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Term Rewriting

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Outline

Knuth–Bendix Completion Procedure (Simple Version)

input: ES $\mathcal{E}$ and reduction order $>$

output: complete TRS $\mathcal{R}$ such that $\leftrightarrow^*_{\mathcal{E}} = \leftrightarrow^*_{\mathcal{R}}$

$\mathcal{R} := \varnothing \quad C := \mathcal{E}$

while $C \neq \varnothing$ **do**

  select $s \approx t \in C$

  $C := C \setminus \{s \approx t\}$

  compute $\mathcal{R}$–normal forms s' and t' of s and t

  if $s' \neq t'$ **then**

    if $s' > t'$ **then** $\mathcal{S} := \{s' \to t'\}$

    else if $t' > s'$ **then** $\mathcal{S} := \{t' \to s'\}$

    else failure

    $C := C \cup \mathrm{CP}(\mathcal{R}, \mathcal{S}) \cup \mathrm{CP}(\mathcal{S}, \mathcal{R}) \cup \mathrm{CP}(\mathcal{S})$

    $C := \mathcal{R} \cup \mathcal{S}$

Three Possibilities

Knuth–Bendix completion procedure may

① terminate without failure $\implies$ $\mathcal{R}$ is complete and $\leftrightarrow^*_{\mathcal{E}} = \leftrightarrow^*_{\mathcal{R}}$

② terminate with failure

③ not terminate (divergence)

Definition

critical pair $t \approx u$ derived from critical peak $t \ {}_p\!\leftarrow s \rightarrow_\epsilon u$ is prime if all proper subterms of $s|_p$ are in normal form

Theorem

terminating TRS is confluent $\iff$ all prime critical pairs are joinable

Theorem

first–order theory of rewriting is decidable for left–linear right–ground TRSs

Outline

Definitions (Weight Function)

- weight function (w, w_0) consists of mapping $w\colon \mathcal{F} \to \mathbb{N}$ and constant $w_0 > 0$ such that $w(c) \geqslant w_0$ for all constants $c \in \mathcal{F}$
- weight of term t:

$$w(t) = \begin{cases} w_0 & \text{if } t \in \mathcal{V} \\ w(f) + \sum_{i=1}^{n} w(t_i) & \text{if } t = f(t_1, \dots, t_n) \end{cases}$$

- weight function (w, w_0) is admissible for precedence $>$ if

$$f > g \quad \text{for all } g \in \mathcal{F} \setminus \{f\}$$

whenever f is unary function symbol in $\mathcal{F}$ with $w(f) = 0$

Example

- rewrite rules

$$\begin{aligned} \mathsf{e} \cdot x &\to x & x \cdot \mathsf{e} &\to x \\ x^- \cdot x &\to \mathsf{e} & x \cdot x^- &\to \mathsf{e} \\ (x \cdot y) \cdot z &\to x \cdot (y \cdot z) & x^{--} &\to x \\ \mathsf{e}^- &\to \mathsf{e} & (x \cdot y)^- &\to y^- \cdot x^- \\ x^- \cdot (x \cdot y) &\to y & x \cdot (x^- \cdot y) &\to y \end{aligned}$$

- weight function: $w(\mathsf{e}) = w(\cdot) = w_0 = 1 \qquad w(^-) = 0$

$$w(\mathsf{e} \cdot x) = 3 \qquad w(x) = 1 \qquad w((x \cdot y)^-) = 3$$

- precedence: $^- > \cdot > \mathsf{e}$
- admissible because $^-$ is maximal in precedence

Definition (Knuth–Bendix Order)

binary relation $>_{\text{kbo}}$ on terms: $s >_{\text{kbo}} t$ if $|s|_x \geqslant |t|_x$ for all $x \in \mathcal{V}$ and either

- $w(s) > w(t)$
- $w(s) = w(t)$ and either
  1. $s = f^n(t)$ for some $n > 0$ and $t \in \mathcal{V}$
  2. $s = f(s_1, \ldots, s_n)$ and $t = f(t_1, \ldots, t_n)$ and for some $1 \leqslant i \leqslant n$
     - **a** $s_j = t_j$ for all $1 \leqslant j < i$
     - **b** $s_i >_{\text{kbo}} t_i$
  3. $s = f(s_1, \ldots, s_n)$ and $t = g(t_1, \ldots, t_m)$ and $f > g$

Theorem

$>_{\text{kbo}}$ is reduction order if $>$ is well–founded and (w, w_0) is admissible for $>$

Example ①

- rewrite rules

$$\begin{aligned} e \cdot x &\to x & x \cdot e &\to x \\ x^- \cdot x &\to e & x \cdot x^- &\to e \\ (x \cdot y) \cdot z &\to x \cdot (y \cdot z) & x^{--} &\to x \\ e^- &\to e & (x \cdot y)^- &\to y^- \cdot x^- \\ x^- \cdot (x \cdot y) &\to y & x \cdot (x^- \cdot y) &\to y \end{aligned}$$

- weight function $\quad w(e) = w(\cdot) = w_0 = 1 \qquad w({}^-) = 0$
- precedence $\quad {}^- > \cdot > e$

$$e \cdot x >_{kbo} x \qquad x^{--} >_{kbo} x \qquad (x \cdot y)^- >_{kbo} y^- \cdot x^-$$

Example ❷

- rewrite rules

$$\mathsf{aa} \to \mathsf{bbb} \qquad\qquad \mathsf{bbbbb} \to \mathsf{aaa}$$

- weight function and precedence

$$w(\mathsf{a}) = 3 \qquad w(\mathsf{b}) = 2 \qquad \mathsf{a} > \mathsf{b}$$

$$w(\mathsf{a}) = 5 \qquad w(\mathsf{b}) = 3 \qquad \mathsf{b} > \mathsf{a}$$

$$w(\mathsf{a}) = 13 \qquad w(\mathsf{b}) = 8$$

Example ❸

- rewrite rules

$$
\begin{array}{lllll}
0+0 \to 0 & 1+0 \to 1 & \cdots & 9+0 \to 9 & 0:x \to x \\
0+1 \to 1 & 1+1 \to 2 & \cdots & 9+1 \to 1:0 & x+(y:z) \to y:(x+z) \\
0+2 \to 2 & 1+2 \to 3 & \cdots & 9+2 \to 1:1 & (x:y)+z \to x:(y+z) \\
0+3 \to 3 & 1+3 \to 4 & \cdots & 9+3 \to 1:2 & x:(y:z) \to (x+y):z \\
0+4 \to 4 & 1+4 \to 5 & \cdots & 9+4 \to 1:3 & \\
0+5 \to 5 & 1+5 \to 6 & \cdots & 9+5 \to 1:4 & \\
0+6 \to 6 & 1+6 \to 7 & \cdots & 9+6 \to 1:5 & \\
0+7 \to 7 & 1+7 \to 8 & \cdots & 9+7 \to 1:6 & \\
0+8 \to 8 & 1+8 \to 9 & \cdots & 9+8 \to 1:7 & \\
0+9 \to 9 & 1+9 \to 1:0 & \cdots & 9+9 \to 1:8 &
\end{array}
$$

- weight function $w(0) = w(1) = w(2) = w(3) = w(4) = w(+) = w_0 = 1 \quad w(:) = 2$

  $w(5) = w(6) = w(7) = w(8) = w(9) = 3$

- precedence $+ > : \quad + > 5 \quad + > 6 \quad + > 7 \quad + > 8$

Example ④

- rewrite rules

$$11 \to 43 \qquad 33 \to 56 \qquad 55 \to 62$$
$$12 \to 21 \qquad 34 \to 11 \qquad 56 \to 12$$
$$22 \to 111 \qquad 44 \to 3 \qquad 66 \to 21$$

- weight function and precedence

$$w(1) = 31 \qquad w(2) = 47 \qquad w(3) = 41$$
$$w(4) = 21 \qquad w(5) = 43 \qquad w(6) = 3$$
$$3 > 5 > 6 > 1 > 4 \qquad 1 > 2$$

Theorem

- if $> \subseteq \sqsupset$ then $>_{\mathsf{kbo}} \subseteq \sqsupset_{\mathsf{kbo}}$ (incrementality)
- if $>$ is total then $>_{\mathsf{kbo}}$ is total on ground terms
- following two problems are decidable:

1. instance: finite TRS $\mathcal{R}$, weight function (w, w_0), precedence $>$

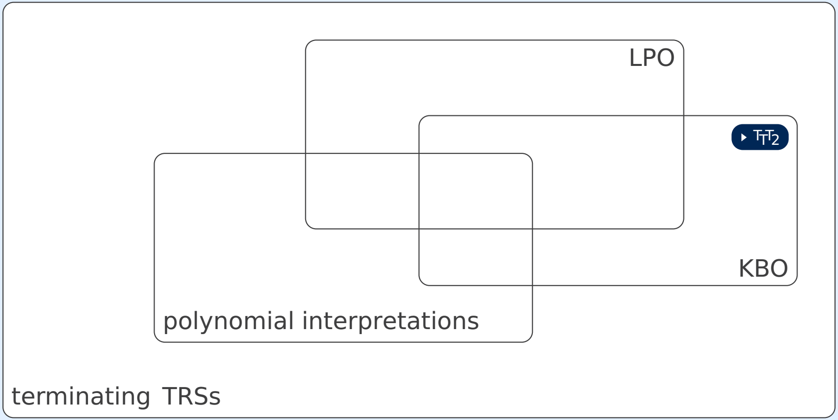
   question: $\mathcal{R} \subseteq >_{\mathsf{kbo}}$?

2. instance: finite TRS $\mathcal{R}$

   question: $\exists$ weight function (w, w_0) and $\exists$ precedence $>$ such that (w, w_0) is admissible for $>$ and $\mathcal{R} \subseteq >_{\mathsf{kbo}}$?

Remark

KBO, LPO and polynomial interpretations are incomparable



Outline

Example

TRS $\mathcal{R} = \{1, 2, 3, 4, 5, 6\}$ TRS $\mathcal{S} = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$

$$
\begin{aligned}
x + 0 &\xrightarrow{1} x & x + \mathsf{s}(y) &\xrightarrow{3} \mathsf{s}(x + y) & \mathsf{p}(\mathsf{s}(x)) &\xrightarrow{5} x \\
x - 0 &\xrightarrow{2} x & x - \mathsf{s}(y) &\xrightarrow{4} \mathsf{p}(x - y) & \mathsf{s}(\mathsf{p}(x)) &\xrightarrow{6} x \\
\mathsf{s}(x + \mathsf{p}(y)) &\xrightarrow{7} x + y & \mathsf{p}(x - \mathsf{p}(y)) &\xrightarrow{8} x - y & & \\
x + \mathsf{p}(y) &\xrightarrow{9} \mathsf{p}(x + y) & x - \mathsf{p}(y) &\xrightarrow{10} \mathsf{s}(x - y) & &
\end{aligned}
$$

rewrite rules 7 and 8 are redundant:

$$\mathsf{s}(x + \mathsf{p}(y)) \xrightarrow{7} x + y \qquad \mathsf{s}(x + \mathsf{p}(y)) \xrightarrow{9} \mathsf{s}(\mathsf{p}(x + y)) \xrightarrow{6} x + y$$

$$\mathsf{p}(x - \mathsf{p}(y)) \xrightarrow{8} x - y \qquad \mathsf{p}(x - \mathsf{p}(y)) \xrightarrow{10} \mathsf{p}(\mathsf{s}(x - y)) \xrightarrow{5} x - y$$

Observation

- fewer rewrite rules $\Longrightarrow$ fewer critical pairs
- TRS without redundancy $=$ reduced TRS

Definition (Canonicity)

- TRS $\mathcal{R}$ is reduced if for all $\ell \to r \in \mathcal{R}$
  ① r is normal form of $\mathcal{R}$
  ② ℓ is normal form of $\mathcal{R} \setminus \{\ell \to r\}$
- reduced complete TRS is canonical

Example

TRS $\mathcal{R} = \{1, 2, 3, 4, 5, 6\}$ TRS $\mathcal{S} = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$

$$x + 0 \xrightarrow{1} x \qquad x + \mathsf{s}(y) \xrightarrow{3} \mathsf{s}(x + y) \qquad \mathsf{p}(\mathsf{s}(x)) \xrightarrow{5} x$$

$$x - 0 \xrightarrow{2} x \qquad x - \mathsf{s}(y) \xrightarrow{4} \mathsf{p}(x - y) \qquad \mathsf{s}(\mathsf{p}(x)) \xrightarrow{6} x$$

$$\mathsf{s}(x + \mathsf{p}(y)) \xrightarrow{7} x + y \qquad \mathsf{p}(x - \mathsf{p}(y)) \xrightarrow{8} x - y$$

$$x + \mathsf{p}(y) \xrightarrow{9} \mathsf{p}(x + y) \qquad x - \mathsf{p}(y) \xrightarrow{10} \mathsf{s}(x - y)$$

- $\mathcal{R}$ is reduced
- $\mathcal{S}$ is not reduced

Definitions (Equivalence)

- TRSs $\mathcal{R}$ and $\mathcal{S}$ are conversion equivalent if $\leftrightarrow^*_{\mathcal{R}} \;=\; \leftrightarrow^*_{\mathcal{S}}$
- TRSs $\mathcal{R}$ and $\mathcal{S}$ are normalization equivalent if $\to^!_{\mathcal{R}} \;=\; \to^!_{\mathcal{S}}$

Example

ARSs

$$\mathcal{A}_1: \quad \mathsf{a} \longrightarrow \mathsf{b} \qquad\qquad \mathcal{B}_1: \quad \mathsf{a} \longleftarrow \mathsf{b}$$

$$\mathcal{A}_2: \quad \mathsf{a} \longrightarrow \mathsf{b} \circlearrowright \qquad\qquad \mathcal{B}_2: \; \circlearrowright \mathsf{a} \qquad \mathsf{b} \circlearrowright$$

- $\mathcal{A}_1$ and $\mathcal{B}_1$ are conversion equivalent but not normalization equivalent
- $\mathcal{A}_2$ and $\mathcal{B}_2$ are normalization equivalent but not conversion equivalent

Lemma

normalization equivalent terminating TRSs are conversion equivalent

Theorem

normalization equivalent reduced TRSs are unique up to literal similarity

Theorem

conversion equivalent canonical TRSs that are compatible with same reduction order are unique up to literal similarity simplification after completion

Notation

$t{\downarrow_{\mathcal{R}}}$ denotes unique normal form t for complete TRS $\mathcal{R}$

Definition

- $\dot{\mathcal{R}} = \{\ell \to r{\downarrow_{\mathcal{R}}} \mid \ell \to r \in \mathcal{R}\}$
- $\ddot{\mathcal{R}} = \{\ell \to r \in \dot{\mathcal{R}} \mid \ell \in \mathsf{NF}(\dot{\mathcal{R}} \setminus \{\ell \to r\})\}$

Theorem

if $\mathcal{R}$ is complete TRS then $\ddot{\mathcal{R}}$ is (normalization) equivalent canonical TRS

more efficient: simplification during completion

Outline

Definition (Abstract Completion)

set of equations $\mathcal{E}$ set of rewrite rules $\mathcal{R}$ reduction order $>$

inference system KB consists of eight rules

delete $$\frac{\mathcal{E} \uplus \{s \approx s\}, \mathcal{R}}{\mathcal{E}, \mathcal{R}}$$

deduce $$\frac{\mathcal{E}, \mathcal{R}}{\mathcal{E} \cup \{s \approx t\}, \mathcal{R}}$$ if $s \;{}_{\mathcal{R}}\!\leftarrow \cdot \rightarrow_{\mathcal{R}} t$

compose $$\frac{\mathcal{E}, \mathcal{R} \uplus \{s \rightarrow t\}}{\mathcal{E}, \mathcal{R} \cup \{s \rightarrow u\}}$$ if $t \rightarrow_{\mathcal{R}} u$

collapse $$\frac{\mathcal{E}, \mathcal{R} \uplus \{t \rightarrow s\}}{\mathcal{E} \cup \{u \approx s\}, \mathcal{R}}$$ if $t \rightarrow_{\mathcal{R}} u$

orient $$\frac{\mathcal{E} \uplus \{s \approx t\}, \mathcal{R}}{\mathcal{E}, \mathcal{R} \cup \{s \rightarrow t\}} \qquad \frac{\mathcal{E} \uplus \{t \approx s\}, \mathcal{R}}{\mathcal{E}, \mathcal{R} \cup \{s \rightarrow t\}}$$ if $s > t$

simplify $$\frac{\mathcal{E} \uplus \{s \approx t\}, \mathcal{R}}{\mathcal{E} \cup \{s \approx u\}, \mathcal{R}} \qquad \frac{\mathcal{E} \uplus \{t \approx s\}, \mathcal{R}}{\mathcal{E} \cup \{u \approx s\}, \mathcal{R}}$$ if $t \rightarrow_{\mathcal{R}} u$

Example

$$\begin{aligned} \mathsf{g(b)} &\approx \mathsf{g(b)} \\ \mathsf{f(b)} &\approx \mathsf{g(f(a))} \end{aligned}$$

$$\begin{aligned} \mathsf{f(f(}x\mathsf{))} &\to \mathsf{g(}x\mathsf{)} \\ \mathsf{g(a)} &\to \mathsf{b} \\ \mathsf{f(g(}x\mathsf{))} &\to \mathsf{g(f(}x\mathsf{))} \\ \mathsf{f(b)} &\to \mathsf{g(f(a))} \end{aligned}$$

- LPO with precedence $\mathsf{f} > \mathsf{g} > \mathsf{b} > \mathsf{a}$
- canonical TRS

Example

$$\mathsf{f}(\mathsf{f}(x)) \approx \mathsf{g}(x) \qquad\qquad \mathsf{g}(x) \to \mathsf{f}(\mathsf{f}(x))$$

$$\mathsf{g}(\mathsf{a}) \approx \mathsf{b} \qquad\qquad \mathsf{b} \to \mathsf{f}(\mathsf{f}(\mathsf{a}))$$

- LPO with precedence $\mathsf{b} > \mathsf{g} > \mathsf{f} > \mathsf{a}$ canonical TRS

Example

$$\mathsf{f}(\mathsf{f}(\mathsf{a})) \approx \mathsf{b} \qquad \mathsf{g}(x) \to \mathsf{f}(\mathsf{f}(x))$$

$$\mathsf{g}(\mathsf{a}) \approx \mathsf{b} \qquad \mathsf{f}(\mathsf{f}(\mathsf{a})) \to \mathsf{b}$$

- LPO with precedence $\mathsf{g} > \mathsf{f} > \mathsf{b} > \mathsf{a}$ canonical TRS

Lemma

if $(\mathcal{E}, \mathcal{R}) \vdash_{\mathsf{KB}}^* (\mathcal{E}', \mathcal{R}')$ then

1. $\leftrightarrow^*_{\mathcal{E} \cup \mathcal{R}} = \leftrightarrow^*_{\mathcal{E}' \cup \mathcal{R}'}$
2. $\mathcal{R} \subseteq > \quad \Longrightarrow \quad \mathcal{R}' \subseteq >$

Definition (Fair Run)

run for given ES $\mathcal{E}$ is finite sequence

$$\mathcal{E}_0, \mathcal{R}_0 \vdash_{\mathsf{KB}} \mathcal{E}_1, \mathcal{R}_1 \vdash_{\mathsf{KB}} \cdots \vdash_{\mathsf{KB}} \mathcal{E}_n, \mathcal{R}_n$$

such that $\mathcal{E}_0 = \mathcal{E}$ and $\mathcal{R}_0 = \varnothing$

- run fails if $\mathcal{E}_n \neq \varnothing$
- run is fair if $\mathsf{PCP}(\mathcal{R}_n) \subseteq \downarrow_{\mathcal{R}_n} \cup \bigcup_{i=0}^{n} \leftrightarrow_{\mathcal{E}_i}$

Theorem

for every fair non–failing run

$$\mathcal{E}_0, \mathcal{R}_0 \vdash_{\mathsf{KB}} \mathcal{E}_1, \mathcal{R}_1 \vdash_{\mathsf{KB}} \cdots \vdash_{\mathsf{KB}} \mathcal{E}_n, \mathcal{R}_n$$

$\mathcal{R}_n$ is complete presentation of $\mathcal{E}_0$

Completion Tools

Waldmeister mkbTT KBCV Maxcomp

Example

- ES $\mathcal{E}$

$$\mathsf{f}(x) \approx \mathsf{f}(\mathsf{a}) \qquad \mathsf{f}(\mathsf{b}) \approx \mathsf{b}$$

- consider reduction order $>$
  - if $\mathsf{f}(\mathsf{b}) \not> \mathsf{b}$ then $(\mathcal{E}, \varnothing)$ is normal form of $\vdash_{\mathsf{KB}}$
  - if $\mathsf{f}(\mathsf{b}) > \mathsf{b}$ then

$$\begin{aligned} \mathcal{E}, \varnothing &\vdash_{\mathsf{KB}} \{\mathsf{f}(x) \approx \mathsf{f}(\mathsf{a})\}, \{\mathsf{f}(\mathsf{b}) \to \mathsf{b}\} \\ &\vdash_{\mathsf{KB}} \{\mathsf{f}(x) \approx \mathsf{f}(\mathsf{a}), \mathsf{b} \approx \mathsf{b}\}, \{\mathsf{f}(\mathsf{b}) \to \mathsf{b}\} \\ &\vdash_{\mathsf{KB}} \{\mathsf{f}(x) \approx \mathsf{f}(\mathsf{a})\}, \{\mathsf{f}(\mathsf{b}) \to \mathsf{b}\} \\ &\vdash_{\mathsf{KB}} \cdots \end{aligned}$$

    so completion fails

- TRS $\mathcal{R}$ consisting of single rule $\mathsf{f}(x) \to \mathsf{b}$ is complete presentation of $\mathcal{E}$

Outline

A team of genetic engineers decides to create cows that produce cola instead of milk. To that end they have to transform the DNA of the milk gene

TAGCTAGCTAGCT

in every fertilized egg into the cola gene

CTGACTGACT

Techniques exist to perform the following DNA substitutions

TCAT $\leftrightarrow$ T GAG $\leftrightarrow$ AG CTC $\leftrightarrow$ TC AGTA $\leftrightarrow$ A TAT $\leftrightarrow$ CT

Recently it has been discovered that the mad cow disease is caused by a retrovirus with the following DNA sequence CTGCTACTGACT. What now, if accidentally cows with this virus are created? According to the engineers there is little risk because this never happened in their experiments, but various action groups demand absolute assurance.

Demo

KBCV developed by Thomas Sternagel

Example (Cola Gene Puzzle)

$\mathcal{E}$ $\quad$ $\mathsf{TCAT} \approx \mathsf{T}$ $\quad$ $\mathsf{GAG} \approx \mathsf{AG}$ $\quad$ $\mathsf{CTC} \approx \mathsf{TC}$ $\quad$ $\mathsf{AGTA} \approx \mathsf{A}$ $\quad$ $\mathsf{TAT} \approx \mathsf{CT}$

$\mathcal{R}$ $\quad$ $\mathsf{GA} \to \mathsf{A}$ $\quad$ $\mathsf{AGT} \to \mathsf{AT}$ $\quad$ $\mathsf{ATA} \to \mathsf{A}$ $\quad$ $\mathsf{CT} \to \mathsf{T}$ $\quad$ $\mathsf{TAT} \to \mathsf{T}$ $\quad$ $\mathsf{TCA} \to \mathsf{TA}$

- $\mathcal{R}$ is canonical presentation of $\mathcal{E}$
- (milk gene) $\mathsf{TAGCTAGCTAGCT} \leftrightarrow^*_{\mathcal{E}} \mathsf{CTGACTGACT}$ (cola gene)

  $$\mathsf{TAGCTAGCTAGCT} \to^!_{\mathcal{R}} \mathsf{T} \; {}_{\mathcal{R}}\!\!\leftarrow^! \mathsf{CTGACTGACT}$$

- (milk gene) $\mathsf{TAGCTAGCTAGCT} \not\leftrightarrow^*_{\mathcal{E}} \mathsf{CTGCTACTGACT}$ (mad cow retrovirus)

  $$\mathsf{TAGCTAGCTAGCT} \to^!_{\mathcal{R}} \mathsf{T} \neq \mathsf{TGT} \; {}_{\mathcal{R}}\!\!\leftarrow^! \mathsf{CTGCTACTGACT}$$

Outline

Homework Exercises for May 11

1. Exercise 4.34. ①
2. Exercise 4.37. ①
3. Exercise 5.22. ③
4. Exercise 5.31. ②
5. Exercise 4.41. ☆☆☆

Outline

Lecture Notes

- Section 1.2 (from Definition 1.2.18)
- Section 4.4
- Section 5.3
- Section 5.4

Additional Literature

- H. Zankl, N. Hirokawa and A. Middeldorp, KBO Orientability, Journal of Automated Reasoning 43(2), pp. 173–201, 2009

Important Concepts

- abstract completion
- admissibility
- canonical
- conversion equivalence
- failing run
- fair run
- Knuth–Bendix order
- normalization equivalence
- reduced
- run
- weight function