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Term Rewriting

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Outline

Definition

- precedence is proper order $>$ on $\mathcal{F}$
- binary relation $>_{\mathrm{lpo}}$ on terms over $\mathcal{F}$: $s >_{\mathrm{lpo}} t$ if $s = f(s_1, \dots, s_n)$ and either

  ① $t = f(t_1, \dots, t_n)$ and for some $1 \leqslant i \leqslant n$

  a $s_j = t_j$ for all $1 \leqslant j < i$

  b $s_i >_{\mathrm{lpo}} t_i$

  c $s >_{\mathrm{lpo}} t_j$ for all $i < j \leqslant n$

  ② $t = g(t_1, \dots, t_m)$ and $f > g$ and $s >_{\mathrm{lpo}} t_j$ for all $1 \leqslant j \leqslant m$

  ③ $s_i >_{\mathrm{lpo}} t$ or $s_i = t$ for some $1 \leqslant i \leqslant n$

Theorem

$>_{\mathrm{lpo}}$ is reduction order if precedence $>$ is well–founded

Theorem

- if $> \subseteq \sqsupset$ then $>_{\text{lpo}} \subseteq \sqsupset_{\text{lpo}}$ (incrementality)
- if $>$ is total then $>_{\text{lpo}}$ is total on ground terms
- following problem is decidable:

  instance: finite TRS $\mathcal{R}$

  question: $\exists$ precedence $>$ such that $\mathcal{R} \subseteq >_{\text{lpo}}$?

Definitions

- $s \mathrel{\dot{\leq}} t \iff s\sigma = t$ for some substitution σ
- $s \mathrel{\dot{<}} t \iff s \mathrel{\dot{\leq}} t \wedge t \mathrel{\dot{\nleq}} s$
- $s \doteq t \iff s \mathrel{\dot{\leq}} t \wedge t \mathrel{\dot{\leq}} s$
- substitution σ is at least as general as τ ($\sigma \mathrel{\dot{\leq}} \tau$) if $\sigma\rho = \tau$ for some substitution ρ

Lemma

$>\!\!\!\cdot$ is well–founded order on terms and substitutions

Definitions

- variable substitution is substitution from $\mathcal{V}$ to $\mathcal{V}$
- renaming is bijective variable substitution
- terms s and t are variants if $s = t\sigma$ for some renaming σ
- terms s and t are unifiable if $s\sigma = t\sigma$ for some substitution σ
- most general unifier (mgu) is at least as general as any other unifier

Lemma

terms s and t are variants $\iff$ $s \doteq t$

Theorem

unification problem

instance: terms s, t

question: are s and t unifiable ?

is decidable (and unification algorithm produces most general unifier)

Definitions

- overlap is triple $\langle \ell_1 \to r_1, p, \ell_2 \to r_2 \rangle$ such that
  1. $\ell_1 \to r_1$ and $\ell_2 \to r_2$ are variants of rewrite rules without common variables
  2. $p \in \mathcal{P}\mathrm{os}_{\mathcal{F}}(\ell_2)$
  3. ℓ_1 and $\ell_2|_p$ are unifiable with most general unifier σ
  4. if $p = \epsilon$ then $\ell_1 \to r_1$ and $\ell_2 \to r_2$ are not variants
- $\ell_2\sigma[r_1\sigma]_p \xleftarrow{p} \ell_2\sigma \xrightarrow{\epsilon} r_2\sigma$ critical peak $\quad \ell_2\sigma[r_1\sigma]_p \approx r_2\sigma$ critical pair
- critical pair $s \approx t$ is joinable if $s \downarrow t$

Critical Pair Lemma

TRS is locally confluent $\iff$ all critical pairs are joinable

Corollary

terminating TRS is confluent $\iff$ all critical pairs are joinable

Outline

Example

TRS $\mathcal{R}$

$$x + 0 \xrightarrow{1} x \qquad x + \mathsf{s}(y) \xrightarrow{3} \mathsf{s}(x + y) \qquad \mathsf{p}(\mathsf{s}(x)) \xrightarrow{5} x$$

$$x - 0 \xrightarrow{2} x \qquad x - \mathsf{s}(y) \xrightarrow{4} \mathsf{p}(x - y) \qquad \mathsf{s}(\mathsf{p}(x)) \xrightarrow{6} x$$

Example

TRS $\mathcal{R}$

$$x + 0 \xrightarrow{1} x \qquad x + \mathsf{s}(y) \xrightarrow{3} \mathsf{s}(x + y) \qquad \mathsf{p}(\mathsf{s}(x)) \xrightarrow{5} x$$

$$x - 0 \xrightarrow{2} x \qquad x - \mathsf{s}(y) \xrightarrow{4} \mathsf{p}(x - y) \qquad \mathsf{s}(\mathsf{p}(x)) \xrightarrow{6} x$$

- SN ?

Example

TRS $\mathcal{R}$

$$x + 0 \xrightarrow{1} x \qquad x + \mathsf{s}(y) \xrightarrow{3} \mathsf{s}(x + y) \qquad \mathsf{p}(\mathsf{s}(x)) \xrightarrow{5} x$$

$$x - 0 \xrightarrow{2} x \qquad x - \mathsf{s}(y) \xrightarrow{4} \mathsf{p}(x - y) \qquad \mathsf{s}(\mathsf{p}(x)) \xrightarrow{6} x$$

- SN (e.g.) LPO with precedence $+ > \mathsf{s}$ and $- > \mathsf{p}$

Example

TRS $\mathcal{R}$

$$x + 0 \xrightarrow{1} x \qquad x + \mathsf{s}(y) \xrightarrow{3} \mathsf{s}(x + y) \qquad \mathsf{p}(\mathsf{s}(x)) \xrightarrow{5} x$$

$$x - 0 \xrightarrow{2} x \qquad x - \mathsf{s}(y) \xrightarrow{4} \mathsf{p}(x - y) \qquad \mathsf{s}(\mathsf{p}(x)) \xrightarrow{6} x$$

- SN (e.g.) LPO with precedence $+ > \mathsf{s}$ and $- > \mathsf{p}$
- WCR ?

Example

TRS $\mathcal{R}$

$$x + 0 \xrightarrow{1} x \qquad x + \mathsf{s}(y) \xrightarrow{3} \mathsf{s}(x + y) \qquad \mathsf{p}(\mathsf{s}(x)) \xrightarrow{5} x$$

$$x - 0 \xrightarrow{2} x \qquad x - \mathsf{s}(y) \xrightarrow{4} \mathsf{p}(x - y) \qquad \mathsf{s}(\mathsf{p}(x)) \xrightarrow{6} x$$

- SN (e.g.) LPO with precedence $+ > \mathsf{s}$ and $- > \mathsf{p}$
- WCR ? 4 critical pairs

$$x + y \xleftarrow{6} x + \mathsf{s}(\mathsf{p}(y)) \xrightarrow{3} \mathsf{s}(x + \mathsf{p}(y))$$

$$x - y \xleftarrow{6} x - \mathsf{s}(\mathsf{p}(y)) \xrightarrow{4} \mathsf{p}(x - \mathsf{p}(y))$$

$$\mathsf{p}(x) \xleftarrow{6} \mathsf{p}(\mathsf{s}(\mathsf{p}(x))) \xrightarrow{5} \mathsf{p}(x)$$

$$\mathsf{s}(x) \xleftarrow{5} \mathsf{s}(\mathsf{p}(\mathsf{s}(x))) \xrightarrow{6} \mathsf{s}(x)$$

Example

TRS $\mathcal{R}$

$$x + 0 \xrightarrow{1} x \qquad x + \mathsf{s}(y) \xrightarrow{3} \mathsf{s}(x + y) \qquad \mathsf{p}(\mathsf{s}(x)) \xrightarrow{5} x$$

$$x - 0 \xrightarrow{2} x \qquad x - \mathsf{s}(y) \xrightarrow{4} \mathsf{p}(x - y) \qquad \mathsf{s}(\mathsf{p}(x)) \xrightarrow{6} x$$

- SN (e.g.) LPO with precedence $+ > \mathsf{s}$ and $- > \mathsf{p}$
- WCR ? 4 critical pairs

$$\overline{x + \underline{\mathsf{s}(\mathsf{p}(y))}} \quad {}_{6}\!\swarrow \; \searrow_{3} \quad x + y \qquad \mathsf{s}(x + \mathsf{p}(y))$$

$$\overline{x - \underline{\mathsf{s}(\mathsf{p}(y))}} \quad {}_{6}\!\swarrow \; \searrow_{4} \quad x - y \qquad \mathsf{p}(x - \mathsf{p}(y))$$

$$\overline{\mathsf{p}(\underline{\mathsf{s}(\mathsf{p}(x))})} \quad {}_{6}\!\swarrow \; \searrow_{5} \quad \mathsf{p}(x) = \mathsf{p}(x)$$

$$\overline{\mathsf{s}(\underline{\mathsf{p}(\mathsf{s}(x))})} \quad {}_{5}\!\swarrow \; \searrow_{6} \quad \mathsf{s}(x) = \mathsf{s}(x)$$

Example

TRS $\mathcal{R}$

$$x + 0 \xrightarrow{1} x \qquad x + \mathsf{s}(y) \xrightarrow{3} \mathsf{s}(x + y) \qquad \mathsf{p}(\mathsf{s}(x)) \xrightarrow{5} x$$

$$x - 0 \xrightarrow{2} x \qquad x - \mathsf{s}(y) \xrightarrow{4} \mathsf{p}(x - y) \qquad \mathsf{s}(\mathsf{p}(x)) \xrightarrow{6} x$$

$$\mathsf{s}(x + \mathsf{p}(y)) \xrightarrow{7} x + y$$

- SN (e.g.) LPO with precedence $+ > \mathsf{s}$ and $- > \mathsf{p}$
- WCR ? 4 critical pairs

$$\overline{x + \underline{\mathsf{s}(\mathsf{p}(y))}} \quad \xrightarrow{6} x + y, \quad \xrightarrow{3} \mathsf{s}(x + \mathsf{p}(y)); \qquad x + y \xleftarrow{7} \mathsf{s}(x + \mathsf{p}(y))$$

$$\overline{x - \underline{\mathsf{s}(\mathsf{p}(y))}} \quad \xrightarrow{6} x - y, \quad \xrightarrow{4} \mathsf{p}(x - \mathsf{p}(y))$$

$$\overline{\mathsf{p}(\underline{\mathsf{s}(\mathsf{p}(x))})} \quad \xrightarrow{6} \mathsf{p}(x), \quad \xrightarrow{5} \mathsf{p}(x); \qquad \mathsf{p}(x) = \mathsf{p}(x)$$

$$\overline{\mathsf{s}(\underline{\mathsf{p}(\mathsf{s}(x))})} \quad \xrightarrow{5} \mathsf{s}(x), \quad \xrightarrow{6} \mathsf{s}(x); \qquad \mathsf{s}(x) = \mathsf{s}(x)$$

Example

TRS $\mathcal{R}$

$$
\begin{array}{rcl@{\qquad}rcl@{\qquad}rcl}
x + 0 & \xrightarrow{1} & x & x + \mathsf{s}(y) & \xrightarrow{3} & \mathsf{s}(x + y) & \mathsf{p}(\mathsf{s}(x)) & \xrightarrow{5} & x \\
x - 0 & \xrightarrow{2} & x & x - \mathsf{s}(y) & \xrightarrow{4} & \mathsf{p}(x - y) & \mathsf{s}(\mathsf{p}(x)) & \xrightarrow{6} & x \\
\mathsf{s}(x + \mathsf{p}(y)) & \xrightarrow{7} & x + y & \mathsf{p}(x - \mathsf{p}(y)) & \xrightarrow{8} & x - y & & &
\end{array}
$$

- SN (e.g.) LPO with precedence $+ > \mathsf{s}$ and $- > \mathsf{p}$
- WCR ? 4 critical pairs

$$
\overline{x + \underline{\mathsf{s}(\mathsf{p}(y))}}: \quad x + y \xleftarrow{6} \cdot \xrightarrow{3} \mathsf{s}(x + \mathsf{p}(y)), \qquad x + y \xleftarrow[7]{} \mathsf{s}(x + \mathsf{p}(y))
$$

$$
\overline{x - \underline{\mathsf{s}(\mathsf{p}(y))}}: \quad x - y \xleftarrow{6} \cdot \xrightarrow{4} \mathsf{p}(x - \mathsf{p}(y)), \qquad x - y \xleftarrow[8]{} \mathsf{p}(x - \mathsf{p}(y))
$$

$$
\overline{\mathsf{p}(\underline{\mathsf{s}(\mathsf{p}(x))})}: \quad \mathsf{p}(x) \xleftarrow{6} \cdot \xrightarrow{5} \mathsf{p}(x), \qquad \mathsf{p}(x) = \mathsf{p}(x)
$$

$$
\overline{\mathsf{s}(\underline{\mathsf{p}(\mathsf{s}(x))})}: \quad \mathsf{s}(x) \xleftarrow{5} \cdot \xrightarrow{6} \mathsf{s}(x), \qquad \mathsf{s}(x) = \mathsf{s}(x)
$$

Example

TRS $\mathcal{R}$

$$
\begin{array}{rclrclrcl}
x + 0 & \xrightarrow{1} & x & \qquad x + \mathsf{s}(y) & \xrightarrow{3} & \mathsf{s}(x + y) & \qquad \mathsf{p}(\mathsf{s}(x)) & \xrightarrow{5} & x \\
x - 0 & \xrightarrow{2} & x & \qquad x - \mathsf{s}(y) & \xrightarrow{4} & \mathsf{p}(x - y) & \qquad \mathsf{s}(\mathsf{p}(x)) & \xrightarrow{6} & x \\
\mathsf{s}(x + \mathsf{p}(y)) & \xrightarrow{7} & x + y & \qquad \mathsf{p}(x - \mathsf{p}(y)) & \xrightarrow{8} & x - y & & &
\end{array}
$$

- SN (e.g.) LPO with precedence $+ > \mathsf{s}$ and $- > \mathsf{p}$
- WCR ? 4 critical pairs

$$x + \mathsf{s}(\mathsf{p}(y)) \xrightarrow{6} x + y, \qquad x + \mathsf{s}(\mathsf{p}(y)) \xrightarrow{3} \mathsf{s}(x + \mathsf{p}(y)), \qquad x + y \xleftarrow{7} \mathsf{s}(x + \mathsf{p}(y))$$

$$x - \mathsf{s}(\mathsf{p}(y)) \xrightarrow{6} x - y, \qquad x - \mathsf{s}(\mathsf{p}(y)) \xrightarrow{4} \mathsf{p}(x - \mathsf{p}(y)), \qquad x - y \xleftarrow{8} \mathsf{p}(x - \mathsf{p}(y))$$

$$\mathsf{p}(\mathsf{s}(\mathsf{p}(x))) \xrightarrow{6} \mathsf{p}(x), \qquad \mathsf{p}(\mathsf{s}(\mathsf{p}(x))) \xrightarrow{5} \mathsf{p}(x), \qquad \mathsf{p}(x) = \mathsf{p}(x)$$

$$\mathsf{s}(\mathsf{p}(\mathsf{s}(x))) \xrightarrow{5} \mathsf{s}(x), \qquad \mathsf{s}(\mathsf{p}(\mathsf{s}(x))) \xrightarrow{6} \mathsf{s}(x), \qquad \mathsf{s}(x) = \mathsf{s}(x)$$

- new rewrite rules preserve termination

Example

TRS $\mathcal{R}$

$$x + 0 \xrightarrow{1} x \qquad x + \mathsf{s}(y) \xrightarrow{3} \mathsf{s}(x + y) \qquad \mathsf{p}(\mathsf{s}(x)) \xrightarrow{5} x$$

$$x - 0 \xrightarrow{2} x \qquad x - \mathsf{s}(y) \xrightarrow{4} \mathsf{p}(x - y) \qquad \mathsf{s}(\mathsf{p}(x)) \xrightarrow{6} x$$

$$\mathsf{s}(x + \mathsf{p}(y)) \xrightarrow{7} x + y \qquad \mathsf{p}(x - \mathsf{p}(y)) \xrightarrow{8} x - y$$

- SN (e.g.) LPO with precedence $+ > \mathsf{s}$ and $- > \mathsf{p}$
- WCR ? 4 critical pairs

$$\overline{x + \underline{\mathsf{s}(\mathsf{p}(y))}} \quad {}_6\swarrow \ \searrow_3 \quad x + y \xleftarrow[7]{} \mathsf{s}(x + \mathsf{p}(y))$$

$$\overline{x - \underline{\mathsf{s}(\mathsf{p}(y))}} \quad {}_6\swarrow \ \searrow_4 \quad x - y \xleftarrow[8]{} \mathsf{p}(x - \mathsf{p}(y))$$

$$\overline{\mathsf{p}(\underline{\mathsf{s}(\mathsf{p}(x))})} \quad {}_6\swarrow \ \searrow_5 \quad \mathsf{p}(x) = \mathsf{p}(x)$$

$$\overline{\mathsf{s}(\underline{\mathsf{p}(\mathsf{s}(x))})} \quad {}_5\swarrow \ \searrow_6 \quad \mathsf{s}(x) = \mathsf{s}(x)$$

- new rewrite rules preserve termination and do not change $\leftrightarrow^*$

Example (cont'd)

- new critical pairs

$$\mathsf{p}(x+y) \;{}_{7}\!\leftarrow\; \mathsf{p}(\mathsf{s}(x+\mathsf{p}(y))) \;\rightarrow_{5}\; x+\mathsf{p}(y)$$

$$\mathsf{s}(x+y) \;{}_{5}\!\leftarrow\; \mathsf{s}(x+\mathsf{p}(\mathsf{s}(y))) \;\rightarrow_{7}\; x+\mathsf{s}(y)$$

$$\mathsf{s}(x-y) \;{}_{8}\!\leftarrow\; \mathsf{s}(\mathsf{p}(x-\mathsf{p}(y))) \;\rightarrow_{6}\; x-\mathsf{p}(y)$$

$$\mathsf{p}(x-y) \;{}_{5}\!\leftarrow\; \mathsf{p}(x-\mathsf{p}(\mathsf{s}(y))) \;\rightarrow_{8}\; x-\mathsf{s}(y)$$

Example (cont'd)

- new critical pairs

$$\overline{\mathsf{p}(\underline{\mathsf{s}(x+\mathsf{p}(y))})} \qquad \mathsf{p}(x+y) \xleftarrow{7} \cdot \xrightarrow{5} x+\mathsf{p}(y)$$

$$\overline{\mathsf{s}(x+\underline{\mathsf{p}(\mathsf{s}(y))})} \qquad \mathsf{s}(x+y) \xleftarrow{5} \cdot \xrightarrow{7} x+\mathsf{s}(y) \qquad \mathsf{s}(x+y) \xleftarrow[3]{} x+\mathsf{s}(y)$$

$$\overline{\mathsf{s}(\underline{\mathsf{p}(x-\mathsf{p}(y))})} \qquad \mathsf{s}(x-y) \xleftarrow{8} \cdot \xrightarrow{6} x-\mathsf{p}(y)$$

$$\overline{\mathsf{p}(x-\underline{\mathsf{p}(\mathsf{s}(y))})} \qquad \mathsf{p}(x-y) \xleftarrow{5} \cdot \xrightarrow{8} x-\mathsf{s}(y) \qquad \mathsf{p}(x-y) \xleftarrow[4]{} x-\mathsf{s}(y)$$

Example (cont'd)

- new critical pairs

$$\overline{\mathsf{p}(\underline{\mathsf{s}(x+\mathsf{p}(y))})} \qquad \mathsf{p}(x+y) \;{}_{7}\!\swarrow \quad \searrow_{5}\; x+\mathsf{p}(y) \qquad \mathsf{p}(x+y) \xleftarrow[9]{} x+\mathsf{p}(y)$$

$$\overline{\mathsf{s}(x+\underline{\mathsf{p}(\mathsf{s}(y))})} \qquad \mathsf{s}(x+y) \;{}_{5}\!\swarrow \quad \searrow_{7}\; x+\mathsf{s}(y) \qquad \mathsf{s}(x+y) \xleftarrow[3]{} x+\mathsf{s}(y)$$

$$\overline{\mathsf{s}(\underline{\mathsf{p}(x-\mathsf{p}(y))})} \qquad \mathsf{s}(x-y) \;{}_{8}\!\swarrow \quad \searrow_{6}\; x-\mathsf{p}(y)$$

$$\overline{\mathsf{p}(x-\underline{\mathsf{p}(\mathsf{s}(y))})} \qquad \mathsf{p}(x-y) \;{}_{5}\!\swarrow \quad \searrow_{8}\; x-\mathsf{s}(y) \qquad \mathsf{p}(x-y) \xleftarrow[4]{} x-\mathsf{s}(y)$$

Example (cont'd)

- new critical pairs

$$\overline{\mathsf{p}(\underline{\mathsf{s}(x+\mathsf{p}(y))})} \quad \swarrow_{7} \quad \searrow_{5}$$

$$\mathsf{p}(x+y) \xleftarrow[9]{} x+\mathsf{p}(y)$$

$$\overline{\mathsf{s}(x+\underline{\mathsf{p}(\mathsf{s}(y))})} \quad \swarrow_{5} \quad \searrow_{7}$$

$$\mathsf{s}(x+y) \xleftarrow[3]{} x+\mathsf{s}(y)$$

$$\overline{\mathsf{s}(\underline{\mathsf{p}(x-\mathsf{p}(y))})} \quad \swarrow_{8} \quad \searrow_{6}$$

$$\mathsf{s}(x-y) \xleftarrow[10]{} x-\mathsf{p}(y)$$

$$\overline{\mathsf{p}(x-\underline{\mathsf{p}(\mathsf{s}(y))})} \quad \swarrow_{5} \quad \searrow_{8}$$

$$\mathsf{p}(x-y) \xleftarrow[4]{} x-\mathsf{s}(y)$$

Example (cont'd)

- new critical pairs

$$\mathsf{p}(x+y) \;\overset{7}{\swarrow}\; \overline{\mathsf{p}(\underline{\mathsf{s}(x+\mathsf{p}(y))})} \;\overset{5}{\searrow}\; x+\mathsf{p}(y) \qquad \mathsf{p}(x+y) \xleftarrow[{\color{red}9}]{} x+\mathsf{p}(y)$$

$$\mathsf{s}(x+y) \;\overset{5}{\swarrow}\; \overline{\mathsf{s}(x+\underline{\mathsf{p}(\mathsf{s}(y))})} \;\overset{7}{\searrow}\; x+\mathsf{s}(y) \qquad \mathsf{s}(x+y) \xleftarrow[3]{} x+\mathsf{s}(y)$$

$$\mathsf{s}(x-y) \;\overset{8}{\swarrow}\; \overline{\mathsf{s}(\underline{\mathsf{p}(x-\mathsf{p}(y))})} \;\overset{6}{\searrow}\; x-\mathsf{p}(y) \qquad \mathsf{s}(x-y) \xleftarrow[{\color{red}10}]{} x-\mathsf{p}(y)$$

$$\mathsf{p}(x-y) \;\overset{5}{\swarrow}\; \overline{\mathsf{p}(x-\underline{\mathsf{p}(\mathsf{s}(y))})} \;\overset{8}{\searrow}\; x-\mathsf{s}(y) \qquad \mathsf{p}(x-y) \xleftarrow[4]{} x-\mathsf{s}(y)$$

- new rewrite rules

$$x+\mathsf{p}(y) \xrightarrow{9} \mathsf{p}(x+y) \qquad\qquad x-\mathsf{p}(y) \xrightarrow{10} \mathsf{s}(x-y)$$

preserve termination (extend LPO precedence with $+ > \mathsf{p}$ and $- > \mathsf{s}$)

Example (cont'd)

- new critical pairs

$$x + (y + z) \xleftarrow{7} \overline{x + \underline{\mathsf{s}(y + \mathsf{p}(z))}} \xrightarrow{3} \mathsf{s}(x + (y + \mathsf{p}(z)))$$

$$x - (y + z) \xleftarrow{7} \overline{x - \underline{\mathsf{s}(y + \mathsf{p}(z))}} \xrightarrow{4} \mathsf{p}(x - (y + \mathsf{p}(z)))$$

$$x + y \xleftarrow{5} \overline{x + \underline{\mathsf{p}(\mathsf{s}(y))}} \xrightarrow{9} \mathsf{p}(x + \mathsf{s}(y))$$

$$\mathsf{s}(x + (y - z)) \xleftarrow{8} \overline{\mathsf{s}(x + \underline{\mathsf{p}(y - \mathsf{p}(z))})} \xrightarrow{7} x + (y - \mathsf{p}(z))$$

$$\mathsf{p}(x - (y - z)) \xleftarrow{8} \overline{\mathsf{p}(x - \underline{\mathsf{p}(y - \mathsf{p}(z))})} \xrightarrow{8} x - (y - \mathsf{p}(z))$$

$$x - y \xleftarrow{5} \overline{x - \underline{\mathsf{p}(\mathsf{s}(y))}} \xrightarrow{10} \mathsf{s}(x - \mathsf{s}(y))$$

Example (cont'd)

- new critical pairs

$$x + (y + z) \xleftarrow{7} \overline{x + \underline{\mathsf{s}(y + \mathsf{p}(z))}} \xrightarrow{3} \mathsf{s}(x + (y + \mathsf{p}(z)))$$
$$x + (y + z) \xleftarrow{7} \mathsf{s}(x + \mathsf{p}(y + z)) \xleftarrow{9} \mathsf{s}(x + (y + \mathsf{p}(z)))$$

$$x - (y + z) \xleftarrow{7} \overline{x - \underline{\mathsf{s}(y + \mathsf{p}(z))}} \xrightarrow{4} \mathsf{p}(x - (y + \mathsf{p}(z)))$$
$$x - (y + z) \xleftarrow{8} \mathsf{p}(x - \mathsf{p}(y + z)) \xleftarrow{9} \mathsf{p}(x - (y + \mathsf{p}(z)))$$

$$x + y \xleftarrow{5} \overline{x + \underline{\mathsf{p}(\mathsf{s}(y))}} \xrightarrow{9} \mathsf{p}(x + \mathsf{s}(y))$$
$$x + y \xleftarrow{5} \mathsf{p}(\mathsf{s}(x + y)) \xleftarrow{3} \mathsf{p}(x + \mathsf{s}(y))$$

$$\mathsf{s}(x + (y - z)) \xleftarrow{8} \overline{\mathsf{s}(x + \underline{\mathsf{p}(y - \mathsf{p}(z))})} \xrightarrow{7} x + (y - \mathsf{p}(z))$$
$$\mathsf{s}(x + (y - z)) \xleftarrow{3} x + \mathsf{s}(y - z) \xleftarrow{10} x + (y - \mathsf{p}(z))$$

$$\mathsf{p}(x - (y - z)) \xleftarrow{8} \overline{\mathsf{p}(x - \underline{\mathsf{p}(y - \mathsf{p}(z))})} \xrightarrow{8} x - (y - \mathsf{p}(z))$$
$$\mathsf{p}(x - (y - z)) \xleftarrow{4} x - \mathsf{s}(y - z) \xleftarrow{10} x - (y - \mathsf{p}(z))$$

$$x - y \xleftarrow{5} \overline{x - \underline{\mathsf{p}(\mathsf{s}(y))}} \xrightarrow{10} \mathsf{s}(x - \mathsf{s}(y))$$
$$x - y \xleftarrow{6} \mathsf{s}(\mathsf{p}(x - y)) \xleftarrow{4} \mathsf{s}(x - \mathsf{s}(y))$$

Example (cont'd)

- new critical pairs

$$x + (y - z) \xleftarrow{8} \overline{x + \underline{\mathsf{p}(y - \mathsf{p}(z))}} \xrightarrow{9} \mathsf{p}(x + (y - \mathsf{p}(z)))$$

$$x - (y - z) \xleftarrow{8} \overline{x - \underline{\mathsf{p}(y - \mathsf{p}(z))}} \xrightarrow{10} \mathsf{s}(x - (y - \mathsf{p}(z)))$$

$$\mathsf{s}(\mathsf{p}(x + y)) \xleftarrow{9} \overline{\mathsf{s}(\underline{x + \mathsf{p}(y)})} \xrightarrow{7} x + y$$

$$\mathsf{p}(\mathsf{s}(x - y)) \xleftarrow{10} \overline{\mathsf{p}(\underline{x - \mathsf{p}(y)})} \xrightarrow{8} x - y$$

Example (cont'd)

- new critical pairs

$$\overline{x + \underline{\mathsf{p}(y - \mathsf{p}(z))}}$$

$$x + (y - z) \;{}_{8}\!\leftarrow\; \overline{x + \underline{\mathsf{p}(y - \mathsf{p}(z))}} \;\rightarrow_{9}\; \mathsf{p}(x + (y - \mathsf{p}(z)))$$

$$\mathsf{p}(x + (y - \mathsf{p}(z))) \rightarrow_{10} \mathsf{p}(x + \mathsf{s}(y - z)) \rightarrow_{3} \mathsf{p}(\mathsf{s}(x + (y - z))) \rightarrow_{5} x + (y - z)$$

$$x - (y - z) \;{}_{8}\!\leftarrow\; \overline{x - \underline{\mathsf{p}(y - \mathsf{p}(z))}} \;\rightarrow_{10}\; \mathsf{s}(x - (y - \mathsf{p}(z)))$$

$$\mathsf{s}(x - (y - \mathsf{p}(z))) \rightarrow_{10} \mathsf{s}(x - \mathsf{s}(y - z)) \rightarrow_{4} \mathsf{s}(\mathsf{p}(x - (y - z))) \rightarrow_{6} x - (y - z)$$

$$\mathsf{s}(\mathsf{p}(x + y)) \;{}_{9}\!\leftarrow\; \overline{\mathsf{s}(\underline{x + \mathsf{p}(y)})} \;\rightarrow_{7}\; x + y \qquad \mathsf{s}(\mathsf{p}(x + y)) \rightarrow_{6} x + y$$

$$\mathsf{p}(\mathsf{s}(x - y)) \;{}_{10}\!\leftarrow\; \overline{\mathsf{p}(\underline{x - \mathsf{p}(y)})} \;\rightarrow_{8}\; x - y \qquad \mathsf{p}(\mathsf{s}(x - y)) \rightarrow_{5} x - y$$

Example (cont'd)

TRS $\mathcal{R} = \{1,\ 2,\ 3,\ 4,\ 5,\ 6\}$

$$x + 0 \xrightarrow{1} x \qquad x + \mathsf{s}(y) \xrightarrow{3} \mathsf{s}(x + y) \qquad \mathsf{p}(\mathsf{s}(x)) \xrightarrow{5} x$$

$$x - 0 \xrightarrow{2} x \qquad x - \mathsf{s}(y) \xrightarrow{4} \mathsf{p}(x - y) \qquad \mathsf{s}(\mathsf{p}(x)) \xrightarrow{6} x$$

Example (cont'd)

TRS $\mathcal{R} = \{1, 2, 3, 4, 5, 6\}$ TRS $\mathcal{S} = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$

$$x + 0 \xrightarrow{1} x \qquad x + \mathsf{s}(y) \xrightarrow{3} \mathsf{s}(x + y) \qquad \mathsf{p}(\mathsf{s}(x)) \xrightarrow{5} x$$

$$x - 0 \xrightarrow{2} x \qquad x - \mathsf{s}(y) \xrightarrow{4} \mathsf{p}(x - y) \qquad \mathsf{s}(\mathsf{p}(x)) \xrightarrow{6} x$$

$$\mathsf{s}(x + \mathsf{p}(y)) \xrightarrow{7} x + y \qquad \mathsf{p}(x - \mathsf{p}(y)) \xrightarrow{8} x - y$$

$$x + \mathsf{p}(y) \xrightarrow{9} \mathsf{p}(x + y) \qquad x - \mathsf{p}(y) \xrightarrow{10} \mathsf{s}(x - y)$$

Example (cont'd)

TRS $\mathcal{R} = \{1, 2, 3, 4, 5, 6\}$ TRS $\mathcal{S} = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$

$$x + 0 \xrightarrow{1} x \qquad x + \mathsf{s}(y) \xrightarrow{3} \mathsf{s}(x + y) \qquad \mathsf{p}(\mathsf{s}(x)) \xrightarrow{5} x$$

$$x - 0 \xrightarrow{2} x \qquad x - \mathsf{s}(y) \xrightarrow{4} \mathsf{p}(x - y) \qquad \mathsf{s}(\mathsf{p}(x)) \xrightarrow{6} x$$

$$\mathsf{s}(x + \mathsf{p}(y)) \xrightarrow{7} x + y \qquad \mathsf{p}(x - \mathsf{p}(y)) \xrightarrow{8} x - y$$

$$x + \mathsf{p}(y) \xrightarrow{9} \mathsf{p}(x + y) \qquad x - \mathsf{p}(y) \xrightarrow{10} \mathsf{s}(x - y)$$

- $\mathcal{S}$ is SN LPO with precedence $+ > \mathsf{s}, \mathsf{p}$ and $- > \mathsf{s}, \mathsf{p}$

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- $\mathcal{S}$ is SN — LPO with precedence $+ > \mathsf{s}, \mathsf{p}$ and $- > \mathsf{s}, \mathsf{p}$
- $\mathcal{S}$ is WCR — all critical pairs of $\mathcal{S}$ are joinable
- $\leftrightarrow^*_{\mathcal{S}} = \leftrightarrow^*_{\mathcal{R}}$

Outline

Knuth–Bendix Completion Procedure

input: ES $\mathcal{E}$ and reduction order $>$

output: complete TRS $\mathcal{R}$ such that $\leftrightarrow_{\mathcal{E}}^{*} = \leftrightarrow_{\mathcal{R}}^{*}$

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Notation

$\mathrm{CP}(\mathcal{R}, \mathcal{S})$ consists of critical pairs originating from overlaps $\langle \alpha, p, \beta \rangle$ with $\alpha \in \mathcal{R}$ and $\beta \in \mathcal{S}$

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Three Possibilities

Knuth–Bendix completion procedure may

① terminate without failure

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③ not terminate (divergence)

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Example

- rewrite rules

$$\mathsf{f}(x,y) \to \mathsf{g}(x) \qquad \mathsf{f}(x,y) \to \mathsf{h}(y)$$

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$$\mathsf{f}(x,y) \to \mathsf{g}(x) \qquad \mathsf{f}(x,y) \to \mathsf{h}(y)$$

- two critical pairs

$$\mathsf{g}(x) \approx \mathsf{h}(y) \qquad \mathsf{h}(y) \approx \mathsf{g}(x)$$

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- no orientation possible $\Longrightarrow$ failure

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$$\mathsf{f}(\mathsf{g}(x)) \to \mathsf{g}(\mathsf{h}(x))$$
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$$\mathsf{g}(\mathsf{a}) \to \mathsf{b} \qquad \mathsf{g}(\mathsf{h}(\mathsf{h}(\mathsf{a}))) \to \mathsf{f}(\mathsf{f}(\mathsf{b})) \qquad \cdots$$

- LPO with precedence $\mathsf{a} > \mathsf{f} > \mathsf{g} > \mathsf{h} > \mathsf{b}$
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$$\mathsf{f}(\mathsf{b}) \approx \mathsf{g}(\mathsf{h}(\mathsf{a})) \qquad \mathsf{f}(\mathsf{f}(\mathsf{b})) \approx \mathsf{g}(\mathsf{h}(\mathsf{h}(\mathsf{a}))) \qquad \mathsf{f}(\mathsf{f}(\mathsf{f}(\mathsf{b}))) \approx \mathsf{g}(\mathsf{h}(\mathsf{h}(\mathsf{h}(\mathsf{a}))))$$

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Example

- rewrite rules

$$\mathsf{f}(\mathsf{g}(x)) \leftarrow \mathsf{g}(\mathsf{h}(x))$$
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- LPO with precedence $\mathsf{g} > \mathsf{f} > \mathsf{b}$
- no critical pairs

Outline

Critical Pair Lemma

TRS is locally confluent $\iff$ all critical pairs are joinable

Corollary

terminating TRS is confluent $\iff$ all critical pairs are joinable

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Definition (Prime Critical Pair)

- critical pair $t \approx u$ derived from critical peak $t\ {}_p{\leftarrow}\ s \rightarrow_\epsilon u$ is prime if all proper subterms of $s|_p$ are in normal form

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- critical pair $t \approx u$ derived from critical peak $t \; {}_p{\leftarrow} \; s \to_\epsilon u$ is prime if all proper subterms of $s|_p$ are in normal form
- PCP($\mathcal{R}$) denotes set of all prime critical pairs of TRS $\mathcal{R}$

Example

TRS

$$\begin{aligned} \mathsf{e} \cdot x &\to x & x \cdot \mathsf{e} &\to x \\ x^- \cdot x &\to \mathsf{e} & x \cdot x^- &\to \mathsf{e} \\ (x \cdot y) \cdot z &\to x \cdot (y \cdot z) & x^{--} &\to x \\ \mathsf{e}^- &\to \mathsf{e} & (x \cdot y)^- &\to y^- \cdot x^- \\ x^- \cdot (x \cdot y) &\to y & x \cdot (x^- \cdot y) &\to y \end{aligned}$$

some critical peaks

1. $y \cdot \mathsf{e} \xleftarrow[2]{} y \cdot (y^- \cdot y^{--}) \xrightarrow[\epsilon]{} y^{--}$
2. $\mathsf{e} \cdot \mathsf{e} \xleftarrow[1]{} \mathsf{e}^- \cdot \mathsf{e} \xrightarrow[\epsilon]{} \mathsf{e}$
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Theorem

terminating TRS is confluent $\iff$ all prime critical pairs are joinable

Outline

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most problems for finite left−linear right−ground TRSs are decidable

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First−Order Theory of Rewriting

- first−order logic over language $\mathcal{L}$ without function symbols

Remark

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$\to^*$	relation $\to^*_{\mathcal{R}}$	$\mathrel{+\!\!\!\!\!\to}$	parallel rewriting
$=$	identity relation	$\to_\epsilon$	one–step rewriting at root
$\to^!$	$s \to^*_{\mathcal{R}} t$ and $t \in \mathsf{NF}(\mathcal{R})$	$\to_{>\epsilon}$	one–step rewriting below root

Examples

- $\forall t\ \forall u\ \forall v\ \left(t \to^* u \wedge t \to^* v \implies \exists w\ (u \to^* w \wedge v \to^* w)\right)$

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- $\exists s\ \exists t\ \big(s \mathrel{+\!\!\!\!\!\!\to} t \land \neg(s \to t) \land \neg(s = t)\big)$

Additional Predicate

- $\mathcal{L}$ is extended with unary predicate symbol $\mathsf{INF}_\circ$

$$\mathsf{INF}_\circ(t) \iff \{u \mid t \circ u\} \text{ is infinite}$$

for every boolean combination $\circ$ of binary predicate symbols in $\mathcal{L}$

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Remark

- properties are restricted to ground terms
- so $\forall t\ \forall u\ \forall v\ (t \to^* u \wedge t \to^* v \implies u \downarrow v)$ stands for ground–confluence

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decision procedure for first-order theory of rewriting for left-linear right-ground TRSs is based on tree automata techniques

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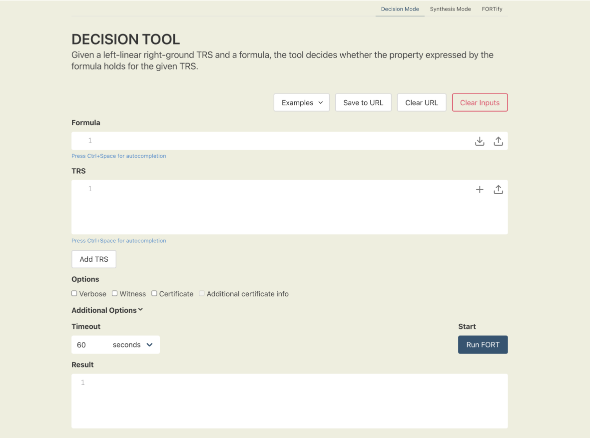
decision procedure for first–order theory of rewriting for left–linear right–ground TRSs is based on tree automata techniques and implemented in FORT



Outline

Homework Exercises for May 4

1. Exercise 5.5. 2
2. Exercise 5.6. 2
3. Exercise 5.14. 3

Outline

Lecture Notes

- Section 5.1 (from Definition 5.1.13)
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Important Concepts

- completion
- divergence
- first–order theory of rewriting
- prime critical pair