universität innsbruck

26S lecture 3

Term Rewriting

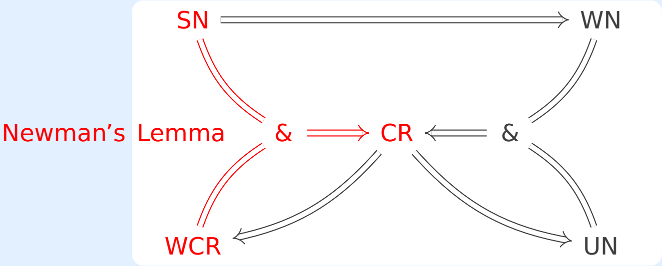
Philipp Dablander and **Aart Middeldorp**

Outline

1. **Summary of Lecture 2**
2. **Multiset Orders**
3. **Equational Reasoning**
4. **Algebras**
5. **Exercises**
6. **Further Reading**

Definitions

- abstract rewrite system (ARS) $\mathcal{A}$ is set A equipped with binary relation $\to$
- normal form is element $x \in A$ such that $x \not\to y$ for all $y \in A$
- ARS is terminating (SN) if there are no infinite rewrite sequences
- ARS is (weakly) normalizing (WN) if every element has normal form
- ARS is confluent (CR) if ${}^*{\leftarrow} \cdot \to^* \subseteq \to^* \cdot {}^*{\leftarrow}$ $\quad (\uparrow \subseteq \downarrow)$
- ARS is locally confluent (WCR) if $\leftarrow \cdot \to \subseteq \to^* \cdot {}^*{\leftarrow}$
- ARS has unique normal forms (UN) if ${}^!{\leftarrow} \cdot \to^! \subseteq =$
- ARS is complete if it is confluent and terminating
- ARS is semi–complete if it is confluent and normalizing
- ARS has diamond property ($\diamond$) if $\leftarrow \cdot \to \subseteq \to \cdot \leftarrow$



Lemma

ARS $\langle A, \to \rangle$ is confluent if $\to \subseteq \rightarrowtail \subseteq \to^*$ for some relation $\rightarrowtail$ on A with diamond property

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   Newman's Lemma
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Definitions (Multisets)

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$\{a, a, a, b, d, d, d\} \in \mathcal{M}(\{a, b, c, d\})$: $\quad a \mapsto 3 \quad b \mapsto 1 \quad c \mapsto 0 \quad d \mapsto 3$

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Lemma

multiset extension of proper order is proper order

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Proof (by contradiction)

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  into infinite finitely branching tree T

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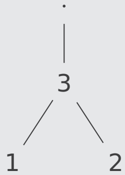

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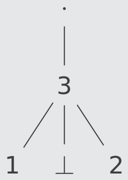

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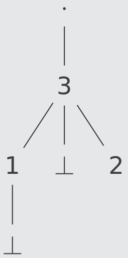

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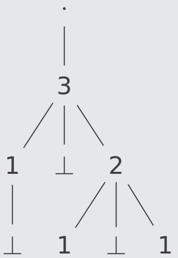

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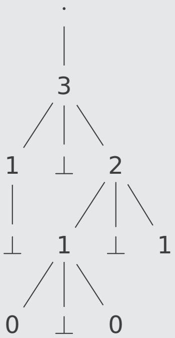

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Newman's Lemma

$\mathsf{SN} \;\&\; \mathsf{WCR} \quad\Longrightarrow\quad \mathsf{CR}$

Newman's Lemma

SN & WCR $\Longrightarrow$ CR

Definition (Conversion)

conversion is finite sequence $(a_1, \ldots, a_n)$ of elements such that

$$a_i \to a_{i+1} \qquad \text{or} \qquad a_i \leftarrow a_{i+1}$$

for all $1 \leqslant i < n$

Newman's Lemma

SN & WCR $\implies$ CR

Third Proof

- given $b \; {}^{*}\!\!\leftarrow a \rightarrow^{*} c$

Newman's Lemma

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Third Proof

- given $b \ {}^*\!\leftarrow a \rightarrow^* c$
- construct sequence of conversions $(C_i)_{i \geqslant 0}$ between b and c

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- given $b \ {}^*\!\!\leftarrow a \rightarrow^* c$
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- $M(C_i)$ is multiset of elements appearing in C_i
- $M(C_i) \ (\rightarrow^+)_{\mathsf{mul}} \ M(C_{i+1})$
- $(\rightarrow^+)_{\mathsf{mul}}$ is well–founded
- hence C_n has no peaks for some $n \geqslant 0$

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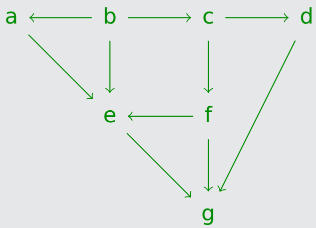
- $M(C_i) \ (\rightarrow^+)_{\text{mul}} \ M(C_{i+1})$
- $(\rightarrow^+)_{\text{mul}}$ is well–founded
- hence C_n has no peaks for some $n \geqslant 0 \quad \implies \quad C_n\!: \ b \rightarrow^* \cdot {}^*\!\leftarrow c$

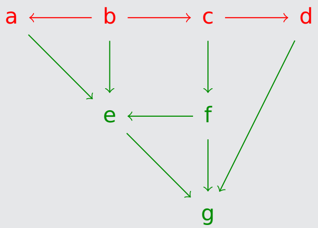
Example

- ARS



Example

- ARS



- conversion

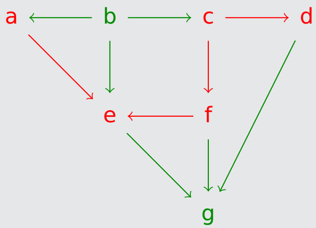
  $a \leftarrow b \rightarrow c \rightarrow d$

  multiset

  $\{a, b, c, d\}$

Example

- ARS

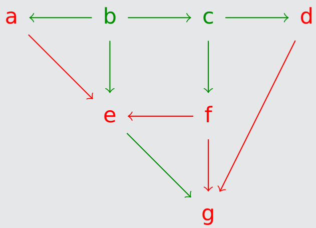


- conversion / multiset

conversion	multiset
$a \leftarrow b \rightarrow c \rightarrow d$	$\{a, b, c, d\}$
$a \rightarrow e \leftarrow f \leftarrow c \rightarrow d$	$\{a, e, f, c, d\}$

Example

- ARS



- conversion multiset

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$a \leftarrow b \rightarrow c \rightarrow d$	$\{a, b, c, d\}$
$a \rightarrow e \leftarrow f \leftarrow c \rightarrow d$	$\{a, e, f, c, d\}$
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Example

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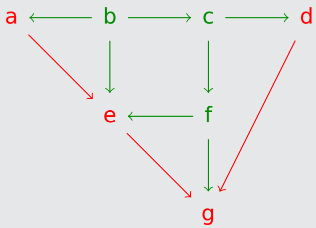

- conversion multiset

conversion	multiset	
$a \leftarrow b \rightarrow c \rightarrow d$	$\{a, b, c, d\}$	
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$a \rightarrow e \leftarrow f \rightarrow g \leftarrow d$	$\{a, e, f, g, d\}$	
$a \rightarrow e \rightarrow g \leftarrow d$	$\{a, e, g, d\}$	rewrite proof

Outline

1. Summary of Lecture 2
2. Multiset Orders
3. **Equational Reasoning**
4. Algebras
5. Exercises
6. Further Reading

Definition (Equational System)

equational system (ES) is pair $(\mathcal{F}, \mathcal{E})$ consisting of

- $\mathcal{F}$ signature
- $\mathcal{E}$ set of equations between terms in $\mathcal{T}(\mathcal{F}, \mathcal{V})$

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Example

ES $(\mathcal{F}, \mathcal{E})$ with signature $\mathcal{F}$

$$0 \ \text{(constant)} \qquad \mathsf{s} \ \text{(unary)} \qquad + \ \text{(binary, infix)}$$

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Notation

$\mathcal{E}$ instead of $(\mathcal{F}, \mathcal{E})$ if $\mathcal{F}$ can be inferred from context

Inference Rules of Equational Logic

r reflexivity $$\frac{}{t \approx t} \qquad \forall\ t$$

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Definition (Derivability)

$s \approx_{\mathcal{E}} t$ if equation $s \approx t$ is derivable from equations in $\mathcal{E}$

Example

ES $\mathcal{E}$

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- $\mathcal{A} = (\mathbb{N}, \{0_\mathcal{A}, \mathsf{s}_\mathcal{A}, +_\mathcal{A}\})$ with $0_\mathcal{A} = 0$ $\mathsf{s}_\mathcal{A}(x) = x + 1$ $+_\mathcal{A}(x, y) = x + y$

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Definition (Interpretation of Ground Terms)

interpretation function $[\,]_{\mathcal{A}}(\cdot)$: $\mathcal{T}(\mathcal{F}) \to A$

$$[\,]_{\mathcal{A}}(f(t_1, \ldots, t_n)) = f_{\mathcal{A}}([\,]_{\mathcal{A}}(t_1), \ldots, [\,]_{\mathcal{A}}(t_n))$$

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$$[\,]_{\mathcal{A}}(\mathsf{s}(\mathsf{s}(\mathsf{s}(\mathsf{s}(0))))) = 4$$

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- assignment $\alpha\colon \mathcal{V} \to A$
- interpretation function $[\alpha]_{\mathcal{A}}(\cdot) : \mathcal{T}(\mathcal{F},\mathcal{V}) \to A$

$$[\alpha]_{\mathcal{A}}(t) = \begin{cases} \alpha(t) & \text{if } t \in \mathcal{V} \\ f_{\mathcal{A}}([\alpha]_{\mathcal{A}}(t_1), \ldots, [\alpha]_{\mathcal{A}}(t_n)) & \text{if } t = f(t_1, \ldots, t_n) \end{cases}$$

Example

- $\mathcal{A} = (\mathbb{N}, \{0_{\mathcal{A}}, \mathsf{s}_{\mathcal{A}}, +_{\mathcal{A}}\})$ with $0_{\mathcal{A}} = 0$ $\quad \mathsf{s}_{\mathcal{A}}(x) = x + 1$ $\quad +_{\mathcal{A}}(x,y) = x + y$
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- $t = \mathsf{s}(\mathsf{s}(x) + \mathsf{s}(x + y))$ $\quad \alpha(x) = 2 \quad \alpha(y) = 3 \quad \beta(x) = 1 \quad \beta(y) = 4$

$$[\alpha]_{\mathcal{A}}(t) = 10 \qquad [\alpha]_{\mathcal{B}}(t) = 15 \qquad [\beta]_{\mathcal{A}}(t) = 9 \qquad [\beta]_{\mathcal{B}}(t) = 12$$

Definitions (Model)

- equation $s \approx t$ is **valid** in algebra $\mathcal{A}$ ($s =_{\mathcal{A}} t$) if

$$[\alpha]_{\mathcal{A}}(s) = [\alpha]_{\mathcal{A}}(t)$$

  for all assignments α

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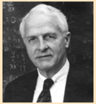
Theorem (Birkhoff)

equational reasoning is sound and complete



$$\forall \text{ ES } \mathcal{E} \quad \approx_{\mathcal{E}} \; = \; =_{\mathcal{E}}$$

Example (Cola Gene Puzzle)

$\mathcal{E}$ TCAT ≈ T GAG ≈ AG CTC ≈ TC AGTA ≈ A TAT ≈ CT

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Validity Problem

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Theorem

validity problem is undecidable for Combinatory Logic

Combinatory Logic

$$\mathsf{I} \cdot x \approx x \qquad (\mathsf{K} \cdot x) \cdot y \approx x \qquad ((\mathsf{S} \cdot x) \cdot y) \cdot z \approx (x \cdot z) \cdot (y \cdot z)$$

Outline

1. Summary of Lecture 2
2. Multiset Orders
3. Equational Reasoning
4. Algebras
5. **Exercises**
6. Further Reading

Homework Exercises for March 23

1. Exercise A.23. 1
2. Exercise A.29. 1
3. Exercise 2.17. 1
4. Exercise 2.29. 2
5. Exercise 2.30. 2

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Lecture Notes

- Section A.3
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Important Concepts

- algebra
- equation
- equational reasoning
- equational system (ES)
- equational theory
- model
- multiset
- multiset order
- validity problem