universität innsbruck

Term Rewriting

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Outline

Definitions

- binary relation $\to_{\mathcal{E}}$ on $\mathcal{T}(\mathcal{F},\mathcal{V})$ for every ES $(\mathcal{F},\mathcal{E})$:

$$s \to_{\mathcal{E}} t \quad\Longleftrightarrow\quad \begin{array}{l} \exists\ p \in \mathcal{P}os(s) \\ \exists\ \ell \approx r \in \mathcal{E} \\ \exists\ \text{substitution } \sigma \end{array} \quad \text{with} \quad \begin{array}{ll} s|_p = \ell\sigma & \text{redex} \\ t = s[r\sigma]_p & \end{array}$$

- rewrite rule $\ell \to r$ is equation $\ell \approx r$ such that $\ell \notin \mathcal{V}$ and $\mathcal{V}ar(r) \subseteq \mathcal{V}ar(\ell)$
- term rewrite system (TRS) is pair $(\mathcal{F},\mathcal{R})$ consisting of
  - $\mathcal{F}$ signature
  - $\mathcal{R}$ set of rewrite rules between terms in $\mathcal{T}(\mathcal{F},\mathcal{V})$

Theorem

$\forall$ ES $\mathcal{E}$ $\quad \leftrightarrow^*_{\mathcal{E}} \;=\; \approx_{\mathcal{E}} \;=\; =_{\mathcal{E}}$

Theorem

validity problem is decidable for ES $\mathcal{E}$ if there exists finite TRS $\mathcal{R}$ such that

1. $\mathcal{R}$ is complete (confluent and terminating)
2. $\leftrightarrow_{\mathcal{E}}^{*} = \leftrightarrow_{\mathcal{R}}^{*}$

Theorem

term rewriting is Turing–complete hence all non–trivial questions are undecidable

Undecidable Problems

instance: (finite) TRS $\mathcal{R}$
question: is $\mathcal{R}$ terminating ?

instance: TRS $\mathcal{R}$
question: is $\mathcal{R}$ confluent ?

instance: TRS $\mathcal{R}$, term t
question: is t terminating ?

...

Outline

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most problems for finite left−linear right−ground TRSs are decidable

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- rewrite rule $\ell \to r$ is left–linear if ℓ is linear
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- rewrite rule $\ell \to r$ is right–ground if r is ground

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Theorem

validity problem is decidable for finite ground ESs congruence closure

Congruence Closure

instance: ground ES $\mathcal{E}$, equation $s \approx t$

question: valid ($s =_{\mathcal{E}} t$) or invalid ($s \neq_{\mathcal{E}} t$) ?

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- **a** put all subterms of terms in $\mathcal{E} \cup \{s \approx t\}$ in separate sets
- **b** merge sets $\{\dots, t_1, \dots\}$ and $\{\dots, t_2, \dots\}$ for all $t_1 \approx t_2$ in $\mathcal{E}$

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- **c** merge sets $\{\dots, f(t_1, \dots, t_n), \dots\}$ and $\{\dots, f(u_1, \dots, u_n), \dots\}$ if t_i and u_i belong to same set for all $1 \leqslant i \leqslant n$, repeatedly

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② if s and t belong to same set then return valid else return invalid

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Remark

(efficient) implementations use graphs for sharing subterms

Example

- ES $\mathcal{E}$

$$\mathsf{f(f(f(a)))} \approx \mathsf{g(f(g(f(b))))} \qquad \mathsf{f(g(f(b)))} \approx \mathsf{f(a)} \qquad \mathsf{g(g(b))} \approx \mathsf{g(f(a))} \qquad \mathsf{g(a)} \approx \mathsf{b}$$

terms $s = \mathsf{f(a)}$ and $t = \mathsf{g(a)}$

Example

- ES $\mathcal{E}$

$$f(f(f(a))) \approx g(f(g(f(b)))) \qquad f(g(f(b))) \approx f(a) \qquad g(g(b)) \approx g(f(a)) \qquad g(a) \approx b$$

  terms $s = f(a)$ and $t = g(a)$

- sets

  1. $\{a\}$
  2. $\{f(a)\}$
  3. $\{b\}$
  4. $\{g(b)\}$
  5. $\{f(f(a))\}$
  6. $\{f(f(f(a)))\}$
  7. $\{f(b)\}$
  8. $\{g(f(b))\}$
  9. $\{f(g(f(b)))\}$
  10. $\{g(f(g(f(b))))\}$
  11. $\{g(g(b))\}$
  12. $\{g(f(a))\}$
  13. $\{g(a)\}$

Example

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$$\mathsf{f}(\mathsf{f}(\mathsf{f}(\mathsf{a}))) \approx \mathsf{g}(\mathsf{f}(\mathsf{g}(\mathsf{f}(\mathsf{b})))) \qquad \mathsf{f}(\mathsf{g}(\mathsf{f}(\mathsf{b}))) \approx \mathsf{f}(\mathsf{a}) \qquad \mathsf{g}(\mathsf{g}(\mathsf{b})) \approx \mathsf{g}(\mathsf{f}(\mathsf{a})) \qquad \mathsf{g}(\mathsf{a}) \approx \mathsf{b}$$

  terms $s = \mathsf{f}(\mathsf{a})$ and $t = \mathsf{g}(\mathsf{a})$

- sets

1. $\{\mathsf{a}\}$
2. $\{\mathsf{f}(\mathsf{a})\}$
3. $\{\mathsf{b}\}$
4. $\{\mathsf{g}(\mathsf{b})\}$
5. $\{\mathsf{f}(\mathsf{f}(\mathsf{a}))\}$
6. $\{\mathsf{f}(\mathsf{f}(\mathsf{f}(\mathsf{a})))\}$
7. $\{\mathsf{f}(\mathsf{b})\}$
8. $\{\mathsf{g}(\mathsf{f}(\mathsf{b}))\}$
9. $\{\mathsf{f}(\mathsf{g}(\mathsf{f}(\mathsf{b})))\}$
10. $\{\mathsf{g}(\mathsf{f}(\mathsf{g}(\mathsf{f}(\mathsf{b}))))\}$
11. $\{\mathsf{g}(\mathsf{g}(\mathsf{b}))\}$
12. $\{\mathsf{g}(\mathsf{f}(\mathsf{a}))\}$
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5. $\{\mathsf{f}(\mathsf{f}(\mathsf{a}))\}$
6. $\{\mathsf{f}(\mathsf{f}(\mathsf{f}(\mathsf{a}))), \mathsf{g}(\mathsf{f}(\mathsf{g}(\mathsf{f}(\mathsf{b}))))\}$
7. $\{\mathsf{f}(\mathsf{b})\}$
8. $\{\mathsf{g}(\mathsf{f}(\mathsf{b}))\}$
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- conclusion: $s \neq_{\mathcal{E}} t$

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rewrite system is terminating if there are no infinite rewrite sequences

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Termination Methods 1967

Knuth–Bendix order

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Termination Methods 1975

Knuth–Bendix order

polynomial interpretations

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Termination Methods

1979

Knuth–Bendix order

multiset order

polynomial interpretations

simple path order

Definition

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Termination Methods 1980s

- Knuth–Bendix order
- multiset order
- polynomial interpretations
- recursive decomposition order
- recursive path order
- semantic path order
- simple path order
- transformation order

Definition

rewrite system is terminating if there are no infinite rewrite sequences

Termination Methods 1990s

dependency pairs

dummy elimination

elementary interpretations

freezing

general path order

Knuth−Bendix order

multiset order

polynomial interpretations

recursive decomposition order

recursive path order

semantic path order

simple path order

top−down labeling

transformation order

type introduction

well−founded monotone algebras

Definition

rewrite system is terminating if there are no infinite rewrite sequences

Termination Methods 2000s

arctic interpretations · bounded increase · context−dependent interpretations · dependency pairs · dummy elimination · elementary interpretations · freezing · general path order · increasing interpretations · Knuth−Bendix order · lexicographic path order · match−bounds · matrix interpretations · monotonic semantic path order · multiset order · multiset path order · ordinal interpretations · polynomial interpretations · predictive labeling · quasi−periodic interpretations · recursive decomposition order · recursive path order · root−labeling · semantic labeling · semantic path order · simple path order · size−change principle · top−down labeling · transformation order · type introduction · uncurrying · weighted path order · well−founded monotone algebras · ...

Definition

rewrite system is **terminating** if there are no infinite rewrite sequences

Termination Methods

dependency pairs

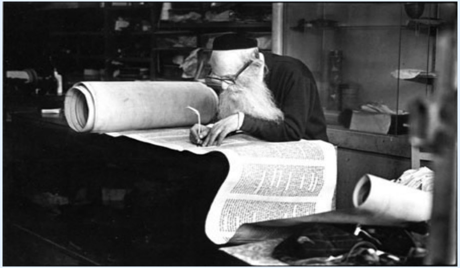
Knuth–Bendix order

lexicographic path order

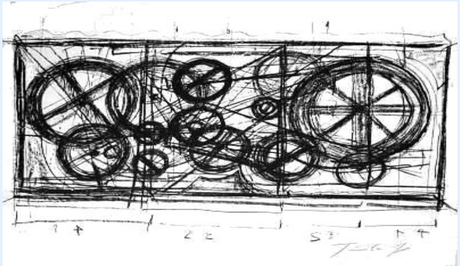
polynomial interpretations

well–founded monotone algebras

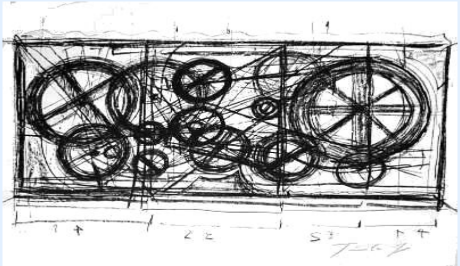
Termination Research



Termination Research



Termination Research



Termination Tools

AProVE | Matchbox | MultumNonMulta | MuTerm | NaTT | NTI | Torpa | $\mathsf{T_TT_2}$ | ...

Termination Competion

https://termination-portal.org/wiki/Termination_Competition

Outline

Lemma

TRS $\mathcal{R}$ is terminating $\iff$ $\exists$ well–founded order $>$ on terms such that $s \to_{\mathcal{R}} t \implies s > t$

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Example

- TRS

$$0 + y \to y \qquad \mathsf{s}(x) + y \to \mathsf{s}(x + y)$$

Lemma

TRS $\mathcal{R}$ is terminating $\iff$ $\exists$ well–founded order $>$ on terms such that

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Example

- TRS

$$0 + y \to y \qquad\qquad \mathsf{s}(x) + y \to \mathsf{s}(x + y)$$

- well–founded order $>$

$$s > t \iff \varphi(s) >_{\mathbb{N}} \varphi(t) \quad \text{with} \quad \varphi(u) = \begin{cases} 1 & \text{if } u = 0 \\ \varphi(v) + 1 & \text{if } u = \mathsf{s}(v) \\ 2\varphi(v) + \varphi(w) & \text{if } u = v + w \\ 0 & \text{otherwise} \end{cases}$$

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TRS $\mathcal{R}$ is terminating $\iff$ $\exists$ well–founded order $>$ on terms such that

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Remark

(very) inconvenient to check all rewrite steps

Definitions (Reduction Order)

- rewrite order is proper order $>$ on terms which is
  - closed under contexts: $s > t \implies C[s] > C[t]$ for all contexts C
  - closed under substitutions: $s > t \implies s\sigma > t\sigma$ for all substitutions σ

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- TRS $\mathcal{R}$ and rewrite order $>$ are compatible if $\ell > r$ for all rules $\ell \to r$ in $\mathcal{R}$

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- TRS $\mathcal{R}$ and rewrite order $>$ are *compatible* if $\ell > r$ for all rules $\ell \to r$ in $\mathcal{R}$
- *reduction order* is well-founded rewrite order

Notation

$\mathcal{R} \subseteq >$ if $\mathcal{R}$ and $>$ are compatible

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- TRS $\mathcal{R}$ and rewrite order $>$ are **compatible** if $\ell > r$ for all rules $\ell \to r$ in $\mathcal{R}$
- **reduction order** is well–founded rewrite order

Notation

$\mathcal{R} \subseteq >$ if $\mathcal{R}$ and $>$ are compatible

Theorem

TRS $\mathcal{R}$ is terminating $\iff$ $\mathcal{R} \subseteq >$ for reduction order $>$

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- $\to_{\mathcal{R}} \subseteq >$:
  - if $s \to_{\mathcal{R}} t$ then $s = C[\ell\sigma]$ and $t = C[r\sigma]$ for some $\ell \to r \in \mathcal{R}$, context C, substitution σ

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- $>$ is well–founded $\implies$ $\mathcal{R}$ is terminating

Definitions (Well–Founded Monotone Algebra)

- well–founded monotone $\mathcal{F}$–algebra $(\mathcal{A}, >)$ consists of non–empty algebra $\mathcal{A} = (A, \{f_{\mathcal{A}}\}_{f \in \mathcal{F}})$ with well–founded order $>$ on A such that every $f_{\mathcal{A}}$ is strictly monotone in all coordinates:

$$f_{\mathcal{A}}(a_1, \dots, a_i, \dots, a_n) > f_{\mathcal{A}}(a_1, \dots, b, \dots, a_n)$$

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used in termination proofs and tools:

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Outline

Example

- TRS

$$0 + y \to y \qquad \mathsf{s}(x) + y \to \mathsf{s}(x + y) \qquad 0 \times y \to 0 \qquad \mathsf{s}(x) \times y \to y + (x \times y)$$

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$$0_{\mathcal{A}} = 1 \qquad s_{\mathcal{A}}(x) = x + 1 \qquad +_{\mathcal{A}}(x, y) = 2x + y \qquad \times_{\mathcal{A}}(x, y) = 2xy + x + y + 1$$

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$$18 > 17 > 7 > 6 > 5 > 3$$

Example

- TRS

$$\partial(\alpha) \to 1 \qquad \partial(x + y) \to \partial(x) + \partial(y) \qquad \partial(x \times y) \to \partial(x) \times y + x \times \partial(y)$$

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- constraints $\forall\, x, y \in \mathbb{N}$

$$x^2+y^2+2xy+12x+12y+33 > x^2+y^2+6x+6y+15 \qquad 13 > 1$$
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$$x^2+y^2+2xy+12x+12y+33 > x^2+y^2+7x+7y+21$$
$$x^2+y^2+2xy+12x+12y+33 > x^2+y^2+7x+9y+27$$

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$$2xy + 6x + 6y + 18 > 0 \qquad 12 > 0$$
$$2xy + 6x + 6y + 18 > 0 \qquad 12 > 0$$
$$2xy + 5x + 5y + 12 > 0$$
$$2xy + 5x + 3y + 6 > 0$$

Definition (Polynomial Termination)

TRS $\mathcal{R}$ is polynomially terminating (over $\mathbb{N}$) if $\mathcal{R} \subseteq >_{\mathcal{A}}$ for some well−founded monotone algebra $(\mathcal{A}, >)$ such that

- carrier of $\mathcal{A}$ is $\mathbb{N}$
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Lemma

$\mathcal{R}$ is polynomially terminating over $\mathbb{N}$ $\iff$

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Notation

$\mathbb{N}_+ = \mathbb{N} \setminus \{0\}$

Example

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$$0 + y \to y \qquad 0 \times y \to 0$$

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- interpretations in $\mathbb{N}$

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- interpretations in $\mathbb{N} \setminus \{0, 1\}$

$$0_{\mathcal{A}} = 2 \qquad\qquad +_{\mathcal{A}}(x, y) = 2x + y$$

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Questions

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Theorem

following problem is undecidable:

instance: polynomial $P \in \mathbb{Z}[x_1, \dots, x_n]$

question: $\forall\, x_1, \dots, x_n \in \mathbb{N} \quad P(x_1, \dots, x_n) > 0$?

Proof

reduction from Hilbert's 10th Problem

instance: polynomial $P \in \mathbb{Z}[x_1, \ldots, x_n]$

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$$\iff \quad \neg \forall\, x_1, \ldots, x_n \in \mathbb{Z} \quad P(x_1, \ldots, x_n)^2 > 0$$

$$\iff \quad \neg \forall\, a_1, \ldots, a_n \in \{-1, 1\} \quad \forall\, y_1, \ldots, y_n \in \mathbb{N} \quad P(a_1 y_1, \ldots, a_n y_n)^2 > 0$$

Proof

reduction from Hilbert's 10th Problem

instance: polynomial $P \in \mathbb{Z}[x_1, \dots, x_n]$

question: $\exists\, x_1, \dots, x_n \in \mathbb{Z} \quad P(x_1, \dots, x_n) = 0$?

$$
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Questions

① how to find suitable polynomials ?

② how to show $P > 0$ for polynomial $P \in \mathbb{Z}[x_1, \ldots, x_n]$?

Theorem

following problem is undecidable:

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Sufficient Condition

all coefficients are non−negative and constant is positive

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Sufficient Condition

all coefficients are non−negative and constant is positive (absolute positiveness)

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Modern Approach

① choose abstract polynomial interpretations (linear, quadratic, . . .)

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① choose abstract polynomial interpretations (linear, quadratic, ...)

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⑤ translate resulting diophantine constraints to SAT or SMT problem

Example

- TRS

$$0 + y \to y \qquad\qquad \mathsf{s}(x) + y \to \mathsf{s}(x + y)$$

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$$0_{\mathcal{A}} = a \qquad \mathsf{s}_{\mathcal{A}}(x) = bx + c \qquad +_{\mathcal{A}}(x, y) = dx + ey + f$$

Example

- TRS

$$0 + y \to y \qquad\qquad s(x) + y \to s(x + y)$$

- interpretations

$$0_{\mathcal{A}} = a \qquad s_{\mathcal{A}}(x) = bx + c \qquad +_{\mathcal{A}}(x, y) = dx + ey + f$$

- polynomial constraints $\quad \forall\, x,\, y \in \mathbb{N}$

$$da + ey + f > y$$

$$d(bx + c) + ey + f > b(dx + ey + f) + c$$

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$$d(bx + c) + ey + f > b(dx + ey + f) + c$$

$$a \geqslant 0 \quad b \geqslant 1 \quad c \geqslant 0 \quad d \geqslant 1 \quad e \geqslant 1 \quad f \geqslant 0$$

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- polynomial constraints $\forall\, x, y \in \mathbb{N}$

$$(e - 1)y + da + f > 0$$

$$(e - be)y + dc + f - bf - c > 0$$

$$a \geqslant 0 \qquad b \geqslant 1 \qquad c \geqslant 0 \qquad d \geqslant 1 \qquad e \geqslant 1 \qquad f \geqslant 0$$

Example

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- diophantine constraints $\exists\ a, b, c, d, e, f \in \mathbb{N}$

$$e - 1 \geqslant 0 \quad da + f > 0$$

$$e - be \geqslant 0 \quad dc + f - bf - c > 0$$

absolute positiveness

$$a \geqslant 0 \quad b \geqslant 1 \quad c \geqslant 0 \quad d \geqslant 1 \quad e \geqslant 1 \quad f \geqslant 0$$

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$$0 + y \to y \qquad\qquad s(x) + y \to s(x + y)$$

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$$a \geqslant 0 \quad b \geqslant 1 \quad c \geqslant 0 \quad d \geqslant 1 \quad e \geqslant 1 \quad f \geqslant 0$$

- possible solution

$$a = 0 \quad b = 1 \quad c = 1 \quad d = 2 \quad e = 1 \quad f = 1$$

Example

- TRS

$$0 + y \to y \qquad \mathsf{s}(x) + y \to \mathsf{s}(x + y)$$

- interpretations

$$0_{\mathcal{A}} = 0 \qquad \mathsf{s}_{\mathcal{A}}(x) = x + 1 \qquad +_{\mathcal{A}}(x, y) = 2x + y + 1$$

- diophantine constraints $\exists\, a, b, c, d, e, f \in \mathbb{N}$

$$e - 1 \geqslant 0 \quad da + f > 0$$

$$e - be \geqslant 0 \quad dc + f - bf - c > 0$$

$$a \geqslant 0 \quad b \geqslant 1 \quad c \geqslant 0 \quad d \geqslant 1 \quad e \geqslant 1 \quad f \geqslant 0$$

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Remark

numerous terminating TRSs are not polynomially terminating

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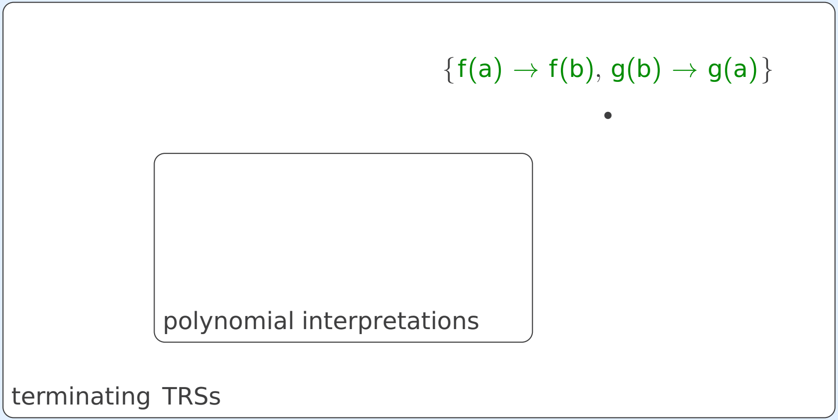

Example

- TRS

$$\mathsf{f}(\mathsf{a}) \to \mathsf{f}(\mathsf{b}) \qquad \mathsf{g}(\mathsf{b}) \to \mathsf{g}(\mathsf{a})$$

- well–founded monotone algebra $(\mathcal{A}, \sqsupset)$ with
  - carrier $A = \{0, 1\} \times \mathbb{N}$

Example

- TRS

$$\mathsf{f(a)} \to \mathsf{f(b)} \qquad\qquad \mathsf{g(b)} \to \mathsf{g(a)}$$

- well–founded monotone algebra $(\mathcal{A}, \sqsupset)$ with
  - carrier $A = \{0, 1\} \times \mathbb{N}$
  - well–founded order $\sqsupset$ on A: $\quad (a, x) \sqsupset (b, y) \quad\Longleftrightarrow\quad a = b$ and $x > y$

Example

- TRS

$$\mathsf{f}(\mathsf{a}) \to \mathsf{f}(\mathsf{b}) \qquad \mathsf{g}(\mathsf{b}) \to \mathsf{g}(\mathsf{a})$$

- well–founded monotone algebra $(\mathcal{A}, \sqsupset)$ with
  - carrier $A = \{0,1\} \times \mathbb{N}$
  - well–founded order $\sqsupset$ on A: $(a,x) \sqsupset (b,y) \iff a = b$ and $x > y$
  - strictly monotone interpretations $\forall\, x \in \mathbb{N}$

$$\begin{aligned}
\mathsf{a}_{\mathcal{A}} &= (0,0) & \mathsf{f}_{\mathcal{A}}((0,x)) &= (1,x+1) & \mathsf{g}_{\mathcal{A}}((0,x)) &= (1,x) \\
\mathsf{b}_{\mathcal{A}} &= (1,0) & \mathsf{f}_{\mathcal{A}}((1,x)) &= (1,x) & \mathsf{g}_{\mathcal{A}}((1,x)) &= (1,x+1)
\end{aligned}$$

Example

- TRS

$$\mathsf{f}(\mathsf{a}) \to \mathsf{f}(\mathsf{b}) \qquad\qquad \mathsf{g}(\mathsf{b}) \to \mathsf{g}(\mathsf{a})$$

- well–founded monotone algebra $(\mathcal{A}, \sqsupset)$ with
  - carrier $A = \{0, 1\} \times \mathbb{N}$
  - well–founded order $\sqsupset$ on A: $(a, x) \sqsupset (b, y) \iff a = b$ and $x > y$
  - strictly monotone interpretations $\forall\, x \in \mathbb{N}$

$$\mathsf{a}_{\mathcal{A}} = (0, 0) \qquad \mathsf{f}_{\mathcal{A}}((0, x)) = (1, x + 1) \qquad \mathsf{g}_{\mathcal{A}}((0, x)) = (1, x)$$

$$\mathsf{b}_{\mathcal{A}} = (1, 0) \qquad \mathsf{f}_{\mathcal{A}}((1, x)) = (1, x) \qquad \mathsf{g}_{\mathcal{A}}((1, x)) = (1, x + 1)$$

- constraints

$$\mathsf{f}_{\mathcal{A}}(\mathsf{a}_{\mathcal{A}}) = (1, 1) \sqsupset (1, 0) = \mathsf{f}_{\mathcal{A}}(\mathsf{b}_{\mathcal{A}}) \qquad \mathsf{g}_{\mathcal{A}}(\mathsf{b}_{\mathcal{A}}) = (1, 1) \sqsupset (1, 0) = \mathsf{g}_{\mathcal{A}}(\mathsf{a}_{\mathcal{A}})$$

Example

- TRS

$$0 + y \to y \qquad \mathsf{s}(x) + y \to \mathsf{s}(x + y) \qquad 0 \times y \to 0 \qquad \mathsf{s}(x) \times y \to y + (x \times y)$$

Example

- TRS

$$0 + y \to y \qquad \mathsf{s}(x) + y \to \mathsf{s}(x + y) \qquad 0 \times y \to 0 \qquad \mathsf{s}(x) \times y \to (x \times y) + y$$

- not polynomially terminating

Example

- TRS

$$0 + y \to y \qquad s(x) + y \to s(x + y) \qquad 0 \times y \to 0 \qquad s(x) \times y \to (x \times y) + y$$

- not polynomially terminating
- interpretations in $\mathbb{N}_+$

$$0_{\mathcal{A}} = 1 \qquad s_{\mathcal{A}}(x) = x + 1 \qquad +_{\mathcal{A}}(x, y) = 2x + y \qquad \times_{\mathcal{A}}(x, y) = 3^x y$$

Example

- TRS

$$0 + y \to y \qquad s(x) + y \to s(x + y) \qquad 0 \times y \to 0 \qquad s(x) \times y \to (x \times y) + y$$

- not polynomially terminating
- interpretations in $\mathbb{N}_+$

$$0_{\mathcal{A}} = 1 \qquad s_{\mathcal{A}}(x) = x + 1 \qquad +_{\mathcal{A}}(x, y) = 2x + y \qquad \times_{\mathcal{A}}(x, y) = 3^x y$$

- constraints $\forall\, x, y \in \mathbb{N}_+$

$$y + 2 > y \qquad 2x + y + 2 > 2x + y + 1 \qquad 3y > 1 \qquad 3^{x+1} y > 2(3^x y) + y$$

Example

- TRS

$$0 + y \to y \qquad \mathsf{s}(x) + y \to \mathsf{s}(x + y) \qquad 0 \times y \to 0 \qquad \mathsf{s}(x) \times y \to (x \times y) + y$$

- not polynomially terminating
- interpretations in $\mathbb{N} \setminus \{0, 1\}$

$$0_{\mathcal{A}} = 2 \qquad \mathsf{s}_{\mathcal{A}}(x) = x + 2 \qquad +_{\mathcal{A}}(x, y) = x + y \qquad \times_{\mathcal{A}}(x, y) = xy$$

Example

- TRS

$$0 + y \to y \qquad \mathsf{s}(x) + y \to \mathsf{s}(x + y) \qquad 0 \times y \to 0 \qquad \mathsf{s}(x) \times y \to (x \times y) + y$$

- not polynomially terminating
- interpretations in $\mathbb{N} \setminus \{0, 1\}$

$$0_{\mathcal{A}} = 2 \qquad \mathsf{s}_{\mathcal{A}}(x) = x + 2 \qquad +_{\mathcal{A}}(x, y) = x + y \qquad \times_{\mathcal{A}}(x, y) = xy$$

- constraints $\forall\, x, y \in \mathbb{N} \setminus \{0, 1\}$

$$y + 2 > y \qquad x + y + 2 > x + y + 2 \qquad 2y > 2 \qquad xy + 2y > xy + y$$

Example

- TRS

$$\mathsf{s}(x) + y \to \mathsf{s}(x + y)$$

- not polynomially terminating
- interpretations in $\mathbb{N} \setminus \{0, 1\}$

$$0_{\mathcal{A}} = 2 \qquad \mathsf{s}_{\mathcal{A}}(x) = x + 2 \qquad +_{\mathcal{A}}(x, y) = x + y \qquad \times_{\mathcal{A}}(x, y) = xy$$

- constraints $\forall\, x, y \in \mathbb{N} \setminus \{0, 1\}$

$$y + 2 > y \qquad x + y + 2 > x + y + 2 \qquad 2y > 2 \qquad xy + 2y > xy + y$$

Example

- TRS

$$\mathsf{s}(x) + y \to \mathsf{s}(x + y)$$

- not polynomially terminating
- interpretations in $\mathbb{N}$

$$\mathsf{s}_{\mathcal{A}}(x) = x + 1 \qquad +_{\mathcal{A}}(x, y) = 2x + y$$

- constraints $\forall\, x,\, y \in \mathbb{N}$

$$2x + y + 2 > 2x + y + 1$$

Example

- TRS
- not polynomially terminating but incremental polynomially terminating
- interpretations in $\mathbb{N}$

$$\mathsf{s}_{\mathcal{A}}(x) = x + 2 \qquad\qquad +_{\mathcal{A}}(x, y) = 2x + y$$

- constraints $\forall\, x,\, y \in \mathbb{N}$

$$2x + y + 2 > 2x + y + 1$$

Outline

Homework Exercises for April 20

1. Exercise 3.13. 1
2. Exercise 4.1. 3
3. Exercise 4.9. 1
4. Exercise 4.15. 2

Outline

1. **Summary of Lecture 4**
2. **Decidability**
3. **Termination**
4. **Well–Founded Monotone Algebras**
5. **Polynomial Interpretations over ℕ**
6. **Exercises**
7. **Further Reading**

Lecture Notes

- Section 3.2 (Theorem 3.2.5 and Example 3.2.6)
- Section 4.1

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- Section 3.2 (Theorem 3.2.5 and Example 3.2.6)
- Section 4.1

Important Concepts

- absolute positiveness
- congruence closure
- polynomial termination
- reduction order
- rewrite order
- well–founded monotone algebra