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Term Rewriting

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Outline

Definitions

- binary relation $\to_{\mathcal{E}}$ on $\mathcal{T}(\mathcal{F},\mathcal{V})$ for every ES $(\mathcal{F},\mathcal{E})$:

$$s \to_{\mathcal{E}} t \quad \Longleftrightarrow \quad \begin{array}{l} \exists\ p \in \mathcal{P}os(s) \\ \exists\ \ell \approx r \in \mathcal{E} \\ \exists\ \text{substitution } \sigma \end{array} \quad \text{with} \quad \begin{array}{ll} s|_p = \ell\sigma & \text{redex} \\ t = s[r\sigma]_p & \end{array}$$

- rewrite rule $\ell \to r$ is equation $\ell \approx r$ such that $\ell \notin \mathcal{V}$ and $\mathcal{V}ar(r) \subseteq \mathcal{V}ar(\ell)$
- term rewrite system (TRS) is pair $(\mathcal{F},\mathcal{R})$ consisting of
  - $\mathcal{F}$ signature
  - $\mathcal{R}$ set of rewrite rules between terms in $\mathcal{T}(\mathcal{F},\mathcal{V})$

Theorem

$\forall$ ES $\mathcal{E}$ $\quad \leftrightarrow^*_{\mathcal{E}} = \approx_{\mathcal{E}} = =_{\mathcal{E}}$

Theorem

validity problem is decidable for ES $\mathcal{E}$ if there exists finite TRS $\mathcal{R}$ such that

1. $\mathcal{R}$ is complete (confluent and terminating)
2. $\leftrightarrow^*_{\mathcal{E}} = \leftrightarrow^*_{\mathcal{R}}$

Theorem

term rewriting is Turing–complete hence all non–trivial questions are undecidable

Undecidable Problems

instance: (finite) TRS $\mathcal{R}$
question: is $\mathcal{R}$ terminating ?

instance: TRS $\mathcal{R}$
question: is $\mathcal{R}$ confluent ?

instance: TRS $\mathcal{R}$, term t
question: is t terminating ?

...

Outline

Theorem

most problems for finite left–linear right–ground TRSs are decidable (FORT, lecture 7)

Definitions

- rewrite rule $\ell \to r$ is left–linear if ℓ is linear
- TRS is left–linear if all rewrite rules are left–linear
- rewrite rule $\ell \to r$ is right–ground if r is ground
- rewrite rule $\ell \to r$ is ground if ℓ and r are ground
- TRS is (right–)ground if all rewrite rules are (right–)ground

Theorem

validity problem is decidable for finite ground ESs congruence closure

Congruence Closure

instance: ground ES $\mathcal{E}$, equation $s \approx t$

question: valid ($s =_{\mathcal{E}} t$) or invalid ($s \neq_{\mathcal{E}} t$) ?

① build congruence classes

a put all subterms of terms in $\mathcal{E} \cup \{s \approx t\}$ in separate sets

b merge sets $\{\dots, t_1, \dots\}$ and $\{\dots, t_2, \dots\}$ for all $t_1 \approx t_2$ in $\mathcal{E}$

c merge sets $\{\dots, f(t_1, \dots, t_n), \dots\}$ and $\{\dots, f(u_1, \dots, u_n), \dots\}$
if t_i and u_i belong to same set for all $1 \leqslant i \leqslant n$, repeatedly

② if s and t belong to same set then return valid else return invalid

Remark

(efficient) implementations use graphs for sharing subterms

Example

- ES $\mathcal{E}$

$$\mathsf{f}(\mathsf{f}(\mathsf{f}(\mathsf{a}))) \approx \mathsf{g}(\mathsf{f}(\mathsf{g}(\mathsf{f}(\mathsf{b})))) \qquad \mathsf{f}(\mathsf{g}(\mathsf{f}(\mathsf{b}))) \approx \mathsf{f}(\mathsf{a}) \qquad \mathsf{g}(\mathsf{g}(\mathsf{b})) \approx \mathsf{g}(\mathsf{f}(\mathsf{a})) \qquad \mathsf{g}(\mathsf{a}) \approx \mathsf{b}$$

  terms $s = \mathsf{f}(\mathsf{a})$ and $t = \mathsf{g}(\mathsf{a})$

- sets

  1. $\{\mathsf{a}\}$
  2. $\{\mathsf{f}(\mathsf{a}), \mathsf{f}(\mathsf{g}(\mathsf{f}(\mathsf{b})))\}$
  3. $\{\mathsf{b}, \mathsf{g}(\mathsf{a})\}$
  4. $\{\mathsf{g}(\mathsf{b})\}$
  5. $\{\mathsf{f}(\mathsf{f}(\mathsf{a}))\}$
  6. $\{\mathsf{f}(\mathsf{f}(\mathsf{f}(\mathsf{a}))), \mathsf{g}(\mathsf{f}(\mathsf{g}(\mathsf{f}(\mathsf{b})))), \mathsf{g}(\mathsf{g}(\mathsf{b})), \mathsf{g}(\mathsf{f}(\mathsf{a}))\}$
  7. $\{\mathsf{f}(\mathsf{b})\}$
  8. $\{\mathsf{g}(\mathsf{f}(\mathsf{b}))\}$

- conclusion: $s \neq_{\mathcal{E}} t$

Outline

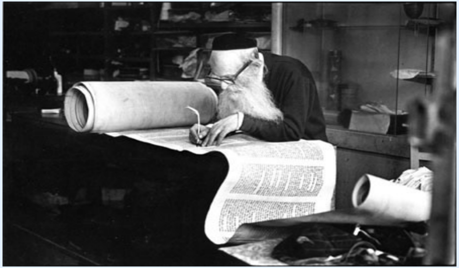
Definition

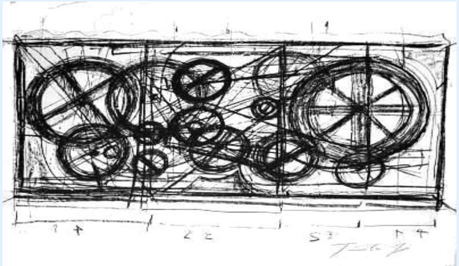
rewrite system is terminating if there are no infinite rewrite sequences

Termination Methods

arctic interpretations · bounded increase · context–dependent interpretations · dependency pairs · dummy elimination · elementary interpretations · freezing · general path order · increasing interpretations · Knuth–Bendix order · lexicographic path order · match–bounds · matrix interpretations · monotonic semantic path order · multiset order · multiset path order · ordinal interpretations · polynomial interpretations · predictive labeling · quasi–periodic interpretations · recursive decomposition order · recursive path order · root–labeling · semantic labeling · semantic path order · simple path order · size–change principle · top–down labeling · transformation order · type introduction · uncurrying · weighted path order · well–founded monotone algebras · ...

Termination Research





Termination Tools

AProVE · Matchbox · MultumNonMulta · MuTerm · NaTT · NTI · Torpa · TTT2 · ...

Termination Competion

https://termination-portal.org/wiki/Termination_Competition

Outline

Lemma

TRS $\mathcal{R}$ is terminating $\iff$ $\exists$ well–founded order $>$ on terms such that

$$s \to_{\mathcal{R}} t \implies s > t$$

Example

- TRS

$$0 + y \to y \qquad \mathsf{s}(x) + y \to \mathsf{s}(x + y)$$

- well–founded order $>$

$$s > t \iff \varphi(s) >_{\mathbb{N}} \varphi(t) \quad \text{with} \quad \varphi(u) = \begin{cases} 1 & \text{if } u = 0 \\ \varphi(v) + 1 & \text{if } u = \mathsf{s}(v) \\ 2\varphi(v) + \varphi(w) & \text{if } u = v + w \\ 0 & \text{otherwise} \end{cases}$$

Remark

(very) inconvenient to check all rewrite steps

Definitions (Reduction Order)

- rewrite order is proper order $>$ on terms which is
  - closed under contexts: $s > t \implies C[s] > C[t]$ for all contexts C
  - closed under substitutions: $s > t \implies s\sigma > t\sigma$ for all substitutions σ
- TRS $\mathcal{R}$ and rewrite order $>$ are compatible if $\ell > r$ for all rules $\ell \to r$ in $\mathcal{R}$
- reduction order is well–founded rewrite order

Notation

$\mathcal{R} \subseteq >$ if $\mathcal{R}$ and $>$ are compatible

Theorem

TRS $\mathcal{R}$ is terminating $\iff$ $\mathcal{R} \subseteq >$ for reduction order $>$

Theorem

TRS $\mathcal{R}$ is terminating $\iff$ $\mathcal{R} \subseteq >$ for some reduction order $>$

Proof ($\Longleftarrow$)

- $\to_{\mathcal{R}} \subseteq >$:
  - if $s \to_{\mathcal{R}} t$ then $s = C[\ell\sigma]$ and $t = C[r\sigma]$ for some $\ell \to r \in \mathcal{R}$, context C, substitution σ
  - $\ell > r$ by assumption
  - $>$ is closed under substitutions $\implies$ $\ell\sigma > r\sigma$
  - $>$ is closed under contexts $\implies$ $s = C[\ell\sigma] > C[r\sigma] = t$
- $>$ is well–founded $\implies$ $\mathcal{R}$ is terminating

Definitions (Well–Founded Monotone Algebra)

- **well–founded monotone $\mathcal{F}$–algebra** $(\mathcal{A}, >)$ consists of non–empty algebra $\mathcal{A} = (A, \{f_{\mathcal{A}}\}_{f \in \mathcal{F}})$ with well–founded order $>$ on A such that every $f_{\mathcal{A}}$ is strictly monotone in all coordinates:

$$f_{\mathcal{A}}(a_1, \ldots, a_i, \ldots, a_n) > f_{\mathcal{A}}(a_1, \ldots, b, \ldots, a_n)$$

  for all $a_1, \ldots, a_n, b \in A$ and $i \in \{1, \ldots, n\}$ with $a_i > b$
- relation $>_{\mathcal{A}}$ on terms: $s >_{\mathcal{A}} t$ if $[\alpha]_{\mathcal{A}}(s) > [\alpha]_{\mathcal{A}}(t)$ for all assignments α

Lemma

$>_{\mathcal{A}}$ is reduction order for every well–founded monotone algebra $(\mathcal{A}, >)$

Theorem

TRS $\mathcal{R}$ is terminating $\iff$ $\mathcal{R} \subseteq >_{\mathcal{A}}$ for some well–founded monotone algebra $(\mathcal{A}, >)$

Theorem

TRS $\mathcal{R}$ is terminating $\iff$ $\mathcal{R} \subseteq >_{\mathcal{A}}$ for some well–founded monotone algebra $(\mathcal{A}, >)$

Proof ($\Longrightarrow$)

- $\mathcal{F}$ is signature of $\mathcal{R}$
- term algebra $\bar{\mathcal{T}} = (\mathcal{T}(\mathcal{F},\mathcal{V}), \{f_{\bar{\mathcal{T}}}\}_{f \in \mathcal{F}})$ with $f_{\bar{\mathcal{T}}}(t_1, \dots, t_n) = f(t_1, \dots, t_n)$ for all n-ary $f \in \mathcal{F}$
- relation $>$ on $\mathcal{T}(\mathcal{F},\mathcal{V})$: $> \; = \; \to_{\mathcal{R}}^{+}$
- $(\bar{\mathcal{T}}, >)$ is well–founded monotone algebra:
  - $>$ is closed under contexts $\Longrightarrow$ every $f_{\bar{\mathcal{T}}}$ is strictly monotone in all coordinates
  - $\mathcal{R}$ is terminating $\Longrightarrow$ $>$ is well–founded order on $\mathcal{T}(\mathcal{F},\mathcal{V})$
- $\ell \to_{\mathcal{R}}^{+} r$ for every $\ell \to r \in \mathcal{R}$ $\Longrightarrow$ $\mathcal{R} \subseteq >$

Well–Founded Monotone Algebras

used in termination proofs and tools:

- polynomial interpretations over $\mathbb{N}$
- polynomial interpretations over $\mathbb{Q}$ and $\mathbb{R}$
- matrix interpretations over $\mathbb{N}$
- matrix interpretations over $\mathbb{N} \cup \{-\infty\}$
- ordinal interpretations
- $\cdots$

Outline

1. Summary of Lecture 4
2. Decidability
3. Termination
4. Well–Founded Monotone Algebras
5. **Polynomial Interpretations over $\mathbb{N}$**
6. Exercises
7. Further Reading

Example

- TRS

$$0 + y \to y \qquad s(x) + y \to s(x + y) \qquad 0 \times y \to 0 \qquad s(x) \times y \to y + (x \times y)$$

- interpretations in $\mathbb{N}$

$$0_{\mathcal{A}} = 1 \qquad s_{\mathcal{A}}(x) = x + 1 \qquad +_{\mathcal{A}}(x, y) = 2x + y \qquad \times_{\mathcal{A}}(x, y) = 2xy + x + y + 1$$

- constraints $\forall\ x, y \in \mathbb{N}$

$$2 > 0 \qquad 1 > 0 \qquad 3y + 1 > 0 \qquad 1 > 0$$

- $s(0) \times s(s(0)) \to s(s(0)) + (0 \times s(s(0))) \to s(s(0)) + 0 \to s(s(0) + 0) \to s(s(0 + 0)) \to s(s(0))$

$$18 > 17 > 7 > 6 > 5 > 3$$

Example

- TRS

$$\partial(\alpha) \to 1 \qquad \partial(x+y) \to \partial(x) + \partial(y) \qquad \partial(x \times y) \to \partial(x) \times y + x \times \partial(y)$$

$$\partial(\beta) \to 0 \qquad \partial(x-y) \to \partial(x) - \partial(y) \qquad \partial(x \div y) \to (\partial(x) \times y - x \times \partial(y)) \div (y \times y)$$

- interpretations in $\mathbb{N}$

$$\alpha_{\mathcal{A}} = \beta_{\mathcal{A}} = 0_{\mathcal{A}} = 1_{\mathcal{A}} = 1 \qquad \partial_{\mathcal{A}}(x) = x^2 + 6x + 6$$

$$+_{\mathcal{A}}(x,y) = -_{\mathcal{A}}(x,y) = \times_{\mathcal{A}}(x,y) = \div_{\mathcal{A}}(x,y) = x + y + 3$$

- constraints $\forall\, x, y \in \mathbb{N}$

$$2xy + 6x + 6y + 18 > 0 \qquad 12 > 0$$

$$2xy + 6x + 6y + 18 > 0 \qquad 12 > 0$$

$$2xy + 5x + 5y + 12 > 0$$

$$2xy + 5x + 3y + 6 > 0$$

Definition (Polynomial Termination)

TRS $\mathcal{R}$ is polynomially terminating (over $\mathbb{N}$) if $\mathcal{R} \subseteq >_{\mathcal{A}}$ for some well–founded monotone algebra $(\mathcal{A}, >)$ such that

- carrier of $\mathcal{A}$ is $\mathbb{N}$
- $>$ is standard order on $\mathbb{N}$
- $f_{\mathcal{A}} \in \mathbb{Z}[x_1, \dots, x_n]$ for every n-ary function symbol f

polynomials with coefficients in $\mathbb{Z}$ and indeterminates $x_1, \dots, x_n$

Lemma

$\mathcal{R}$ is polynomially terminating over $\mathbb{N}$ $\iff$
$\mathcal{R}$ is polynomially terminating over $\{n \in \mathbb{N} \mid n \geqslant N\}$ for some $N \geqslant 0$

Notation

$\mathbb{N}_+ = \mathbb{N} \setminus \{0\}$

Example

- TRS

$$0 + y \to y \qquad 0 \times y \to 0$$
$$\mathsf{s}(x) + y \to \mathsf{s}(x + y) \qquad \mathsf{s}(x) \times y \to y + (x \times y)$$

- interpretations in $\mathbb{N} \setminus \{0, 1\}$

$$0_{\mathcal{A}} = 2 \qquad +_{\mathcal{A}}(x, y) = 2x + y$$
$$\mathsf{s}_{\mathcal{A}}(x) = x + 3 \qquad \times_{\mathcal{A}}(x, y) = xy$$

Questions

① how to find suitable polynomials ?

② how to show $P > 0$ for polynomial $P \in \mathbb{Z}[x_1, \ldots, x_n]$?

Theorem

following problem is undecidable:

instance: polynomial $P \in \mathbb{Z}[x_1, \ldots, x_n]$

question: $\forall\, x_1, \ldots, x_n \in \mathbb{N} \quad P(x_1, \ldots, x_n) > 0$?

Sufficient Condition

all coefficients are non−negative and constant is positive (absolute positiveness)

Proof

reduction from Hilbert's 10th Problem

instance: polynomial $P \in \mathbb{Z}[x_1, \ldots, x_n]$

question: $\exists\, x_1, \ldots, x_n \in \mathbb{Z} \quad P(x_1, \ldots, x_n) = 0$?

$$\exists\, x_1, \ldots, x_n \in \mathbb{Z} \quad P(x_1, \ldots, x_n) = 0$$

$$\iff \quad \neg \forall\, x_1, \ldots, x_n \in \mathbb{Z} \quad P(x_1, \ldots, x_n) \neq 0$$

$$\iff \quad \neg \forall\, x_1, \ldots, x_n \in \mathbb{Z} \quad P(x_1, \ldots, x_n)^2 > 0$$

$$\iff \quad \neg \forall\, a_1, \ldots, a_n \in \{-1, 1\} \quad \forall\, y_1, \ldots, y_n \in \mathbb{N} \quad P(a_1 y_1, \ldots$$

$\in \mathbb{Z}[y_1, \ldots, y_n]$

$$\iff \quad \exists\, a_1, \ldots, a_n \in \{-1, 1\} \quad \neg \forall\, y_1, \ldots, y_n \in \mathbb{N} \quad P(a_1 y_1, \ldots, a_n y_n)^2 > 0$$

Questions

① how to find suitable polynomials ?

② how to show $P > 0$ for polynomial $P \in \mathbb{Z}[x_1, \ldots, x_n]$?

Theorem

polynomial termination over $\mathbb{N}$ is undecidable

Modern Approach

① choose abstract polynomial interpretations (linear, quadratic, ...)

② transform rewrite rules into polynomial ordering constraints

③ add monotonicity and well–definedness constraints

④ eliminate universally quantified variables using absolute positiveness

⑤ translate resulting diophantine constraints to SAT or SMT problem

Example

- TRS

$$0 + y \to y \qquad \mathsf{s}(x) + y \to \mathsf{s}(x + y)$$

- interpretations

$$0_{\mathcal{A}} = 0 \qquad \mathsf{s}_{\mathcal{A}}(x) = x + 1 \qquad +_{\mathcal{A}}(x, y) = 2x + y + 1$$

- diophantine constraints $\exists\, a, b, c, d, e, f \in \mathbb{N}$

$$e - 1 \geqslant 0 \quad da + f > 0$$

$$e - be \geqslant 0 \quad dc + f - bf - c > 0$$

absolute positiveness

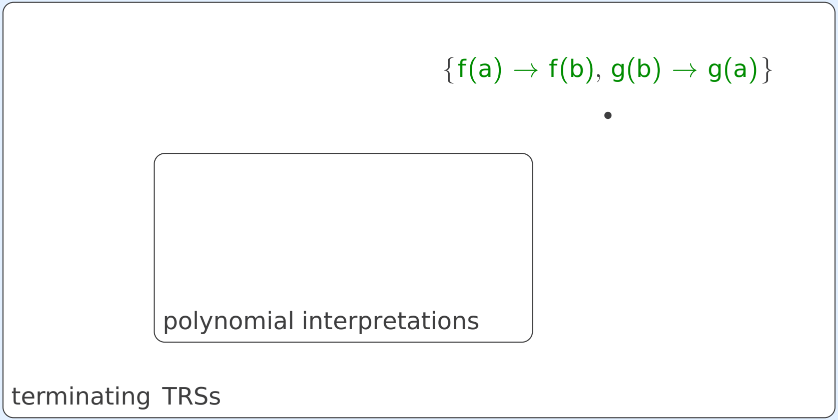
$$a \geqslant 0 \quad b \geqslant 1 \quad c \geqslant 0 \quad d \geqslant 1 \quad e \geqslant 1 \quad f \geqslant 0$$

- possible solution

$$a = 0 \quad b = 1 \quad c = 1 \quad d = 2 \quad e = 1 \quad f = 1$$

Remark

numerous terminating TRSs are not polynomially terminating



Example

- TRS

$$\mathsf{f(a)} \to \mathsf{f(b)} \qquad \mathsf{g(b)} \to \mathsf{g(a)}$$

- well–founded monotone algebra $(\mathcal{A}, \sqsupset)$ with
  - carrier $A = \{0,1\} \times \mathbb{N}$
  - well–founded order $\sqsupset$ on A: $(a,x) \sqsupset (b,y) \iff a = b$ and $x > y$
  - strictly monotone interpretations $\forall\, x \in \mathbb{N}$

$$\begin{aligned} \mathsf{a}_{\mathcal{A}} &= (0,0) & \mathsf{f}_{\mathcal{A}}((0,x)) &= (1,x+1) & \mathsf{g}_{\mathcal{A}}((0,x)) &= (1,x) \\ \mathsf{b}_{\mathcal{A}} &= (1,0) & \mathsf{f}_{\mathcal{A}}((1,x)) &= (1,x) & \mathsf{g}_{\mathcal{A}}((1,x)) &= (1,x+1) \end{aligned}$$

- constraints

$$\mathsf{f}_{\mathcal{A}}(\mathsf{a}_{\mathcal{A}}) = (1,1) \sqsupset (1,0) = \mathsf{f}_{\mathcal{A}}(\mathsf{b}_{\mathcal{A}}) \qquad \mathsf{g}_{\mathcal{A}}(\mathsf{b}_{\mathcal{A}}) = (1,1) \sqsupset (1,0) = \mathsf{g}_{\mathcal{A}}(\mathsf{a}_{\mathcal{A}})$$

Example

- TRS

$$0 + y \to y \qquad s(x) + y \to s(x + y) \qquad 0 \times y \to 0 \qquad s(x) \times y \to (x \times y) + y$$

- not polynomially terminating but incremental polynomially terminating
- interpretations in $\mathbb{N}_+$

$$0_{\mathcal{A}} = 1 \qquad s_{\mathcal{A}}(x) = x + 1 \qquad +_{\mathcal{A}}(x, y) = 2x + y \qquad \times_{\mathcal{A}}(x, y) = 3^x y$$

- constraints $\forall\, x, y \in \mathbb{N}_+$

$$y + 2 > y \qquad 2x + y + 2 > 2x + y + 1 \qquad 3y > 1 \qquad 3^{x+1} y > 2(3^x y) + y$$

Outline

Homework Exercises for April 20

1. Exercise 3.13. 1
2. Exercise 4.1. 3
3. Exercise 4.9. 1
4. Exercise 4.15. 2

Outline

1. **Summary of Lecture 4**
2. **Decidability**
3. **Termination**
4. **Well–Founded Monotone Algebras**
5. **Polynomial Interpretations over $\mathbb{N}$**
6. **Exercises**
7. **Further Reading**

Lecture Notes

- Section 3.2 (Theorem 3.2.5 and Example 3.2.6)
- Section 4.1

Important Concepts

- absolute positiveness
- congruence closure
- polynomial termination
- reduction order
- rewrite order
- well–founded monotone algebra