universität innsbruck

26S lecture 6

Term Rewriting

Philipp Dablander and **Aart Middeldorp**

Outline

1. **Summary of Lecture 5**
2. **Lexicographic Path Order**
3. **Unification**
4. **Critical Pairs**
5. **Exercises**
6. **Further Reading**

Theorem

validity problem for finite ground ESs is decidable congruence closure

Definitions

- rewrite order is proper order $>$ on terms which is
  - closed under contexts $s > t \implies C[s] > C[t]$ for all contexts C
  - closed under substitutions $s > t \implies s\sigma > t\sigma$ for all substitutions σ
- TRS $\mathcal{R}$ and rewrite order $>$ are compatible if $\ell > r$ for all rules $\ell \to r$ in $\mathcal{R}$
- reduction order is well–founded rewrite order

Theorem

TRS $\mathcal{R}$ is terminating $\iff$ $\mathcal{R} \subseteq >$ for some reduction order $>$

Definitions

- well–founded monotone $\mathcal{F}$–algebra $(\mathcal{A}, >)$ consists of non–empty algebra $\mathcal{A} = (A, \{f_\mathcal{A}\}_{f \in \mathcal{F}})$ with well–founded order $>$ on A such that every $f_\mathcal{A}$ is strictly monotone in all coordinates:

$$f_\mathcal{A}(a_1, \ldots, a_i, \ldots, a_n) > f_\mathcal{A}(a_1, \ldots, b, \ldots, a_n)$$

  for all $a_1, \ldots, a_n, b \in A$ and $i \in \{1, \ldots, n\}$ with $a_i > b$
- relation $>_\mathcal{A}$ on terms: $s >_\mathcal{A} t$ if $[\alpha]_\mathcal{A}(s) > [\alpha]_\mathcal{A}(t)$ for all assignments α

Lemma

$>_\mathcal{A}$ is reduction order for every well–founded monotone algebra $(\mathcal{A}, >)$

Theorem

TRS $\mathcal{R}$ is terminating $\iff$ $\mathcal{R} \subseteq >_\mathcal{A}$ for well–founded monotone algebra $(\mathcal{A}, >)$

Definition

TRS $\mathcal{R}$ is polynomially terminating (over $\mathbb{N}$) if $\mathcal{R} \subseteq >_{\mathcal{A}}$ for some well–founded monotone algebra $(\mathcal{A}, >)$ such that

- carrier of $\mathcal{A}$ is $\mathbb{N}$
- $>$ is standard order on $\mathbb{N}$
- $f_{\mathcal{A}} \in \mathbb{Z}[x_1, \dots, x_n]$ for every n-ary f

Lemma

$\mathcal{R}$ is polynomially terminating over $\mathbb{N}$ $\iff$

$\mathcal{R}$ is polynomially terminating over $\{n \in \mathbb{N} \mid n \geqslant N\}$ for some $N \geqslant 0$

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Definition (Lexicographic Path Order)

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    a $s_j = t_j$ for all $1 \leqslant j < i$

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② $t = g(t_1, \ldots, t_m)$ and $f > g$ and $s >_{\text{lpo}} t_j$ for all $1 \leqslant j \leqslant m$

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Theorem

$>_{\mathsf{lpo}}$ is reduction order if precedence $>$ is well−founded

Examples

① $$\begin{aligned} 0 + y &\to y & 0 \times y &\to 0 \\ \mathsf{s}(x) + y &\to \mathsf{s}(x + y) & \mathsf{s}(x) \times y &\to (x \times y) + y \end{aligned}$$

Examples

① $0 + y \to y \qquad 0 \times y \to 0 \qquad\qquad \times > + > \mathsf{s}$

$\mathsf{s}(x) + y \to \mathsf{s}(x + y) \qquad \mathsf{s}(x) \times y \to (x \times y) + y$

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② $$\begin{aligned} \mathsf{ack}(0, y) &\to \mathsf{s}(y) \\ \mathsf{ack}(\mathsf{s}(x), 0) &\to \mathsf{ack}(x, \mathsf{s}(0)) \\ \mathsf{ack}(\mathsf{s}(x), \mathsf{s}(y)) &\to \mathsf{ack}(x, \mathsf{ack}(\mathsf{s}(x), y)) \end{aligned}$$

Examples

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$$\mathsf{ack} > \mathsf{s}$$

③

$$\begin{aligned} \mathsf{e} \cdot x &\to x & x \cdot \mathsf{e} &\to x \\ x^{-} \cdot x &\to \mathsf{e} & x \cdot x^{-} &\to \mathsf{e} \\ (x \cdot y) \cdot z &\to x \cdot (y \cdot z) & x^{--} &\to x \\ \mathsf{e}^{-} &\to \mathsf{e} & (x \cdot y)^{-} &\to y^{-} \cdot x^{-} \\ x^{-} \cdot (x \cdot y) &\to y & x \cdot (x^{-} \cdot y) &\to y \end{aligned}$$

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Examples

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$$\begin{aligned} 0 + y &\to y & 0 \times y &\to 0 \\ \mathsf{s}(x) + y &\to \mathsf{s}(x + y) & \mathsf{s}(x) \times y &\to (x \times y) + y \end{aligned}$$

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Example

$$\dfrac{\mathsf{ack} > \mathsf{s} \qquad \overline{\mathsf{ack}(0,y) >_{\mathsf{lpo}} y}\;\text{③}}{\mathsf{ack}(0,y) >_{\mathsf{lpo}} \mathsf{s}(y)}\;\text{②}$$

$$\dfrac{\text{③}\;\overline{\mathsf{s}(x) >_{\mathsf{lpo}} x} \qquad \dfrac{\mathsf{ack} > \mathsf{s} \qquad \overline{\mathsf{ack}(\mathsf{s}(x),0) >_{\mathsf{lpo}} 0}\;\text{③}}{\mathsf{ack}(\mathsf{s}(x),0) >_{\mathsf{lpo}} \mathsf{s}(0)}\;\text{②}}{\mathsf{ack}(\mathsf{s}(x),0) >_{\mathsf{lpo}} \mathsf{ack}(x,\mathsf{s}(0))}\;\text{①}$$

$$\dfrac{\text{③}\;\overline{\mathsf{s}(x) >_{\mathsf{lpo}} x} \qquad \dfrac{\mathsf{s}(x) = \mathsf{s}(x) \qquad \overline{\mathsf{s}(y) >_{\mathsf{lpo}} y}\;\text{③}}{\mathsf{ack}(\mathsf{s}(x),\mathsf{s}(y)) >_{\mathsf{lpo}} \mathsf{ack}(\mathsf{s}(x),y)}\;\text{①}}{\mathsf{ack}(\mathsf{s}(x),\mathsf{s}(y)) >_{\mathsf{lpo}} \mathsf{ack}(x,\mathsf{ack}(\mathsf{s}(x),y))}\;\text{①}$$

Theorem

- if $> \subseteq \sqsupset$ then $>_{\mathsf{lpo}} \subseteq \sqsupset_{\mathsf{lpo}}$ (incrementality)

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- following two problems are decidable:

  ❶ instance: finite TRS $\mathcal{R}$, precedence $>$
  question: $\mathcal{R} \subseteq >_{\mathsf{lpo}}$?

Theorem

- if $> \subseteq \sqsupset$ then $>_{\mathsf{lpo}} \subseteq \sqsupset_{\mathsf{lpo}}$ (incrementality)
- if $>$ is total then $>_{\mathsf{lpo}}$ is total on ground terms
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❶ instance: finite TRS $\mathcal{R}$, precedence $>$

question: $\mathcal{R} \subseteq >_{\mathsf{lpo}}$?

❷ instance: finite TRS $\mathcal{R}$

question: $\exists$ precedence $>$ such that $\mathcal{R} \subseteq >_{\mathsf{lpo}}$?

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LPO and polynomial interpretations are incomparable

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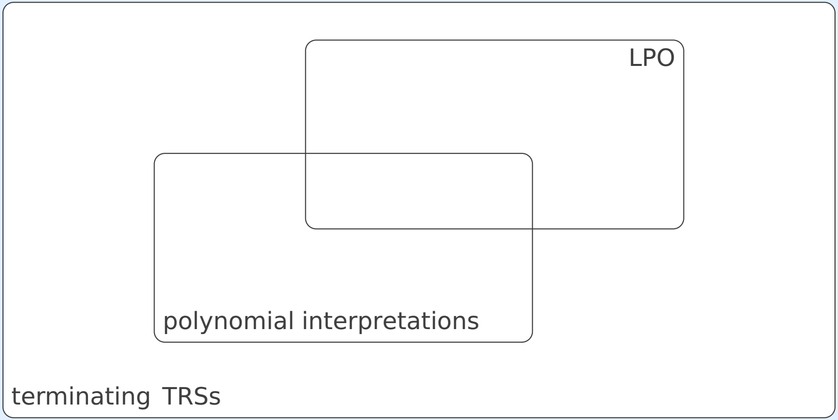
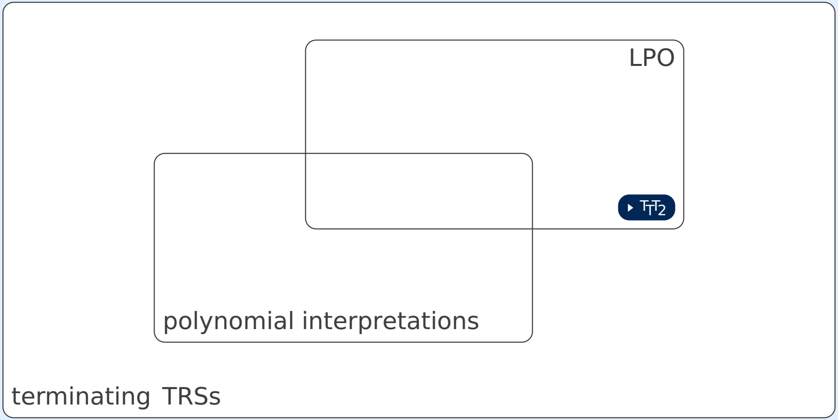


Remark

LPO and polynomial interpretations are incomparable



LPO

▸ TTT2

polynomial interpretations

terminating TRSs

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composition of substitutions σ and τ: $\sigma\tau = \{x \mapsto \sigma(x)\tau \mid x \in \mathcal{V}\}$

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Lemma (Associativity)

$(\rho\sigma)\tau = \rho(\sigma\tau)$ for all substitutions ρ, σ, τ

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Theorem

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instance: terms s, t

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Definition (Most General Unifier)

most general unifier (mgu) is at least as general as any other unifier

Definition (Unification Rules)

d decomposition

$$\frac{\{f(s_1,\ldots,s_n) \approx f(t_1,\ldots,t_n)\} \uplus E}{\{s_1 \approx t_1,\ldots,s_n \approx t_n\} \cup E}$$

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v variable elimination $(x \in \mathcal{V})$

$$\frac{\{x \approx t\} \uplus E}{E\sigma} \quad \text{and} \quad \frac{\{t \approx x\} \uplus E}{E\sigma}$$

if $\underbrace{x \notin \mathcal{V}\mathrm{ar}(t)}_{\text{occurs check}}$ and $\sigma = \{x \mapsto t\}$

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mgu $\quad \{x \mapsto \mathsf{s}(z)\}\{y \mapsto z\} = \{x \mapsto \mathsf{s}(z),\ y \mapsto z\}$

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Optional Failure Rules

$$\frac{\{f(s_1, \dots, s_n) \approx g(t_1, \dots, t_m)\} \uplus E}{\bot} \qquad \frac{\{x \approx t\} \uplus E}{\bot} \qquad \frac{\{t \approx x\} \uplus E}{\bot}$$

if $x \in \mathcal{V}\mathrm{ar}(t)$ and $x \neq t$

Outline

1. **Summary of Lecture 5**
2. **Lexicographic Path Order**
3. **Unification**
4. **Critical Pairs**
5. **Exercises**
6. **Further Reading**

Newman's Lemma

$\mathrm{SN}\ \&\ \mathrm{WCR} \quad \Longrightarrow \quad \mathrm{CR}$

Newman’s Lemma

SN & WCR $\Longrightarrow$ CR

Definition (WCR)

$$\begin{tikzcd}
\forall\, a, b, c & & a \arrow[dl] \arrow[dr] & \\
& b \arrow[dr, dotted, two heads] & & c \arrow[dl, dotted, two heads] \\
\exists\, d & & d &
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Question

how to prove WCR ?

Example

rewrite rules

$$\mathsf{f}(\mathsf{a}, \mathsf{g}(x)) \to \mathsf{f}(x, x) \qquad \mathsf{g}(\mathsf{b}) \to \mathsf{c}$$

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three local peaks

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& \mathsf{f}(\underline{\mathsf{g}(\mathsf{b})}, \overline{\mathsf{g}(\mathsf{b})}) \arrow[dl, "1"'] \arrow[dr, "2"] & \\
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parallel redexes

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variable overlap

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$$f(a, g(x)) \to f(x, x) \qquad\qquad g(b) \to c$$

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overlapping redexes

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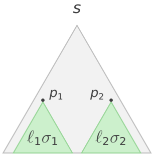
parallel redexes	variable overlap	overlapping redexes
non−critical	non−critical	critical

local peak $t \xleftarrow[p_1 \mid \ell_1 \to r_1 \mid \sigma_1]{} s \xrightarrow[p_2 \mid \ell_2 \to r_2 \mid \sigma_2]{} u$ case 1: parallel redexes



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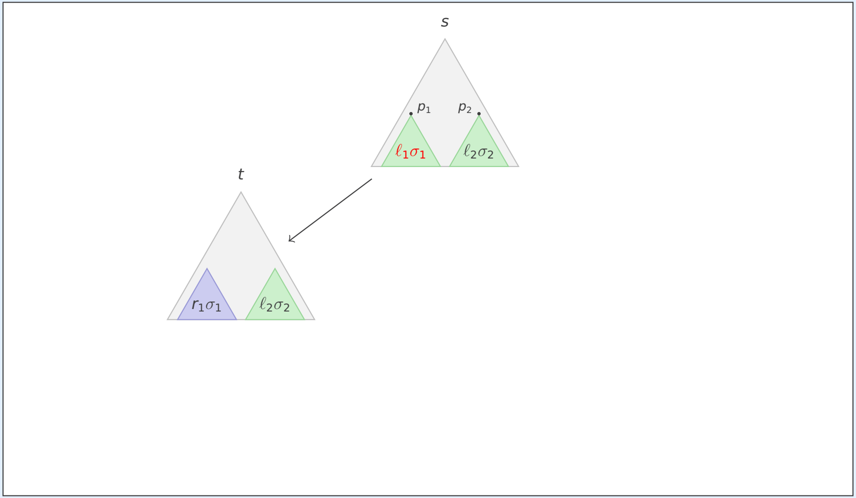

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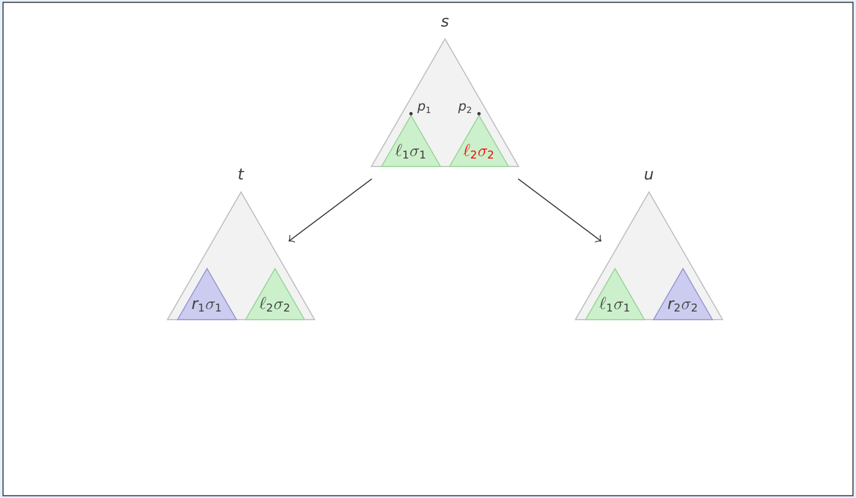

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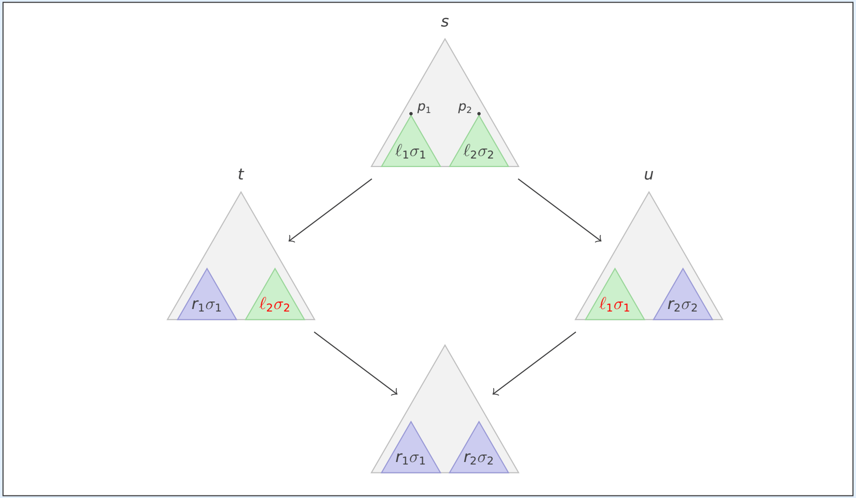

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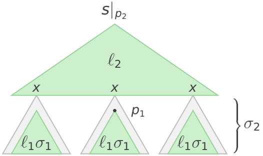

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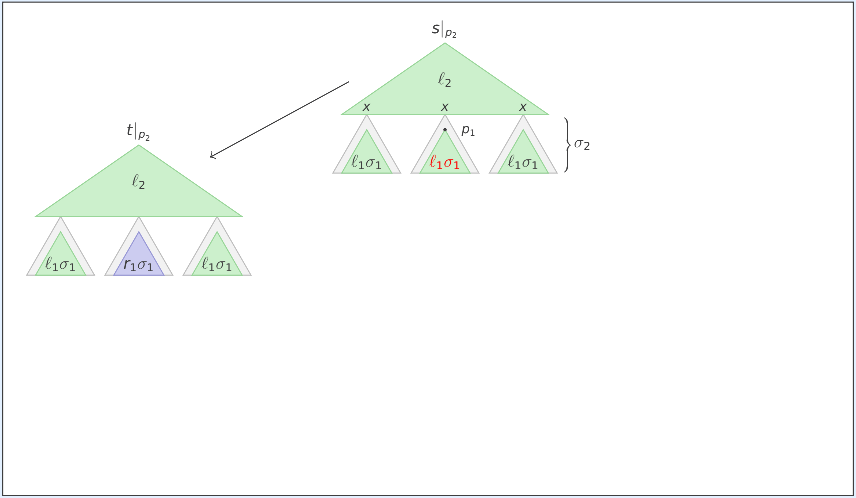

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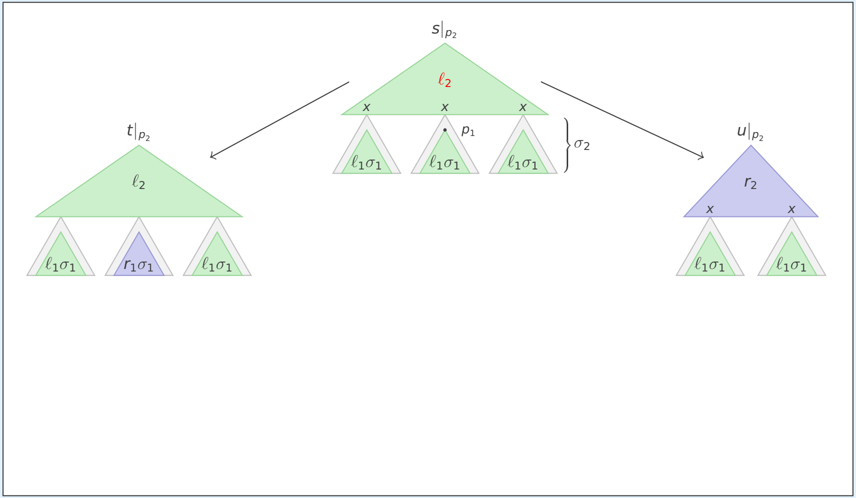

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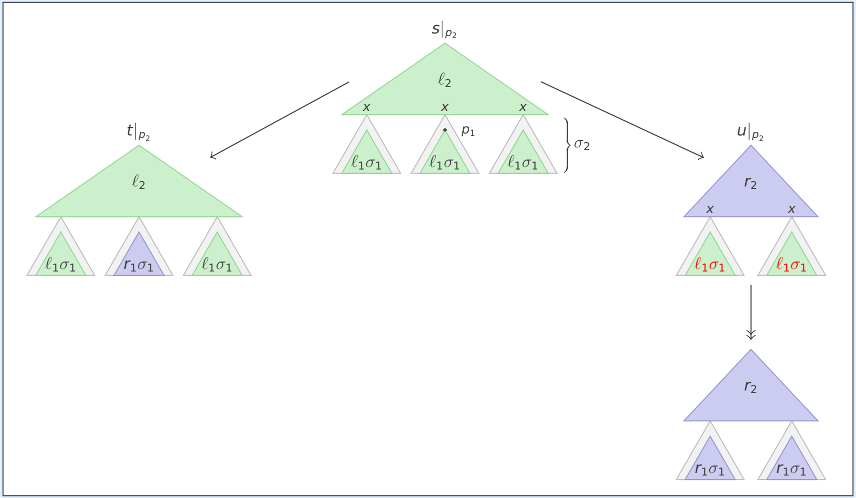

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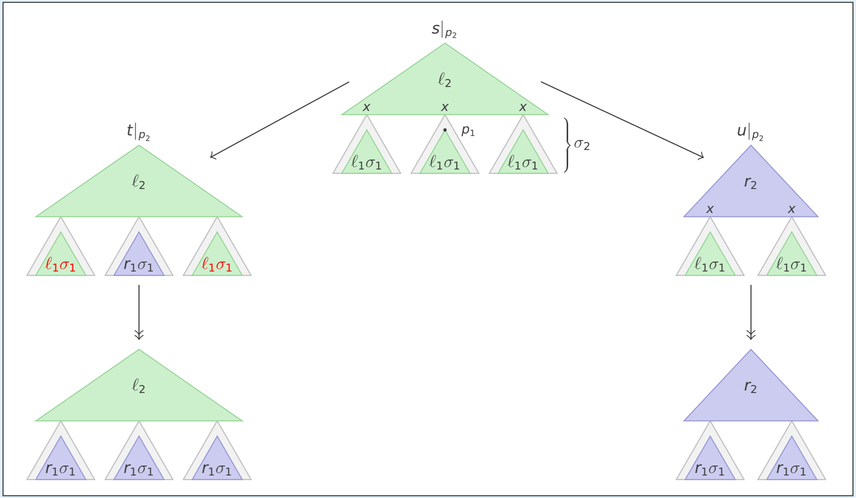

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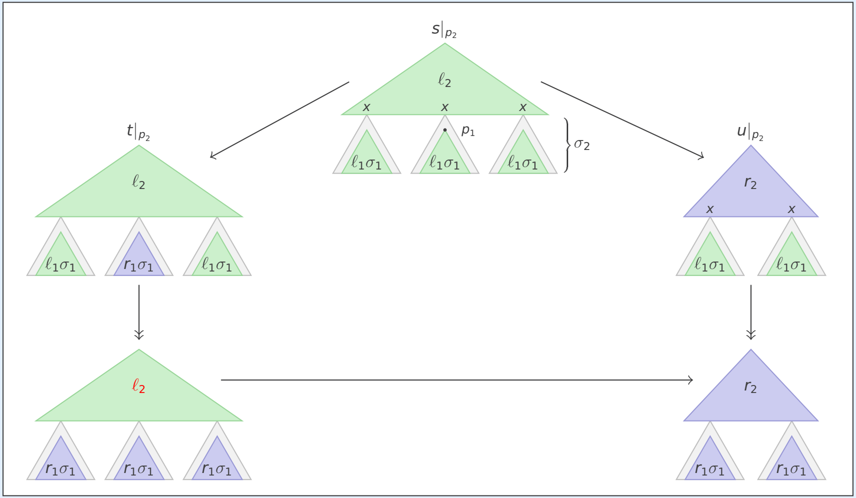

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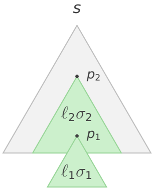

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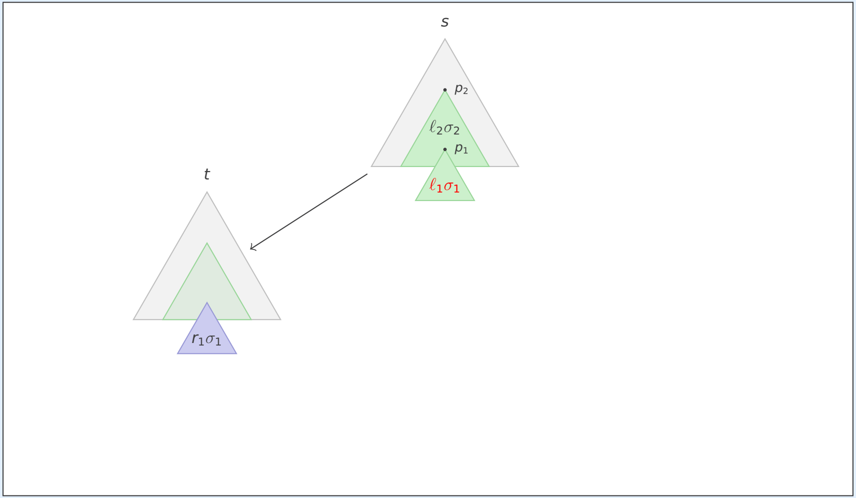

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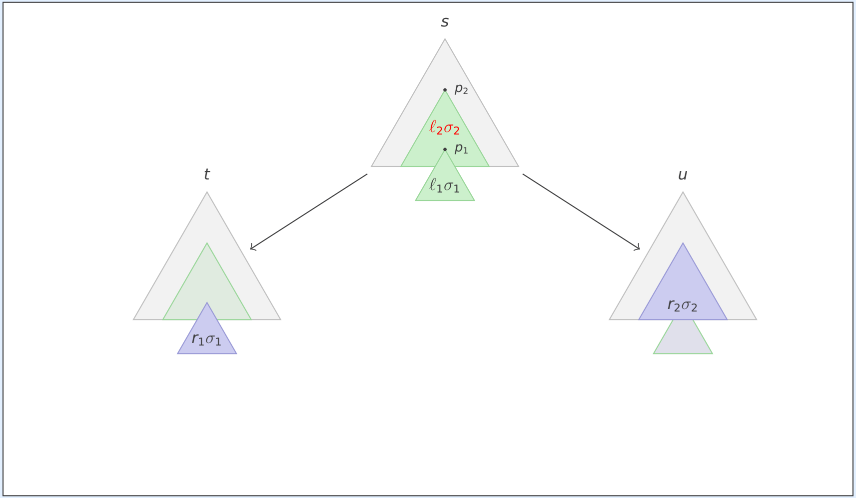

Definitions (Critical Pair)

- overlap of TRS $\mathcal{R}$ is triple $\langle \ell_1 \to r_1, p, \ell_2 \to r_2 \rangle$ such that

  ① $\ell_1 \to r_1$ and $\ell_2 \to r_2$ are variants of rewrite rules in $\mathcal{R}$ without common variables

  ② $p \in \mathcal{P}\mathrm{os}_{\mathcal{F}}(\ell_2)$

  ③ ℓ_1 and $\ell_2|_p$ are unifiable

  ④ if $p = \epsilon$ then $\ell_1 \to r_1$ and $\ell_2 \to r_2$ are not variants

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Critical Pair Lemma

TRS is locally confluent $\iff$ all critical pairs are joinable

Example

$$\mathsf{e} \cdot x \to x \qquad x^{-} \cdot x \to \mathsf{e} \qquad (x \cdot y) \cdot z \to x \cdot (y \cdot z)$$

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overlaps

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Example

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Example

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overlaps

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2. $\langle u^- \cdot u \to \mathsf{e},\ 1,\ (x \cdot y) \cdot z \to x \cdot (y \cdot z) \rangle$
3. $\langle (u \cdot v) \cdot w \to u \cdot (v \cdot w),\ 1,\ (x \cdot y) \cdot z \to x \cdot (y \cdot z) \rangle$

critical peaks

1. $u \cdot z \xleftarrow{1} (\mathsf{e} \cdot u) \cdot z \to \mathsf{e} \cdot (u \cdot z)$
2. $\mathsf{e} \cdot z \xleftarrow{1} (u^- \cdot u) \cdot z \to u^- \cdot (u \cdot z)$
3. $(u \cdot (v \cdot w)) \cdot z \xleftarrow{1} ((u \cdot v) \cdot w) \cdot z \to (u \cdot v) \cdot (w \cdot z)$

Example

$$\mathsf{e} \cdot x \to x \qquad x^{-} \cdot x \to \mathsf{e} \qquad (x \cdot y) \cdot z \to x \cdot (y \cdot z)$$

overlaps

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❸ $(u \cdot (v \cdot w)) \cdot z \xleftarrow{1} ((u \cdot v) \cdot w) \cdot z \to (u \cdot v) \cdot (w \cdot z)$

Theorem (Knuth & Bendix 1970)

terminating TRS is confluent $\iff$ all critical pairs are joinable

Outline

1. Summary of Lecture 5
2. Lexicographic Path Order
3. Unification
4. Critical Pairs
5. **Exercises**
6. Further Reading

Homework Exercises for April 27

① Exercise 2.44. ①

② Prove the termination of the TRS of Exercise 3.5. ①

③ Exercise 4.28. ②

④ Exercise 5.3. ②

⑤ Exercise 5.7. ①

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Lecture Notes

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- Section 2.4 (Definition 2.4.6 — Example 2.4.17)
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Important Concepts

- critical pair
- critical pair lemma
- critical peak
- incrementality
- lexicographic path order (LPO)
- literal similarity
- mgu
- overlap
- precedence
- renaming
- subsumption
- unification algorithm
- unifier
- variant