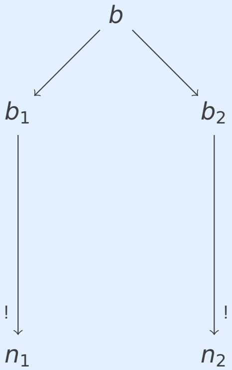
- it suffices to show UN
- suppose $B = \{a \in A \mid \neg\mathrm{UN}(a)\} \neq \varnothing$
- let $b \in B$ be minimal element (with respect to $\to$)

Newman's Lemma

SN & WCR $\Longrightarrow$ CR

Second Proof

- it suffices to show UN
- suppose $B = \{a \in A \mid \neg\mathrm{UN}(a)\} \neq \varnothing$
- let $b \in B$ be minimal element (with respect to $\rightarrow$)
- $b \rightarrow^! n_1$ and $b \rightarrow^! n_2$ with $n_1 \neq n_2$

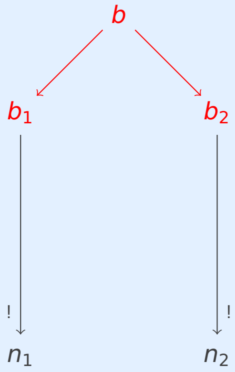


Newman's Lemma

SN & WCR $\Longrightarrow$ CR

Second Proof

- it suffices to show UN
- suppose $B = \{a \in A \mid \neg\mathrm{UN}(a)\} \neq \varnothing$
- let $b \in B$ be minimal element (with respect to $\rightarrow$)
- $b \rightarrow^! n_1$ and $b \rightarrow^! n_2$ with $n_1 \neq n_2$

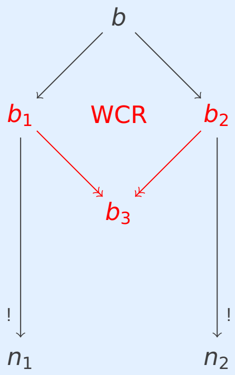


Newman's Lemma

SN & WCR $\implies$ CR

Second Proof

- it suffices to show UN
- suppose $B = \{a \in A \mid \neg\mathrm{UN}(a)\} \neq \varnothing$
- let $b \in B$ be minimal element (with respect to $\to$)
- $b \to^! n_1$ and $b \to^! n_2$ with $n_1 \neq n_2$

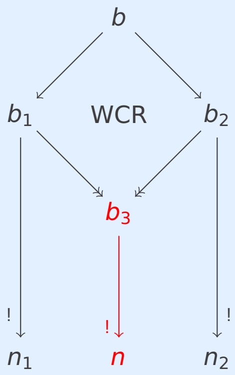


Newman's Lemma

SN & WCR $\Longrightarrow$ CR

Second Proof

- it suffices to show UN
- suppose $B = \{a \in A \mid \neg \mathrm{UN}(a)\} \neq \varnothing$
- let $b \in B$ be minimal element (with respect to $\rightarrow$)
- $b \rightarrow^! n_1$ and $b \rightarrow^! n_2$ with $n_1 \neq n_2$
- SN $\Longrightarrow$ $\exists\, n \;\; b_3 \rightarrow^! n$

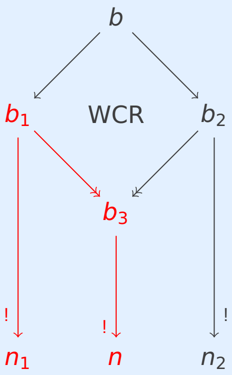


Newman's Lemma

SN & WCR $\Longrightarrow$ CR

Second Proof

- it suffices to show UN
- suppose $B = \{a \in A \mid \neg \text{UN}(a)\} \neq \varnothing$
- let $b \in B$ be minimal element (with respect to $\rightarrow$)
- $b \rightarrow^! n_1$ and $b \rightarrow^! n_2$ with $n_1 \neq n_2$
- SN $\Longrightarrow$ $\exists\, n\ \ b_3 \rightarrow^! n$

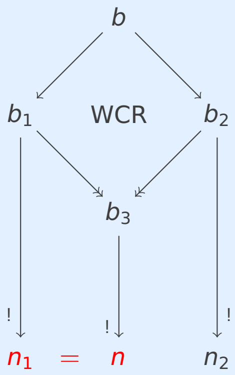


Newman's Lemma

SN & WCR $\implies$ CR

Second Proof

- it suffices to show UN
- suppose $B = \{a \in A \mid \neg\mathrm{UN}(a)\} \neq \varnothing$
- let $b \in B$ be minimal element (with respect to $\to$)
- $b \to^! n_1$ and $b \to^! n_2$ with $n_1 \neq n_2$
- SN $\implies$ $\exists\, n \;\; b_3 \to^! n$
- $b_1 \notin B \implies \mathrm{UN}(b_1) \implies n_1 = n$

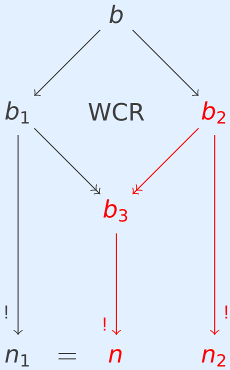


Newman's Lemma

SN & WCR $\implies$ CR

Second Proof

- it suffices to show UN
- suppose $B = \{a \in A \mid \neg\mathrm{UN}(a)\} \neq \varnothing$
- let $b \in B$ be minimal element (with respect to $\to$)
- $b \to^! n_1$ and $b \to^! n_2$ with $n_1 \neq n_2$
- SN $\implies$ $\exists n\ b_3 \to^! n$
- $b_1 \notin B \implies \mathrm{UN}(b_1) \implies n_1 = n$

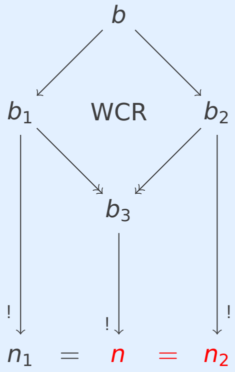


Newman's Lemma

SN & WCR $\Longrightarrow$ CR

Second Proof

- it suffices to show UN
- suppose $B = \{a \in A \mid \neg\mathrm{UN}(a)\} \neq \varnothing$
- let $b \in B$ be minimal element (with respect to $\rightarrow$)
- $b \rightarrow^! n_1$ and $b \rightarrow^! n_2$ with $n_1 \neq n_2$
- SN $\Longrightarrow$ $\exists\, n\;\; b_3 \rightarrow^! n$
- $b_1 \notin B \Longrightarrow \mathrm{UN}(b_1) \Longrightarrow n_1 = n$
- $b_2 \notin B \Longrightarrow \mathrm{UN}(b_2) \Longrightarrow n = n_2$

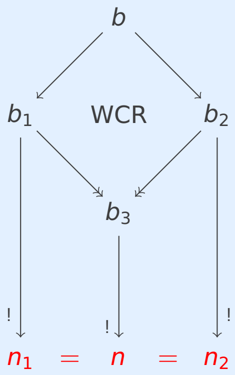


Newman's Lemma

SN & WCR $\Longrightarrow$ CR

Second Proof

- it suffices to show UN
- suppose $B = \{a \in A \mid \neg\mathrm{UN}(a)\} \neq \varnothing$
- let $b \in B$ be minimal element (with respect to $\to$)
- $b \to^! n_1$ and $b \to^! n_2$ with $n_1 \neq n_2$
- SN $\Longrightarrow$ $\exists\, n \;\; b_3 \to^! n$
- $b_1 \notin B \Longrightarrow \mathrm{UN}(b_1) \Longrightarrow n_1 = n$
- $b_2 \notin B \Longrightarrow \mathrm{UN}(b_2) \Longrightarrow n = n_2$ ↯



Outline

Homework Exercises for March 16

1. Exercise 1.4. 2
2. Exercise 1.5. 2
3. Complete the table on slide 18. Motivate your answers. 2
4. Download FORT–s and execute the following commands

```
./fort-s -S "a 0 b 0 c 0" "UN & ~CR"
./fort-s -a "0 0" "WN & ~SN"
./fort-s -a "0 0 0 0" -r 4 "WCR & ~CR"
```

   and compare the output with the ARSs on slide 16. 1
5. Exercise 1.19. ☆☆

Outline

1. Summary of Lecture 1
2. Abstract Rewrite Systems
3. Newman's Lemma
4. Exercises
5. **Further Reading**

Lecture Notes

- Section 1.1
- Section 1.2 (until Lemma 1.2.13)
- Section 1.3
- Appendix A.1
- Appendix A.2

Lecture Notes

- Section 1.1
- Section 1.2 (until Lemma 1.2.13)
- Section 1.3
- Appendix A.1
- Appendix A.2

Important Concepts

- abstract rewrite system
- completeness
- confluence
- conversion
- diamond property
- joinability
- local confluence
- Newman's Lemma
- normal form
- normalization
- reduct
- semi−completeness
- termination
- unique normal forms
- well−founded induction