universität innsbruck

26S lecture 6

Term Rewriting

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Outline

1. **Summary of Lecture 5**
2. **Lexicographic Path Order**
3. **Unification**
4. **Critical Pairs**
5. **Exercises**
6. **Further Reading**

Theorem

validity problem for finite ground ESs is decidable — congruence closure

Definitions

- rewrite order is proper order $>$ on terms which is
  - closed under contexts $\quad s > t \implies C[s] > C[t]$ for all contexts C
  - closed under substitutions $\quad s > t \implies s\sigma > t\sigma$ for all substitutions σ
- TRS $\mathcal{R}$ and rewrite order $>$ are compatible if $\ell > r$ for all rules $\ell \to r$ in $\mathcal{R}$
- reduction order is well–founded rewrite order

Theorem

TRS $\mathcal{R}$ is terminating $\iff$ $\mathcal{R} \subseteq >$ for some reduction order $>$

Definitions

- well–founded monotone $\mathcal{F}$–algebra $(\mathcal{A}, >)$ consists of non–empty algebra $\mathcal{A} = (A, \{f_\mathcal{A}\}_{f \in \mathcal{F}})$ with well–founded order $>$ on A such that every $f_\mathcal{A}$ is strictly monotone in all coordinates:

$$f_\mathcal{A}(a_1, \ldots, a_i, \ldots, a_n) > f_\mathcal{A}(a_1, \ldots, b, \ldots, a_n)$$

  for all $a_1, \ldots, a_n, b \in A$ and $i \in \{1, \ldots, n\}$ with $a_i > b$
- relation $>_\mathcal{A}$ on terms: $s >_\mathcal{A} t$ if $[\alpha]_\mathcal{A}(s) > [\alpha]_\mathcal{A}(t)$ for all assignments α

Lemma

$>_\mathcal{A}$ is reduction order for every well–founded monotone algebra $(\mathcal{A}, >)$

Theorem

TRS $\mathcal{R}$ is terminating $\iff$ $\mathcal{R} \subseteq >_\mathcal{A}$ for well–founded monotone algebra $(\mathcal{A}, >)$

Definition

TRS $\mathcal{R}$ is polynomially terminating (over $\mathbb{N}$) if $\mathcal{R} \subseteq >_{\mathcal{A}}$ for some well–founded monotone algebra $(\mathcal{A}, >)$ such that

- carrier of $\mathcal{A}$ is $\mathbb{N}$
- $>$ is standard order on $\mathbb{N}$
- $f_{\mathcal{A}} \in \mathbb{Z}[x_1, \ldots, x_n]$ for every n-ary f

Lemma

$\mathcal{R}$ is polynomially terminating over $\mathbb{N}$ $\iff$
$\mathcal{R}$ is polynomially terminating over $\{n \in \mathbb{N} \mid n \geqslant N\}$ for some $N \geqslant 0$

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Definition (Lexicographic Path Order)

- precedence is proper order $>$ on $\mathcal{F}$
- binary relation $>_{\text{lpo}}$ on terms over $\mathcal{F}$: $s >_{\text{lpo}} t$ if $s = f(s_1, \ldots, s_n)$ and either
  - ① $t = f(t_1, \ldots, t_n)$ and for some $1 \leqslant i \leqslant n$
    - **a** $s_j = t_j$ for all $1 \leqslant j < i$
    - **b** $s_i >_{\text{lpo}} t_i$
    - **c** $s >_{\text{lpo}} t_j$ for all $i < j \leqslant n$
  - ② $t = g(t_1, \ldots, t_m)$ and $f > g$ and $s >_{\text{lpo}} t_j$ for all $1 \leqslant j \leqslant m$
  - ③ $s_i >_{\text{lpo}} t$ or $s_i = t$ for some $1 \leqslant i \leqslant n$

Theorem

$>_{\text{lpo}}$ is reduction order if precedence $>$ is well–founded

Examples

① $0 + y \to y$ $\quad 0 \times y \to 0$ $\quad \times > + > \mathsf{s}$
$\mathsf{s}(x) + y \to \mathsf{s}(x + y)$ $\quad \mathsf{s}(x) \times y \to (x \times y) + y$

② $\mathsf{ack}(0, y) \to \mathsf{s}(y)$ $\quad \mathsf{ack} > \mathsf{s}$
$\mathsf{ack}(\mathsf{s}(x), 0) \to \mathsf{ack}(x, \mathsf{s}(0))$
$\mathsf{ack}(\mathsf{s}(x), \mathsf{s}(y)) \to \mathsf{ack}(x, \mathsf{ack}(\mathsf{s}(x), y))$

③ $\mathsf{e} \cdot x \to x$ $\quad x \cdot \mathsf{e} \to x$ $\quad {}^{-} > \cdot > \mathsf{e}$
$x^{-} \cdot x \to \mathsf{e}$ $\quad x \cdot x^{-} \to \mathsf{e}$
$(x \cdot y) \cdot z \to x \cdot (y \cdot z)$ $\quad x^{--} \to x$
$\mathsf{e}^{-} \to \mathsf{e}$ $\quad (x \cdot y)^{-} \to y^{-} \cdot x^{-}$
$x^{-} \cdot (x \cdot y) \to y$ $\quad x \cdot (x^{-} \cdot y) \to y$

Example

$$\dfrac{\mathsf{ack} > \mathsf{s} \qquad \dfrac{}{\mathsf{ack}(0,y) >_{\mathsf{lpo}} y}\,③}{\mathsf{ack}(0,y) >_{\mathsf{lpo}} \mathsf{s}(y)}\,②$$

$$\dfrac{③\ \dfrac{}{\mathsf{s}(x) >_{\mathsf{lpo}} x} \qquad \dfrac{\mathsf{ack} > \mathsf{s} \qquad \dfrac{}{\mathsf{ack}(\mathsf{s}(x),0) >_{\mathsf{lpo}} 0}\,③}{\mathsf{ack}(\mathsf{s}(x),0) >_{\mathsf{lpo}} \mathsf{s}(0)}\,②}{\mathsf{ack}(\mathsf{s}(x),0) >_{\mathsf{lpo}} \mathsf{ack}(x,\mathsf{s}(0))}\,①$$

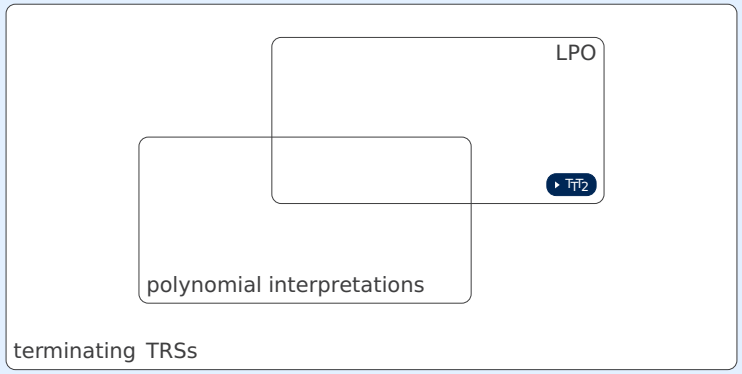
$$\dfrac{③\ \dfrac{}{\mathsf{s}(x) >_{\mathsf{lpo}} x} \qquad \dfrac{\mathsf{s}(x) = \mathsf{s}(x) \qquad \dfrac{}{\mathsf{s}(y) >_{\mathsf{lpo}} y}\,③}{\mathsf{ack}(\mathsf{s}(x),\mathsf{s}(y)) >_{\mathsf{lpo}} \mathsf{ack}(\mathsf{s}(x),y)}\,①}{\mathsf{ack}(\mathsf{s}(x),\mathsf{s}(y)) >_{\mathsf{lpo}} \mathsf{ack}(x,\mathsf{ack}(\mathsf{s}(x),y))}\,①$$

Theorem

- if $> \subseteq \sqsupset$ then $>_{\mathsf{lpo}} \subseteq \sqsupset_{\mathsf{lpo}}$ (incrementality)
- if $>$ is total then $>_{\mathsf{lpo}}$ is total on ground terms
- following two problems are decidable:
  1. instance: finite TRS $\mathcal{R}$, precedence $>$
     question: $\mathcal{R} \subseteq >_{\mathsf{lpo}}$?
  2. instance: finite TRS $\mathcal{R}$
     question: $\exists$ precedence $>$ such that $\mathcal{R} \subseteq >_{\mathsf{lpo}}$?

Remark

LPO and polynomial interpretations are incomparable



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Definition (Composition of Substitutions)

composition of substitutions σ and τ: $\sigma\tau = \{x \mapsto \sigma(x)\tau \mid x \in \mathcal{V}\}$

Example

$\sigma = \{x \mapsto \mathsf{s}(y), y \mapsto x + \mathsf{s}(0)\}$ $\tau = \{x \mapsto \mathsf{s}(0), z \mapsto \mathsf{s}(\mathsf{s}(y))\}$

- $\sigma\tau = \{x \mapsto \mathsf{s}(y), y \mapsto \mathsf{s}(0) + \mathsf{s}(0), z \mapsto \mathsf{s}(\mathsf{s}(y))\}$
- $\tau\sigma = \{x \mapsto \mathsf{s}(0), y \mapsto x + \mathsf{s}(0), z \mapsto \mathsf{s}(\mathsf{s}(x + \mathsf{s}(0)))\}$

Lemma (Associativity)

$(\rho\sigma)\tau = \rho(\sigma\tau)$ for all substitutions ρ, σ, τ

Definitions (Subsumption)

- $\trianglelefteq$ subsumption
  $s \trianglelefteq t \iff s\sigma = t$ for some substitution σ "s subsumes t" "t is instance of s"
- $\triangleleft$ proper subsumption
  $s \triangleleft t \iff s \trianglelefteq t \wedge t \not\trianglelefteq s$

Example

$x + y \trianglelefteq \mathsf{s}(y) + \mathsf{s}(0)$ $\mathsf{s}(x) + y \not\trianglelefteq x + \mathsf{s}(0)$ $\mathsf{s}(x) + y \trianglelefteq \mathsf{s}(x) + x$

Theorem

$\trianglerighteq$ is well–founded order on terms

Definitions (Renaming)

- $\doteq$ literal similarity
  $s \doteq t \iff s \trianglelefteq t \wedge t \trianglelefteq s$
- variable substitution is substitution from $\mathcal{V}$ to $\mathcal{V}$
- renaming is bijective variable substitution
- terms s and t are variants if $s = t\sigma$ for some renaming σ

Lemma

terms s and t are variants $\iff$ $s \doteq t$

Example

$\mathsf{s}(x) + \mathsf{s}(y + 0) \doteq \mathsf{s}(y) + \mathsf{s}(z + 0)$ $\mathsf{s}(x) + \mathsf{s}(y + 0) \not\doteq \mathsf{s}(x) + \mathsf{s}(x + 0)$

Definition (Unification Problem)

instance: terms s, t

question: $\exists$ $\underbrace{\text{substitution } \sigma}_{\text{unifier}}$ such that $s\sigma = t\sigma$?

Definition

substitution σ is at least as general as τ ($\sigma \trianglelefteq \tau$) if $\sigma\rho = \tau$ for some substitution ρ

Theorem

$\trianglerighteq$ is well–founded order on substitutions

Definition (Most General Unifier)

most general unifier (mgu) is at least as general as any other unifier

Definition (Unification Rules)

d decomposition

$$\frac{\{f(s_1, \ldots, s_n) \approx f(t_1, \ldots, t_n)\} \uplus E}{\{s_1 \approx t_1, \ldots, s_n \approx t_n\} \cup E}$$

t removal of trivial equations $(x \in \mathcal{V})$

$$\frac{\{x \approx x\} \uplus E}{E}$$

v variable elimination $(x \in \mathcal{V})$

$$\frac{\{x \approx t\} \uplus E}{E\sigma} \quad \text{and} \quad \frac{\{t \approx x\} \uplus E}{E\sigma}$$

if $\underbrace{x \notin \mathcal{V}\mathrm{ar}(t)}_{\text{occurs check}}$ and $\sigma = \{x \mapsto t\}$

Example

$$\begin{array}{c}
\{x + (0 + \mathsf{s}(y)) \approx \mathsf{s}(z) + (0 + x)\} \\
\mathsf{d} \Downarrow \\
\{x \approx \mathsf{s}(z),\ 0 + \mathsf{s}(y) \approx 0 + x\} \\
\mathsf{v} \Downarrow\ x \mapsto \mathsf{s}(z) \\
\{0 + \mathsf{s}(y) \approx 0 + \mathsf{s}(z)\} \\
\mathsf{d} \Downarrow \\
\{0 \approx 0,\ \mathsf{s}(y) \approx \mathsf{s}(z)\} \\
\mathsf{d} \Downarrow \\
\{\mathsf{s}(y) \approx \mathsf{s}(z)\} \\
\mathsf{d} \Downarrow \\
\{y \approx z\} \\
\mathsf{v} \Downarrow\ y \mapsto z \\
\varnothing
\end{array}$$

mgu $\{x \mapsto \mathsf{s}(z)\}\{y \mapsto z\} = \{x \mapsto \mathsf{s}(z),\ y \mapsto z\}$

Theorem

- there are no infinite derivations

$$\{s \approx t\} \Longrightarrow_{\sigma_1} E_1 \Longrightarrow_{\sigma_2} E_2 \Longrightarrow_{\sigma_3} \cdots$$

- if s and t are unifiable then for every maximal derivation

$$\{s \approx t\} \Longrightarrow_{\sigma_1} E_1 \Longrightarrow_{\sigma_2} E_2 \Longrightarrow_{\sigma_3} \cdots \Longrightarrow_{\sigma_n} E_n$$

  - $E_n = \varnothing$
  - $\sigma_1\sigma_2\sigma_3\cdots\sigma_n$ is mgu of s and t

Optional Failure Rules

$$\frac{\{f(s_1, \ldots, s_n) \approx g(t_1, \ldots, t_m)\} \uplus E}{\bot} \qquad \frac{\{x \approx t\} \uplus E}{\bot} \quad \frac{\{t \approx x\} \uplus E}{\bot}$$

if $x \in \mathcal{V}\mathrm{ar}(t)$ and $x \neq t$

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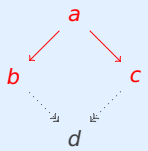
Newman's Lemma

SN & WCR $\Longrightarrow$ CR

Definition (WCR)

$\forall\, a, b, c$



local peak

$\exists\, d$

Question

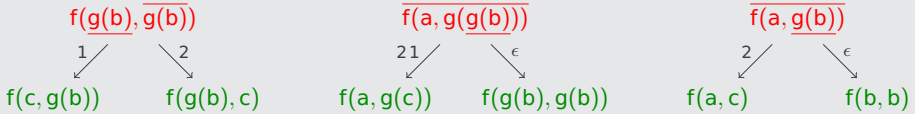
how to prove WCR ?

Example

rewrite rules

$$\mathsf{f}(\mathsf{a}, \mathsf{g}(x)) \to \mathsf{f}(x, x) \qquad \mathsf{g}(\mathsf{b}) \to \mathsf{c}$$

three local peaks

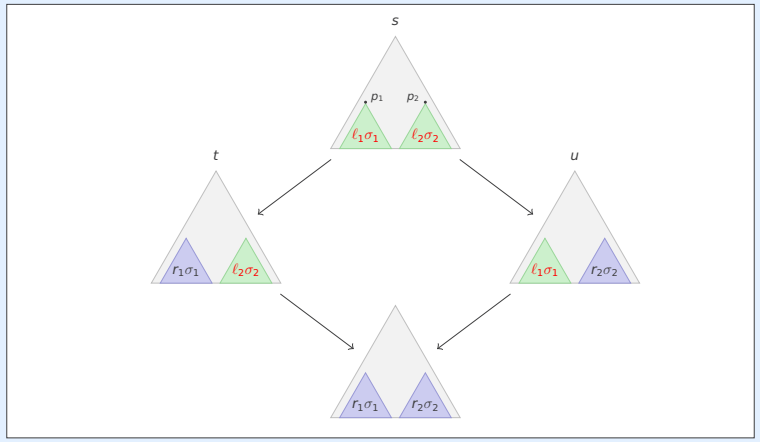


parallel redexes — non−critical

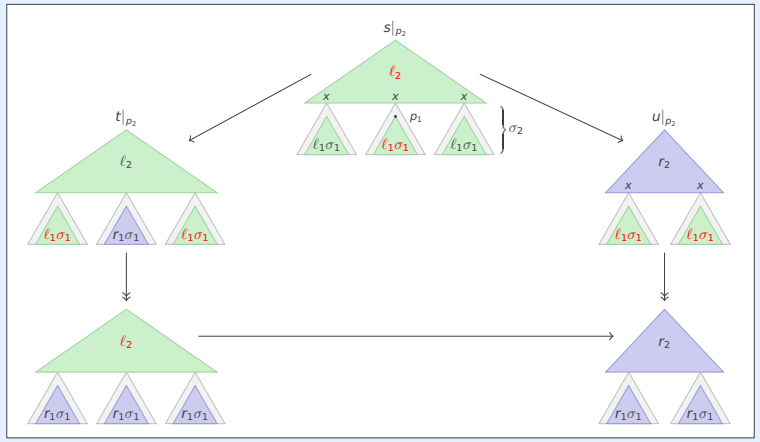
variable overlap — non−critical

overlapping redexes — critical

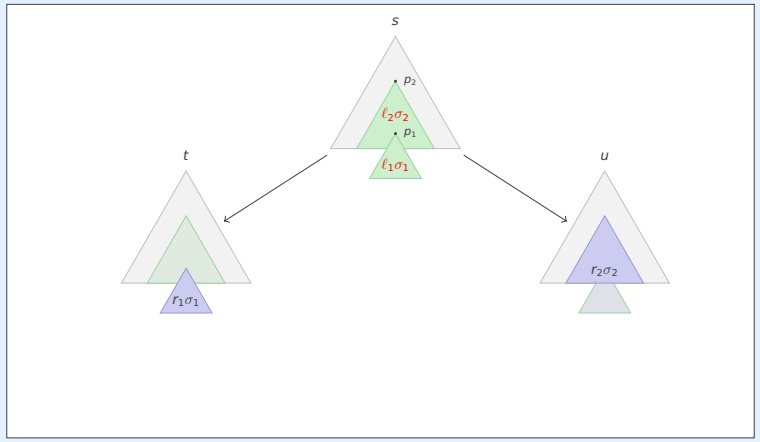
local peak $t \xleftarrow[p_1 \,|\, \ell_1 \to r_1 \,|\, \sigma_1]{} s \xrightarrow[p_2 \,|\, \ell_2 \to r_2 \,|\, \sigma_2]{} u$ case 1: parallel redexes



local peak $t \xleftarrow[p_1 \,|\, \ell_1 \to r_1 \,|\, \sigma_1]{} s \xrightarrow[p_2 \,|\, \ell_2 \to r_2 \,|\, \sigma_2]{} u$ case 2: variable overlap



local peak $t \xleftarrow[p_1 \mid \ell_1 \to r_1 \mid \sigma_1]{} s \xrightarrow[p_2 \mid \ell_2 \to r_2 \mid \sigma_2]{} u$ case 3: overlapping redexes



Definitions (Critical Pair)

- overlap of TRS $\mathcal{R}$ is triple $\langle \ell_1 \to r_1, p, \ell_2 \to r_2 \rangle$ such that
  1. $\ell_1 \to r_1$ and $\ell_2 \to r_2$ are variants of rewrite rules in $\mathcal{R}$ without common variables
  2. $p \in \mathcal{P}\mathrm{os}_{\mathcal{F}}(\ell_2)$
  3. ℓ_1 and $\ell_2|_p$ are unifiable with most general unifier σ
  4. if $p = \epsilon$ then $\ell_1 \to r_1$ and $\ell_2 \to r_2$ are not variants
- $\ell_2\sigma[r_1\sigma]_p \leftarrow \ell_2\sigma[\ell_1\sigma]_p = \ell_2\sigma \to r_2\sigma$ critical peak
- $\ell_2\sigma[r_1\sigma]_p \approx r_2\sigma$ critical pair
- critical pair $s \approx t$ is joinable if $s \downarrow t$

Critical Pair Lemma

TRS is locally confluent $\iff$ all critical pairs are joinable

Example

$\mathsf{e} \cdot x \to x$ $x^{-} \cdot x \to \mathsf{e}$ $(x \cdot y) \cdot z \to x \cdot (y \cdot z)$

overlaps

1. $\langle \mathsf{e} \cdot u \to u, 1, (x \cdot y) \cdot z \to x \cdot (y \cdot z) \rangle$
2. $\langle u^{-} \cdot u \to \mathsf{e}, 1, (x \cdot y) \cdot z \to x \cdot (y \cdot z) \rangle$
3. $\langle (u \cdot v) \cdot w \to u \cdot (v \cdot w), 1, (x \cdot y) \cdot z \to x \cdot (y \cdot z) \rangle$

critical peaks

1. $u \cdot z \overset{1}{\leftarrow} (\mathsf{e} \cdot u) \cdot z \to \mathsf{e} \cdot (u \cdot z)$
2. $\mathsf{e} \cdot z \overset{1}{\leftarrow} (u^{-} \cdot u) \cdot z \to u^{-} \cdot (u \cdot z)$
3. $(u \cdot (v \cdot w)) \cdot z \overset{1}{\leftarrow} ((u \cdot v) \cdot w) \cdot z \to (u \cdot v) \cdot (w \cdot z)$

Theorem (Knuth & Bendix 1970)

terminating TRS is confluent $\iff$ all critical pairs are joinable

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Homework Exercises for April 27

1. Exercise 2.44. (1)
2. Prove the termination of the TRS of Exercise 3.5. (1)
3. Exercise 4.28. (2)
4. Exercise 5.3. (2)
5. Exercise 5.7. (1)

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Lecture Notes

- Section 4.3 (except Lemma 4.3.8 — Corollary 4.3.10)
- Section 2.4 (Definition 2.4.6 — Example 2.4.17)
- Section 2.5
- Section 5.1 (until Corollary 5.1.12)

Important Concepts

- critical pair
- critical pair lemma
- critical peak
- incrementality
- lexicographic path order (LPO)
- literal similarity
- mgu
- overlap
- precedence
- renaming
- subsumption
- unification algorithm
- unifier
- variant