universität innsbruck

26S lecture 2

Term Rewriting

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Outline

1. **Summary of Lecture 1**
2. **Abstract Rewrite Systems**
3. **Newman's Lemma**
4. **Exercises**
5. **Further Reading**

Definitions

- $\mathcal{T}(\mathcal{F},\mathcal{V})$ is set of terms built from signature (function symbols with arities) $\mathcal{F}$ and infinitely many variables $\mathcal{V}$ with $\mathcal{F} \cap \mathcal{V} \neq \varnothing$
- context is term with one hole: element of $\mathcal{T}(\mathcal{F} \cup \{\Box\},\mathcal{V})$ that contains exactly one occurrence of special constant $\Box$
- $C[t]$ denotes result of replacing hole in context C by term t
- binary relation R on terms is closed under contexts if for all terms s, t

$$s \mathrel{R} t \quad \Longrightarrow \quad C[s] \mathrel{R} C[t] \quad \text{for all contexts } C$$

- substitution is mapping $\sigma\colon \mathcal{V} \to \mathcal{T}(\mathcal{F},\mathcal{V})$ such that its domain

$$\mathcal{D}\mathrm{om}(\sigma) = \{x \in \mathcal{V} \mid \sigma(x) \neq x\}$$

is finite

Definitions

- application of substitution σ to term t

$$t\sigma = \begin{cases} \sigma(t) & \text{if } t \in \mathcal{V} \\ f(t_1\sigma, \dots, t_n\sigma) & \text{if } t = f(t_1, \dots, t_n) \end{cases}$$

- binary relation R on terms is closed under substitutions if for all terms s, t

$$s \mathrel{R} t \quad \Longrightarrow \quad s\sigma \mathrel{R} t\sigma \quad \text{for all substitutions } \sigma$$

Theorem

matching problem

instance: terms s, t

question: $\exists$ substitution σ: $s\sigma = t$? (t is instance of s, $t \geqslant s$)

is decidable (in linear time)

Outline

Motivation

concrete rewrite formalisms

- string rewriting
- term rewriting
- graph rewriting
- λ–calculus
- interaction nets
- $\cdots$

abstract rewriting

- no structure on objects that are rewritten
- uniform presentation of properties and proofs

Outline

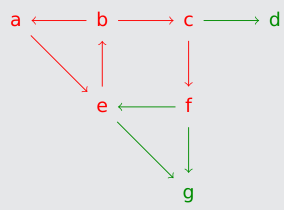
Definitions (Abstract Rewrite System)

- abstract rewrite system (ARS) is set A equipped with binary relation $\to$
- elements of $\to$ are called rewrite steps
- (finite) rewrite sequence is sequence of $(a_0, \ldots, a_n)$ such that $a_i \to a_{i+1}$ for all $0 \leqslant i < n$

Example

- ARS $\mathcal{A} = \langle A, \to \rangle$ with $A = \{\mathsf{a}, \mathsf{b}, \mathsf{c}, \mathsf{d}, \mathsf{e}, \mathsf{f}, \mathsf{g}\}$ and

$$\to \; = \left\{ \begin{matrix} (\mathsf{a},\mathsf{e}), (\mathsf{b},\mathsf{a}), (\mathsf{b},\mathsf{c}), (\mathsf{c},\mathsf{d}), (\mathsf{c},\mathsf{f}) \\ (\mathsf{e},\mathsf{b}), (\mathsf{e},\mathsf{g}), (\mathsf{f},\mathsf{e}), (\mathsf{f},\mathsf{g}) \end{matrix} \right\}$$



- rewrite sequences
  - finite $\mathsf{a} \to \mathsf{e} \to \mathsf{b} \to \mathsf{c} \to \mathsf{f}$
  - empty a
  - infinite $\mathsf{a} \to \mathsf{e} \to \mathsf{b} \to \mathsf{a} \to \mathsf{e} \to \mathsf{b} \to \cdots$

Definitions (Derived Relations)

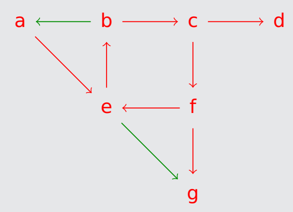
- $\leftarrow$ inverse of $\rightarrow$
- $\rightarrow^*$ transitive and reflexive closure of $\rightarrow$
- ${}^*\!\leftarrow$ inverse of $\rightarrow^*$
- $\downarrow$ joinability $\downarrow \,=\, \rightarrow^* \cdot {}^*\!\leftarrow$
- $\leftrightarrow$ symmetric closure of $\rightarrow$
- $\leftrightarrow^*$ conversion (equivalence relation generated by $\rightarrow$)
- $\rightarrow^+$ transitive closure of $\rightarrow$
- $\rightarrow^=$ reflexive closure of $\rightarrow$
- $\uparrow$ meetability $\uparrow \,=\, {}^*\!\leftarrow \cdot \rightarrow^*$

$\cdot$ denotes relation composition: $R \cdot S = \{(a,c) \mid a\ R\ b \text{ and } b\ S\ c\}$

Terminology

- if $x \rightarrow^* y$ then x rewrites to y and y is reduct of x
- if $x \rightarrow^* z\ {}^*\!\leftarrow y$ then z is common reduct of x and y
- if $x \leftrightarrow^* y$ then x and y are convertible

Example



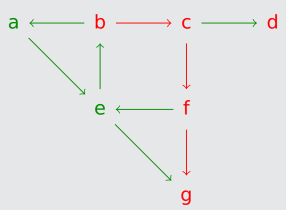
- $\mathsf{a} \rightarrow^* \mathsf{f}$
- $\mathsf{e} \downarrow \mathsf{f}$ $\quad \mathsf{f} \downarrow \mathsf{d}$ $\quad \mathsf{g} \not\downarrow \mathsf{d}$
- $\mathsf{g} \leftrightarrow^* \mathsf{d}$

Definitions (Normal Forms)

- normal form is element x such that $x \not\rightarrow y$ for all y
- $\mathrm{NF}(\mathcal{A})$ denotes set of normal forms of ARS $\mathcal{A}$
- $x \rightarrow^! y$ if $x \rightarrow^* y$ for normal form y (x has normal form y)

Example

- ARS $\mathcal{A} = \langle A, \rightarrow \rangle$
- d is normal form
- $\mathrm{NF}(\mathcal{A}) = \{\mathsf{d}, \mathsf{g}\}$
- $\mathsf{b} \rightarrow^! \mathsf{g}$



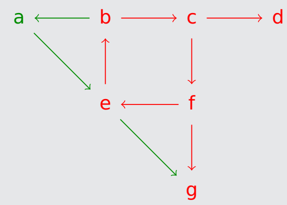
Outline

Definitions (Properties)

- SN strong normalization termination
  - no infinite rewrite sequences
- WN (weak) normalization
  - every element has at least one normal form
  - $\forall\, a\ \exists\, b\ \ a \to^! b$
- UN unique normal forms
  - no element has more than one normal form
  - $\forall\, a, b, c$ if $a \to^! b$ and $a \to^! c$ then $b = c$
  - ${}^!\!\leftarrow \cdot \to^! \subseteq =$

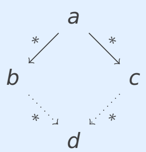
Example

- ARS $\mathcal{A} = \langle A, \to \rangle$
- not SN: b $\to^+$ b
- WN
- not UN: c $\to^!$ d and c $\to^!$ g
- not CR: d $\uparrow$ g and d $\not\downarrow$ g
- WCR



Definitions (Properties)

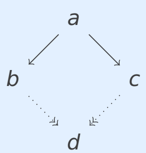
- CR confluence
  - $\uparrow \subseteq \downarrow$
  - $\forall\, a, b, c$ … $\exists\, d$



in diagrams: $\twoheadrightarrow$ for $\to^*$

Wikimedia

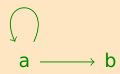
- WCR local confluence weak Church–Rosser property
  - $\leftarrow \cdot \to \subseteq \downarrow$
  - $\forall\, a, b, c$ … $\exists\, d$



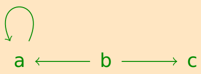
Outline

Lemmata

- SN $\Longrightarrow$ WN
- WN $\not\Longrightarrow$ SN



- CR $\Longleftrightarrow$ $\leftrightarrow^* \subseteq \downarrow$ $\Longleftrightarrow$ $\leftrightarrow^* = \downarrow$ Church–Rosser property
- CR $\Longrightarrow$ UN
- UN $\not\Longrightarrow$ CR



- WN & UN $\Longrightarrow$ CR
- CR $\Longrightarrow$ WCR
- WCR $\not\Longrightarrow$ CR

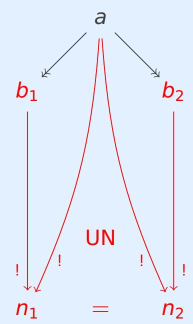
- SN & WCR $\Longrightarrow$ CR Newman's Lemma

Lemma

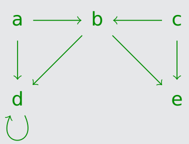
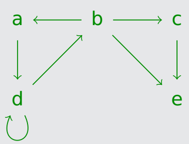
WN & UN $\Longrightarrow$ CR

Proof

- WN $\Longrightarrow$ $\exists\, n_1, n_2$ such that $b_1 \to^! n_1$ and $b_2 \to^! n_2$
- $\to^* \cdot \to^! \subseteq \to^!$
- UN $\Longrightarrow$ $n_1 = n_2$ $\Longrightarrow$ $b_1 \downarrow b_2$

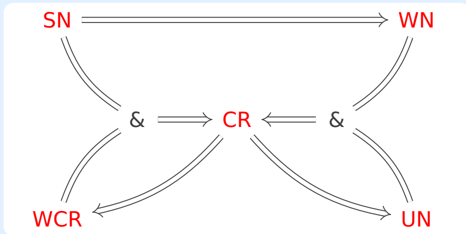


Examples

ARS	SN	WN	CR	WCR	UN
	🙁	🙁	🙁	🙁	🙂
	🙁	🙂	🙂	🙂	🙂

$\langle \mathbb{N}, \to \rangle$ with $\to\ = \{(0,2), (1,3)\} \cup \{(x^2, x), (x+3, x) \mid x \in \mathbb{N}\}$

SN	WN	CR	WCR	UN
?	?	?	?	?



Definitions (Properties)

- semi–completeness
  - CR & WN
  - every element has unique normal form
- completeness
  - CR & SN

Definition (Diamond Property)

- diamond property $\diamond$
  - $\leftarrow \cdot \rightarrow \;\subseteq\; \rightarrow \cdot \leftarrow$
  - $\forall\, a, b, c \quad \exists\, d$

$$\begin{array}{ccccc} & & a & & \\ & \swarrow & & \searrow & \\ b & & & & c \\ & \searrow & & \swarrow & \\ & & d & & \end{array}$$

Lemmata

- every ARS with diamond property is confluent
- ARS $\langle A, \rightarrow \rangle$ is confluent $\iff$ ARS $\langle A, \rightarrowtail \rangle$ with $\rightarrowtail^* = \rightarrow^*$ is confluent

Corollary

ARS $\langle A, \rightarrow \rangle$ is confluent if $\rightarrow \;\subseteq\; \rightarrowtail \;\subseteq\; \rightarrow^*$ for some relation $\rightarrowtail$ with diamond property

Outline

Well–Founded Induction

given

- property P of ARSs that satisfies $\quad \mathrm{P}(\mathcal{A}) \iff \forall\, a \;\; \mathrm{P}(a)$
- terminating ARS $\mathcal{A} = \langle A, \rightarrow \rangle$

to conclude

- $\mathrm{P}(\mathcal{A})$

it is sufficient to prove

- if $\underbrace{\mathrm{P}(b) \text{ for every } b \text{ with } a \rightarrow b}_{\text{induction hypothesis}}$ then $\mathrm{P}(a)$

for arbitrary element $a \in A$

$$\Big(\forall\, a\; \big(\forall\, b\; (a \rightarrow b \implies \mathrm{P}(b))\big) \implies \mathrm{P}(a)\Big) \implies \forall\, a \;\; \mathrm{P}(a)$$

Definitions (Properties of Elements)

ARS $\mathcal{A} = \langle A, \rightarrow \rangle$ and $a \in A$

- $\mathrm{CR}(a) \iff b \downarrow c$ whenever $b\; {}^*\!\!\leftarrow a \rightarrow^* c$
- $\mathrm{WCR}(a) \iff b \downarrow c$ whenever $b \leftarrow a \rightarrow c$
- $\mathrm{SN}(a) \iff$ there are no infinite rewrite sequences starting from a
- $\mathrm{WN}(a) \iff a \rightarrow^! b$ for some b
- $\mathrm{UN}(a) \iff b = c$ whenever $b\; {}^!\!\!\leftarrow a \rightarrow^! c$

Lemma

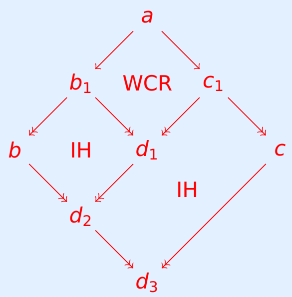
ARS $\mathcal{A} = \langle A, \rightarrow \rangle$

- $\mathrm{CR}(\mathcal{A}) \iff \forall\, a \in A \quad \mathrm{CR}(a)$
- ...

Newman's Lemma

SN & WCR $\Longrightarrow$ CR

First Proof



induction hypothesis $\mathrm{CR}(b_1)$ & $\mathrm{CR}(c_1)$

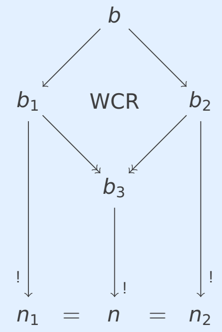
$\forall\, a'$ if $a \to a'$ then $\mathrm{CR}(a')$

Newman's Lemma

SN & WCR $\Longrightarrow$ CR

Second Proof

- it suffices to show UN because SN $\Longrightarrow$ WN and WN & UN $\Longrightarrow$ CR
- suppose $B = \{a \in A \mid \neg\mathrm{UN}(a)\} \neq \varnothing$
- let $b \in B$ be minimal element (with respect to $\to$)
- $b \to^! n_1$ and $b \to^! n_2$ with $n_1 \neq n_2$
- SN $\Longrightarrow$ $\exists\, n\ \ b_3 \to^! n$
- $b_1 \notin B$ $\Longrightarrow$ $\mathrm{UN}(b_1)$ $\Longrightarrow$ $n_1 = n$
- $b_2 \notin B$ $\Longrightarrow$ $\mathrm{UN}(b_2)$ $\Longrightarrow$ $n = n_2$ ↯



Outline

Homework Exercises for March 16

① Exercise 1.4. ❷

② Exercise 1.5. ❷

③ Complete the table on slide 18. Motivate your answers. ❷

④ Download FORT–s and execute the following commands

```
./fort-s -S "a 0 b 0 c 0" "UN & ~CR"
./fort-s -a "0 0" "WN & ~SN"
./fort-s -a "0 0 0 0" -r 4 "WCR & ~CR"
```

and compare the output with the ARSs on slide 16. ❶

⑤ Exercise 1.19. ☆☆

Outline

Lecture Notes

- Section 1.1
- Section 1.2 (until Lemma 1.2.13)
- Section 1.3
- Appendix A.1
- Appendix A.2

Important Concepts

- abstract rewrite system
- completeness
- confluence
- conversion
- diamond property
- joinability
- local confluence
- Newman's Lemma
- normal form
- normalization
- reduct
- semi−completeness
- termination
- unique normal forms
- well−founded induction