universität innsbruck

Term Rewriting

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Outline

Definitions

- weight function (w, w_0) consists of mapping $w\colon \mathcal{F} \to \mathbb{N}$ and constant $w_0 > 0$ such that $w(c) \geqslant w_0$ for all constants $c \in \mathcal{F}$
- weight of term t: $w(t) = w_0$ if $t \in \mathcal{V}$ and $w(t) = w(f) + \sum_{i=1}^{n} w(t_i)$ if $t = f(t_1, \ldots, t_n)$
- weight function (w, w_0) is admissible for precedence $>$ if $f > g$ for all $g \in \mathcal{F} \setminus \{f\}$, whenever f is unary function symbol in $\mathcal{F}$ with $w(f) = 0$
- binary relation $>_{\text{kbo}}$ on terms (Knuth–Bendix order): $s >_{\text{kbo}} t$ if $|s|_x \geqslant |t|_x$ for all $x \in \mathcal{V}$ and either $w(s) > w(t)$ or both $w(s) = w(t)$ and either

  ① $s = f^n(t)$ for some $n > 0$ and $t \in \mathcal{V}$

  ② $s = f(s_1, \ldots, s_n)$ and $t = f(t_1, \ldots, t_n)$ and for some $1 \leqslant i \leqslant n$

  a $s_j = t_j$ for all $1 \leqslant j < i$

  b $s_i >_{\text{kbo}} t_i$

  ③ $s = f(s_1, \ldots, s_n)$ and $t = g(t_1, \ldots, t_m)$ and $f > g$

Theorem

- $>_{\text{kbo}}$ is reduction order if $>$ is well–founded and (w, w_0) is admissible for $>$
- if $> \subseteq \sqsupset$ then $>_{\text{kbo}} \subseteq \sqsupset_{\text{kbo}}$ (incrementality)
- if $>$ is total then $>_{\text{kbo}}$ is total on ground terms
- following problem is decidable:

  instance: finite TRS $\mathcal{R}$

  question: $\exists$ weight function (w, w_0) / $\exists$ precedence $>$ such that (w, w_0) is admissible for $>$ and $\mathcal{R} \subseteq >_{\text{kbo}}$?

Definitions

- TRS $\mathcal{R}$ is reduced if $r \in \text{NF}(\mathcal{R})$ and $\ell \in \text{NF}(\mathcal{R} \setminus \{\ell \to r\})$ for all $\ell \to r \in \mathcal{R}$
- reduced complete TRS is canonical

Definitions

- TRSs $\mathcal{R}$ and $\mathcal{S}$ are **conversion equivalent** if $\leftrightarrow^*_{\mathcal{R}} = \leftrightarrow^*_{\mathcal{S}}$
- TRSs $\mathcal{R}$ and $\mathcal{S}$ are **normalization equivalent** if $\rightarrow^!_{\mathcal{R}} = \rightarrow^!_{\mathcal{S}}$

Theorem

conversion equivalent canonical TRSs that are compatible with same reduction order are unique up to literal similarity

Definitions

- $\dot{\mathcal{R}} = \{\ell \to r{\downarrow_{\mathcal{R}}} \mid \ell \to r \in \mathcal{R}\}$
- $\ddot{\mathcal{R}} = \{\ell \to r \in \dot{\mathcal{R}} \mid \ell \in \mathsf{NF}(\dot{\mathcal{R}} \setminus \{\ell \to r\})\}$

Theorem

if $\mathcal{R}$ is complete TRS then $\ddot{\mathcal{R}}$ is (normalization) equivalent canonical TRS

Definition

set of equations $\mathcal{E}$ set of rewrite rules $\mathcal{R}$ reduction order $>$

inference system KB consists of eight rules

delete $\dfrac{\mathcal{E} \uplus \{s \approx s\}, \mathcal{R}}{\mathcal{E}, \mathcal{R}}$

deduce $\dfrac{\mathcal{E}, \mathcal{R}}{\mathcal{E} \cup \{s \approx t\}, \mathcal{R}}$ if $s \;{}_{\mathcal{R}}{\leftarrow} \cdot \rightarrow_{\mathcal{R}} t$

compose $\dfrac{\mathcal{E}, \mathcal{R} \uplus \{s \rightarrow t\}}{\mathcal{E}, \mathcal{R} \cup \{s \rightarrow u\}}$ if $t \rightarrow_{\mathcal{R}} u$

collapse $\dfrac{\mathcal{E}, \mathcal{R} \uplus \{t \rightarrow s\}}{\mathcal{E} \cup \{u \approx s\}, \mathcal{R}}$ if $t \rightarrow_{\mathcal{R}} u$

orient $\dfrac{\mathcal{E} \uplus \{s \approx t\}, \mathcal{R}}{\mathcal{E}, \mathcal{R} \cup \{s \rightarrow t\}}$ $\dfrac{\mathcal{E} \uplus \{t \approx s\}, \mathcal{R}}{\mathcal{E}, \mathcal{R} \cup \{s \rightarrow t\}}$ if $s > t$

simplify $\dfrac{\mathcal{E} \uplus \{s \approx t\}, \mathcal{R}}{\mathcal{E} \cup \{s \approx u\}, \mathcal{R}}$ $\dfrac{\mathcal{E} \uplus \{t \approx s\}, \mathcal{R}}{\mathcal{E} \cup \{u \approx s\}, \mathcal{R}}$ if $t \rightarrow_{\mathcal{R}} u$

Definition

run for ES $\mathcal{E}$ is finite sequence

$$\mathcal{E}_0, \mathcal{R}_0 \vdash_{\mathsf{KB}} \mathcal{E}_1, \mathcal{R}_1 \vdash_{\mathsf{KB}} \cdots \vdash_{\mathsf{KB}} \mathcal{E}_n, \mathcal{R}_n$$

such that $\mathcal{E}_0 = \mathcal{E}$ and $\mathcal{R}_0 = \varnothing$

- run fails if $\mathcal{E}_n \neq \varnothing$
- run is fair if $\mathsf{PCP}(\mathcal{R}_n) \subseteq \downarrow_{\mathcal{R}_n} \cup \bigcup_{i=0}^{n} \leftrightarrow_{\mathcal{E}_i}$

Theorem

$\mathcal{R}_n$ is complete presentation of $\mathcal{E}_0$ for every fair non–failing run $(\mathcal{E}_0, \mathcal{R}_0) \vdash_{\mathsf{KB}} \cdots \vdash_{\mathsf{KB}} (\mathcal{E}_n, \mathcal{R}_n)$

Outline

Example

TRS $\mathcal{R}$ modeling Sieve of Eratosthenes for generating list of prime numbers

$$\begin{aligned}
\mathsf{primes} &\to \mathsf{sieve}(\mathsf{from}(\mathsf{s}(\mathsf{s}(0)))) & \mathsf{sieve}(0 : y) &\to \mathsf{sieve}(y)\\
\mathsf{from}(x) &\to x : \mathsf{from}(\mathsf{s}(x)) & \mathsf{sieve}(\mathsf{s}(x) : y) &\to \mathsf{s}(x) : \mathsf{sieve}(\mathsf{filter}(x, y, x))\\
\mathsf{head}(x : y) &\to x & \mathsf{filter}(0, y : z, w) &\to 0 : \mathsf{filter}(w, z, w)\\
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- $\mathcal{R}$ is confluent but not terminating:

$$\mathsf{from}(0) \to 0 : \mathsf{from}(\mathsf{s}(0)) \to 0 : (\mathsf{s}(0) : \mathsf{from}(\mathsf{s}(\mathsf{s}(0)))) \to \cdots$$

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- $\exists$ non−terminating terms with (unique) normal form

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- how to compute normal forms in $\mathcal{R}$? strategy (lectures 10, 11)

Confluence Methods

- critical pair closing systems
- decreasing diagrams
- development closed critical pairs
- discrimination pairs
- joinable critical pairs for terminating systems
- orthogonality
- parallel closed critical pairs
- parallel critical pairs
- redundant rules
- rule labeling
- simultaneous critical pairs
- source labeling
- strongly closed critical pairs
- tree automata
- weak orthogonality
- Z property
- ⋯

Confluence Methods

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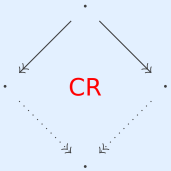
Confluence

every two co−initial rewrite sequences can be joined

- · · · yields uniqueness of normal forms



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Confluence

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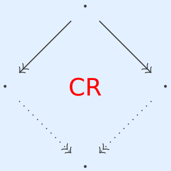

Confluence Methods

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joinable critical pairs for terminating systems

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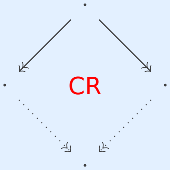
Confluence

every two co−initial rewrite sequences can be joined

- · · · yields uniqueness of normal forms
- · · · is decidable for terminating TRSs
- · · · what about non−terminating TRSs ?



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$$\begin{aligned} \mathsf{f}(x,x) &\to \mathsf{a} \\ \mathsf{g}(x) &\to \mathsf{f}(x,\mathsf{g}(x)) \\ \mathsf{c} &\to \mathsf{g}(\mathsf{c}) \end{aligned}$$

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control interference of rewrite rules

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Confluence via Orthogonality

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orthogonal TRS is left–linear and lacks critical pairs

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$$\begin{aligned} \mathsf{I} \cdot x &\to x \\ (\mathsf{K} \cdot x) \cdot y &\to x \\ ((\mathsf{S} \cdot x) \cdot y) \cdot z &\to (x \cdot z) \cdot (y \cdot z) \end{aligned}$$

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\end{aligned}
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$$
\begin{aligned}
\mathsf{ack}(0, y) &\to \mathsf{s}(y) \\
\mathsf{ack}(\mathsf{s}(x), 0) &\to \mathsf{ack}(x, \mathsf{s}(0)) \\
\mathsf{ack}(\mathsf{s}(x), \mathsf{s}(y)) &\to \mathsf{ack}(x, \mathsf{ack}(\mathsf{s}(x), y))
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$$(\mathsf{K} \cdot x) \cdot y \to x \qquad \mathsf{ack}(\mathsf{s}(x), 0) \to \mathsf{ack}(x, \mathsf{s}(0))$$

$$((\mathsf{S} \cdot x) \cdot y) \cdot z \to (x \cdot z) \cdot (y \cdot z) \qquad \mathsf{ack}(\mathsf{s}(x), \mathsf{s}(y)) \to \mathsf{ack}(x, \mathsf{ack}(\mathsf{s}(x), y))$$

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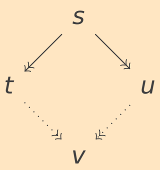

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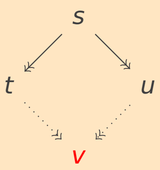

$\exists\, v$

Observation

for orthogonal TRSs there is canonical way to compute common reduct v

Definition (Parallel Rewriting)

parallel rewriting $\dashrightarrow\!\!\!\!\!\!\!+\!\!\!\!+$ is inductively defined as follows:

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$$\frac{x \in \mathcal{V}}{x \mathrel{+\!\!\!+\!\!\!\!\rightarrow} x} \qquad \frac{s \to t}{s \mathrel{+\!\!\!+\!\!\!\!\rightarrow} t} \qquad \frac{s_1 \mathrel{+\!\!\!+\!\!\!\!\rightarrow} t_1 \quad \cdots \quad s_n \mathrel{+\!\!\!+\!\!\!\!\rightarrow} t_n}{f(s_1, \dots, s_n) \mathrel{+\!\!\!+\!\!\!\!\rightarrow} f(t_1, \dots, t_n)}$$

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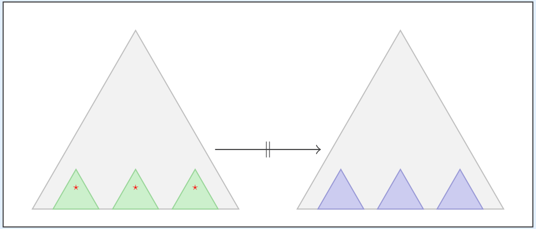
Lemma

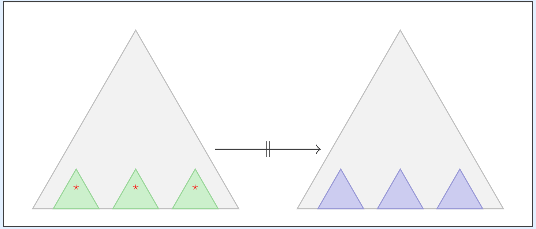
$s \mathrel{+\!\!\!\!\!\!\longrightarrow} t \iff$ $\exists$ context C with $n \geqslant 0$ holes, $\exists$ terms $s_1, \dots, s_n, t_1, \dots, t_n$ such that

$s = C[s_1, \dots, s_n]$

$t = C[t_1, \dots, t_n]$

$s_i \to t_i$ for all $1 \leqslant i \leqslant n$





Example

- TRS

$$0 + y \to y \qquad 0 \times y \to 0$$
$$\mathsf{s}(x) + y \to \mathsf{s}(x + y) \qquad \mathsf{s}(x) \times y \to (x \times y) + y$$

- rewrite sequences

$$\mathsf{s}(0 \times 0) + \mathsf{s}(0) \times (0 + \mathsf{s}(0)) \to^* \mathsf{s}(0) + (0 \times (0 + \mathsf{s}(0)) + (0 + \mathsf{s}(0)))$$
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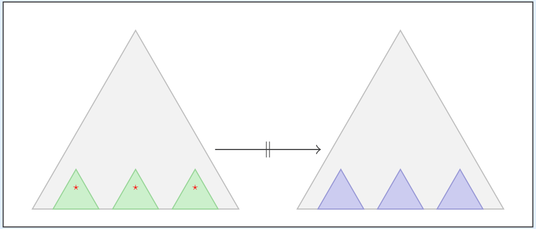

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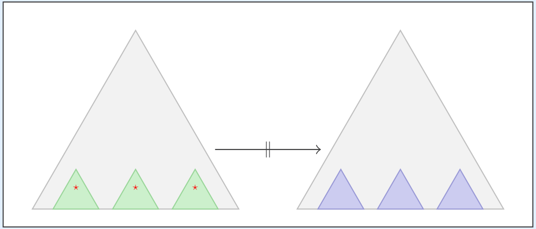

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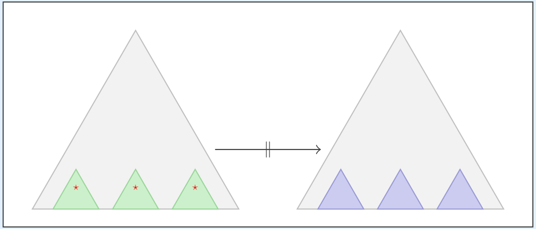

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Lemma

$\rightarrow \; \subseteq \; \mathrel{+\!\!\!\!\!+\!\!\!\!\rightarrow} \; \subseteq \; \rightarrow^*$

Lemma

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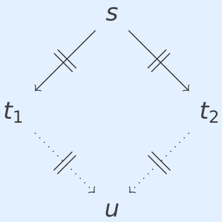
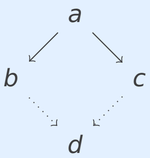
Parallel Moves Lemma



Definition (Diamond Property)

- diamond property $\diamond$
  - $\leftarrow \cdot \rightarrow \; \subseteq \; \rightarrow \cdot \leftarrow$
  - $\forall\, a, b, c$

    $\exists\, d$

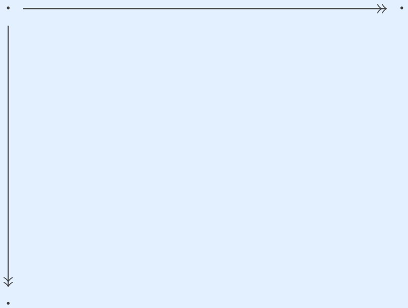
Lemma

ARS $\langle A, \rightarrow \rangle$ is confluent if $\rightarrow \; \subseteq \; \rightarrowtail \; \subseteq \; \rightarrow^*$ for some relation $\rightarrowtail$ on A with diamond property

Corollary

orthogonal TRSs are confluent

Proof



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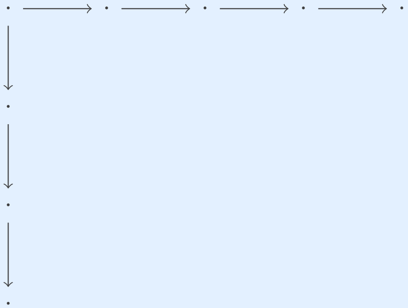

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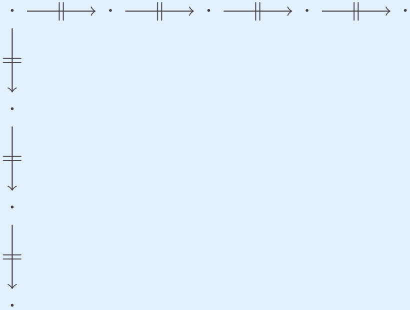

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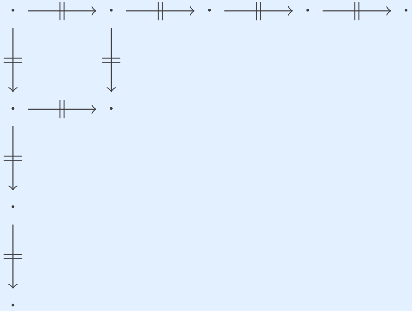

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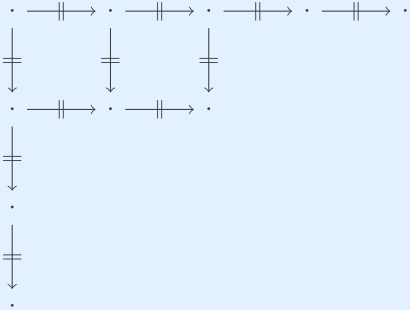

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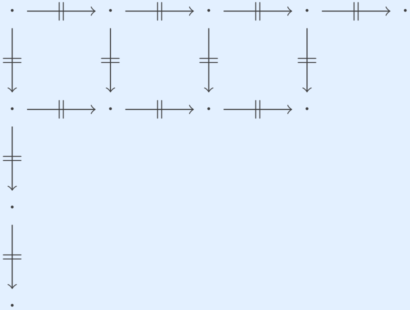

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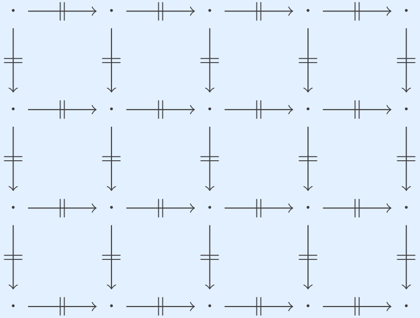

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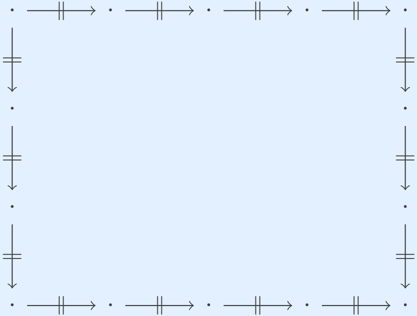

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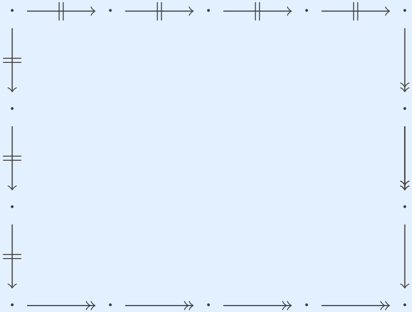

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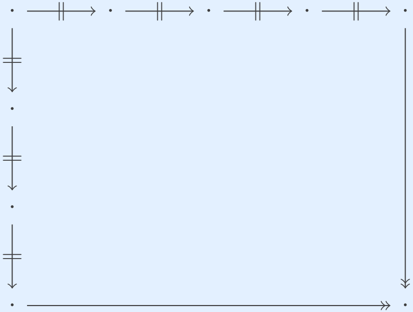

Outline

Definition (Multi–Step Rewriting)

multi–step relation $\multimap\!\!\!\rightarrow$ is inductively defined as follows:

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Example

TRS

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$$\dfrac{\dfrac{\dfrac{\mathsf{0} \times y \to \mathsf{0} \qquad \dfrac{}{\mathsf{0} \mathrel{\circ\!\!\!\!\!\!\longrightarrow} \mathsf{0}}\,(2)}{\mathsf{0} \times \mathsf{0} \mathrel{\circ\!\!\!\!\!\!\longrightarrow} \mathsf{0}}\,(3)}{\mathsf{s}(\mathsf{0} \times \mathsf{0}) \mathrel{\circ\!\!\!\!\!\!\longrightarrow} \mathsf{s}(\mathsf{0})}\,(2) \qquad \dfrac{\mathsf{s}(x) \times y \to (x \times y) + y \qquad \dfrac{}{\mathsf{0} \mathrel{\circ\!\!\!\!\!\!\longrightarrow} \mathsf{0}}\,(2) \qquad \dfrac{\mathsf{0} + y \to y \qquad \dfrac{\dfrac{}{\mathsf{0} \mathrel{\circ\!\!\!\!\!\!\longrightarrow} \mathsf{0}}\,(2)}{\mathsf{s}(\mathsf{0}) \mathrel{\circ\!\!\!\!\!\!\longrightarrow} \mathsf{s}(\mathsf{0})}\,(2)}{\mathsf{0} + \mathsf{s}(\mathsf{0}) \mathrel{\circ\!\!\!\!\!\!\longrightarrow} \mathsf{s}(\mathsf{0})}\,(3)}{\mathsf{s}(\mathsf{0}) \times (\mathsf{0} + \mathsf{s}(\mathsf{0})) \mathrel{\circ\!\!\!\!\!\!\longrightarrow} (\mathsf{0} \times \mathsf{s}(\mathsf{0})) + \mathsf{s}(\mathsf{0})}\,(3)}{\mathsf{s}(\mathsf{0} \times \mathsf{0}) + \mathsf{s}(\mathsf{0}) \times (\mathsf{0} + \mathsf{s}(\mathsf{0})) \mathrel{\circ\!\!\!\!\!\!\longrightarrow} \mathsf{s}(\mathsf{0}) + ((\mathsf{0} \times \mathsf{s}(\mathsf{0})) + \mathsf{s}(\mathsf{0}))}\,(2)$$

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TRS

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$$\dfrac{\dfrac{\dfrac{0 \times y \to 0 \qquad \dfrac{}{0 \mathrel{\circ\!\!\!\!\!\longrightarrow} 0}\,②}{0 \times 0 \mathrel{\circ\!\!\!\!\!\longrightarrow} 0}\,③}{\mathsf{s}(0 \times 0) \mathrel{\circ\!\!\!\!\!\longrightarrow} \mathsf{s}(0)}\,② \qquad \dfrac{\mathsf{s}(x) \times y \to (x \times y) + y \qquad \dfrac{}{0 \mathrel{\circ\!\!\!\!\!\longrightarrow} 0}\,② \qquad \dfrac{0 + y \to y \qquad \dfrac{\dfrac{}{0 \mathrel{\circ\!\!\!\!\!\longrightarrow} 0}\,②}{\mathsf{s}(0) \mathrel{\circ\!\!\!\!\!\longrightarrow} \mathsf{s}(0)}\,②}{0 + \mathsf{s}(0) \mathrel{\circ\!\!\!\!\!\longrightarrow} \mathsf{s}(0)}\,③}{\mathsf{s}(0) \times (0 + \mathsf{s}(0)) \mathrel{\circ\!\!\!\!\!\longrightarrow} (0 \times \mathsf{s}(0)) + \mathsf{s}(0)}\,③}{\mathsf{s}(0 \times 0) + \mathsf{s}(0) \times (0 + \mathsf{s}(0)) \mathrel{\circ\!\!\!\!\!\longrightarrow} \mathsf{s}(0) + ((0 \times \mathsf{s}(0)) + \mathsf{s}(0))}\,②$$

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$$

$$
\dfrac{
\dfrac{
\dfrac{0 \times y \rightarrow 0 \qquad \dfrac{}{0 \mathrel{\circ\!\!\!\!\rightarrow} 0}\,②}{0 \times 0 \mathrel{\circ\!\!\!\!\rightarrow} 0}\,③
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\qquad
\dfrac{
\mathsf{s}(x) \times y \rightarrow (x \times y) + y \qquad \dfrac{}{0 \mathrel{\circ\!\!\!\!\rightarrow} 0}\,② \qquad
\dfrac{0 + y \rightarrow y \qquad \dfrac{\dfrac{}{0 \mathrel{\circ\!\!\!\!\rightarrow} 0}\,②}{\mathsf{s}(0) \mathrel{\circ\!\!\!\!\rightarrow} \mathsf{s}(0)}\,②}{0 + \mathsf{s}(0) \mathrel{\circ\!\!\!\!\rightarrow} \mathsf{s}(0)}\,③
}{\mathsf{s}(0) \times (0 + \mathsf{s}(0)) \mathrel{\circ\!\!\!\!\rightarrow} (0 \times \mathsf{s}(0)) + \mathsf{s}(0)}\,③
}{\mathsf{s}(0 \times 0) + \mathsf{s}(0) \times (0 + \mathsf{s}(0)) \mathrel{\circ\!\!\!\!\rightarrow} \mathsf{s}(0) + ((0 \times \mathsf{s}(0)) + \mathsf{s}(0))}\,②
$$

① $x \mathrel{-\!\!\circ\!\!\!\to} x$ for all variables x

② $f(s_1, \dots, s_n) \mathrel{-\!\!\circ\!\!\!\to} f(t_1, \dots, t_n)$ if $s_i \mathrel{-\!\!\circ\!\!\!\to} t_i$ for all $1 \leqslant i \leqslant n$

③ $\ell\sigma \mathrel{-\!\!\circ\!\!\!\to} r\tau$ if $\ell \to r \in \mathcal{R}$ and $\sigma \mathrel{-\!\!\circ\!\!\!\to} \tau$

Example

TRS

$$0 + y \to y \qquad 0 \times y \to 0$$

$$\mathsf{s}(x) + y \to \mathsf{s}(x + y) \qquad \mathsf{s}(x) \times y \to (x \times y) + y$$

$$\dfrac{\dfrac{\dfrac{0 \times y \to 0 \qquad \dfrac{}{0 \mathrel{-\!\!\circ\!\!\!\to} 0}\,②}{0 \times 0 \mathrel{-\!\!\circ\!\!\!\to} 0}\,③}{\mathsf{s}(0 \times 0) \mathrel{-\!\!\circ\!\!\!\to} \mathsf{s}(0)}\,② \qquad \dfrac{\mathsf{s}(x) \times y \to (x \times y) + y \qquad \dfrac{}{0 \mathrel{-\!\!\circ\!\!\!\to} 0}\,② \qquad \dfrac{0 + y \to y \qquad \dfrac{\dfrac{}{0 \mathrel{-\!\!\circ\!\!\!\to} 0}\,②}{\mathsf{s}(0) \mathrel{-\!\!\circ\!\!\!\to} \mathsf{s}(0)}\,②}{0 + \mathsf{s}(0) \mathrel{-\!\!\circ\!\!\!\to} \mathsf{s}(0)}\,③}{\mathsf{s}(0) \times (0 + \mathsf{s}(0)) \mathrel{-\!\!\circ\!\!\!\to} (0 \times \mathsf{s}(0)) + \mathsf{s}(0)}\,③}{\mathsf{s}(0 \times 0) + \mathsf{s}(0) \times (0 + \mathsf{s}(0)) \mathrel{-\!\!\circ\!\!\!\to} \mathsf{s}(0) + ((0 \times \mathsf{s}(0)) + \mathsf{s}(0))}\,②$$

① $x \mathrel{\circ\!\!\!\rightarrow} x$ for all variables x

② $f(s_1, \ldots, s_n) \mathrel{\circ\!\!\!\rightarrow} f(t_1, \ldots, t_n)$ if $s_i \mathrel{\circ\!\!\!\rightarrow} t_i$ for all $1 \leqslant i \leqslant n$

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Example

TRS

$$0 + y \to y \qquad 0 \times y \to 0$$

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$$\dfrac{\dfrac{\dfrac{0 \times y \to 0 \qquad \dfrac{}{0 \mathrel{\circ\!\!\!\rightarrow} 0}\,②}{0 \times 0 \mathrel{\circ\!\!\!\rightarrow} 0}\,③}{\mathsf{s}(0 \times 0) \mathrel{\circ\!\!\!\rightarrow} \mathsf{s}(0)}\,② \qquad \dfrac{\mathsf{s}(x) \times y \to (x \times y) + y \qquad \dfrac{}{0 \mathrel{\circ\!\!\!\rightarrow} 0}\,② \qquad \dfrac{0 + y \to y \qquad \dfrac{\dfrac{}{0 \mathrel{\circ\!\!\!\rightarrow} 0}\,②}{\mathsf{s}(0) \mathrel{\circ\!\!\!\rightarrow} \mathsf{s}(0)}\,②}{0 + \mathsf{s}(0) \mathrel{\circ\!\!\!\rightarrow} \mathsf{s}(0)}\,③}{\mathsf{s}(0) \times (0 + \mathsf{s}(0)) \mathrel{\circ\!\!\!\rightarrow} (0 \times \mathsf{s}(0)) + \mathsf{s}(0)}\,③}{\mathsf{s}(0 \times 0) + \mathsf{s}(0) \times (0 + \mathsf{s}(0)) \mathrel{\circ\!\!\!\rightarrow} \mathsf{s}(0) + ((0 \times \mathsf{s}(0)) + \mathsf{s}(0))}\,②$$

Lemma

$\to \;\subseteq\; \mathrel{-\!\!\Vert\!\!\to} \;\subseteq\; \mathrel{-\!\!\circ\!\!\to} \;\subseteq\; \to^*$

Lemma

$\to \;\subseteq\; \mathrel{+\!\!\!\!\!\!\;|\!\!\to} \;\subseteq\; \mathrel{\circ\!\!\!\to} \;\subseteq\; \to^*$

Definition (Maximal Multi-Step Rewriting)

maximal multi-step relation $\mathrel{\bullet\!\!\!\to}$ is inductively defined as follows:

① $x \mathrel{\bullet\!\!\!\to} x$ for all variables x

② $f(s_1, \ldots, s_n) \mathrel{\bullet\!\!\!\to} f(t_1, \ldots, t_n)$ if $s_i \mathrel{\bullet\!\!\!\to} t_i$ for all $1 \leqslant i \leqslant n$ and $f(s_1, \ldots, s_n)$ is no redex

③ $\ell\sigma \mathrel{\bullet\!\!\!\to} r\tau$ if $\ell \to r \in \mathcal{R}$ and $\sigma \mathrel{\bullet\!\!\!\to} \tau$

Lemma

$\rightarrow \subseteq \;\not\Vdash\!\!\rightarrow \subseteq \;\multimap\!\!\!\rightarrow \subseteq \rightarrow^*$

Definition (Maximal Multi–Step Rewriting)

maximal multi–step relation $\twoheadrightarrow\!\!\!\bullet$ is inductively defined as follows:

① $x \,{-\!\!\bullet\!\!\!\rightarrow}\, x$ for all variables x

② $f(s_1, \ldots, s_n) \,{-\!\!\bullet\!\!\!\rightarrow}\, f(t_1, \ldots, t_n)$ if $s_i \,{-\!\!\bullet\!\!\!\rightarrow}\, t_i$ for all $1 \leqslant i \leqslant n$ and $f(s_1, \ldots, s_n)$ is no redex

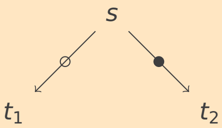
③ $\ell\sigma \,{-\!\!\bullet\!\!\!\rightarrow}\, r\tau$ if $\ell \rightarrow r \in \mathcal{R}$ and $\sigma \,{-\!\!\bullet\!\!\!\rightarrow}\, \tau$

Remark

${-\!\!\bullet\!\!\!\rightarrow}$ is deterministic for orthogonal TRSs

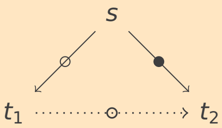
Lemma

for orthogonal TRSs



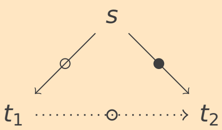
Lemma

for orthogonal TRSs **triangle property**



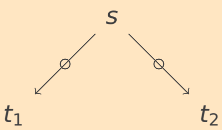
Lemma

for orthogonal TRSs triangle property



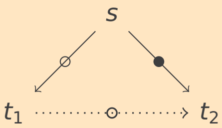
Corollary

for orthogonal TRSs



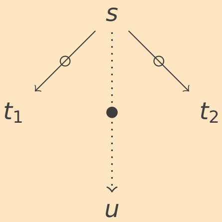
Lemma

for orthogonal TRSs triangle property



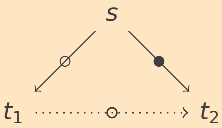
Corollary

for orthogonal TRSs



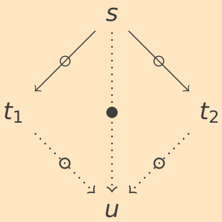
Lemma

for orthogonal TRSs — triangle property



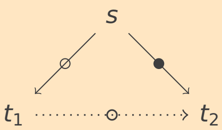
Corollary

for orthogonal TRSs



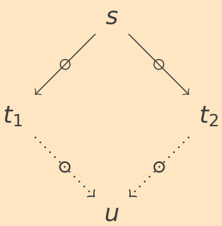
Lemma

for orthogonal TRSs — triangle property



Corollary

for orthogonal TRSs



Outline

Definition

TRS is strongly closed if $t \to^{=} \cdot\ {}^{*}\!\leftarrow u$ and $t \to^{*} \cdot\ {}^{=}\!\leftarrow u$ for every critical pair $t \approx u$

Definition

TRS is strongly closed if $t \to^{=} \cdot {}^{*}\!\leftarrow u$ and $t \to^{*} \cdot {}^{=}\!\leftarrow u$ for every critical pair $t \approx u$

Theorem

linear strongly closed TRSs are confluent

Definition

TRS is strongly closed if $t \rightarrow^{=} \cdot {}^{*}\!\leftarrow u$ and $t \rightarrow^{*} \cdot {}^{=}\!\leftarrow u$ for every critical pair $t \approx u$

Theorem

linear strongly closed TRSs are confluent

Example ①

- TRS $\mathcal{R}$

$$\mathsf{f}(\mathsf{f}(x,y),z) \rightarrow \mathsf{f}(x,\mathsf{f}(y,z)) \qquad \mathsf{f}(x,y) \rightarrow \mathsf{f}(y,x)$$

Definition

TRS is strongly closed if $t \to^= \cdot\, {}^*\!\leftarrow u$ and $t \to^* \cdot\, {}^=\!\leftarrow u$ for every critical pair $t \approx u$

Theorem

linear strongly closed TRSs are confluent

Example ①

- TRS $\mathcal{R}$

$$\mathsf{f}(\mathsf{f}(x,y),z) \to \mathsf{f}(x,\mathsf{f}(y,z)) \qquad \mathsf{f}(x,y) \to \mathsf{f}(y,x)$$

- 4 critical pairs

$$\mathsf{f}(\mathsf{f}(x,\mathsf{f}(y,z)),v) \approx \mathsf{f}(\mathsf{f}(x,y),\mathsf{f}(z,v)) \qquad \mathsf{f}(x,\mathsf{f}(y,z)) \approx \mathsf{f}(z,\mathsf{f}(x,y))$$

$$\mathsf{f}(z,\mathsf{f}(x,y)) \approx \mathsf{f}(x,\mathsf{f}(y,z)) \qquad \mathsf{f}(\mathsf{f}(y,x),z) \approx \mathsf{f}(x,\mathsf{f}(y,z))$$

Definition

TRS is strongly closed if $t \to^{=} \cdot {}^{*}\!\leftarrow u$ and $t \to^{*} \cdot {}^{=}\!\leftarrow u$ for every critical pair $t \approx u$

Theorem

linear strongly closed TRSs are confluent

Example 1

- TRS $\mathcal{R}$

$$\mathsf{f}(\mathsf{f}(x,y),z) \to \mathsf{f}(x,\mathsf{f}(y,z)) \qquad \mathsf{f}(x,y) \to \mathsf{f}(y,x)$$

- 4 critical pairs

$$\mathsf{f}(\mathsf{f}(x,\mathsf{f}(y,z)),v) \approx \mathsf{f}(\mathsf{f}(x,y),\mathsf{f}(z,v)) \qquad \mathsf{f}(x,\mathsf{f}(y,z)) \approx \mathsf{f}(z,\mathsf{f}(x,y))$$

$$\mathsf{f}(z,\mathsf{f}(x,y)) \approx \mathsf{f}(x,\mathsf{f}(y,z)) \qquad \mathsf{f}(\mathsf{f}(y,x),z) \approx \mathsf{f}(x,\mathsf{f}(y,z))$$

- $\mathcal{R}$ is linear and strongly closed $\implies$ $\mathcal{R}$ is confluent

Example ❷

- linear TRS $\mathcal{R}$

$$(x + y) + z \to x + (y + z) \qquad x + (y + z) \to (x + y) + z \qquad x + y \to y + x$$

Example ❷

- linear TRS $\mathcal{R}$

$$(x+y)+z \to x+(y+z) \qquad x+(y+z) \to (x+y)+z \qquad x+y \to y+x$$

- 12 critical pairs

$$(x+(y+z))+w \approx (x+y)+(z+w) \qquad (y+x)+z \approx x+(y+z)$$
$$((x+y)+z)+w \approx x+((y+z)+w) \qquad z+(y+x) \approx x+(y+z)$$
$$((x+y)+z)+w \approx x+(y+(z+w)) \qquad x+(y+z) \approx z+(y+x)$$
$$x+(y+(z+w)) \approx ((x+y)+z)+w \qquad x+(z+y) \approx (x+y)+z$$
$$x+(y+(z+w)) \approx (x+(y+z))+w \qquad (y+z)+x \approx (x+y)+z$$
$$x+((y+z)+w) \approx (x+y)+(z+w) \qquad (x+y)+z \approx (y+z)+x$$

Example ❷

- linear TRS $\mathcal{R}$

$$(x+y)+z \to x+(y+z) \qquad x+(y+z) \to (x+y)+z \qquad x+y \to y+x$$

- 12 critical pairs

$$\begin{aligned}
(x+(y+z))+w &\approx (x+y)+(z+w) & (y+x)+z &\approx x+(y+z) \\
((x+y)+z)+w &\approx x+((y+z)+w) & z+(y+x) &\approx x+(y+z) \\
((x+y)+z)+w &\approx x+(y+(z+w)) & x+(y+z) &\approx z+(y+x) \\
x+(y+(z+w)) &\approx ((x+y)+z)+w & x+(z+y) &\approx (x+y)+z \\
x+(y+(z+w)) &\approx (x+(y+z))+w & (y+z)+x &\approx (x+y)+z \\
x+((y+z)+w) &\approx (x+y)+(z+w) & (x+y)+z &\approx (y+z)+x
\end{aligned}$$

- $\mathcal{R}$ is strongly closed

Example ❷

- linear TRS $\mathcal{R}$

$$(x+y)+z \to x+(y+z) \qquad x+(y+z) \to (x+y)+z \qquad x+y \to y+x$$

- 12 critical pairs

$$
\begin{array}{ll}
(x+(y+z))+w \approx (x+y)+(z+w) & (y+x)+z \approx x+(y+z) \\
((x+y)+z)+w \approx x+((y+z)+w) & z+(y+x) \approx x+(y+z) \\
((x+y)+z)+w \approx x+(y+(z+w)) & x+(y+z) \approx z+(y+x) \\
x+(y+(z+w)) \approx ((x+y)+z)+w & x+(z+y) \approx (x+y)+z \\
x+(y+(z+w)) \approx (x+(y+z))+w & (y+z)+x \approx (x+y)+z \\
x+((y+z)+w) \approx (x+y)+(z+w) & (x+y)+z \approx (y+z)+x
\end{array}
$$

- $\mathcal{R}$ is strongly closed

$$(x+(y+z))+w \leftrightarrows ((x+y)+z)+w \leftrightarrows (x+y)+(z+w)$$

Remark

linearity cannot be weakened to left–linearity

Remark

linearity cannot be weakened to left–linearity

Example

- left–linear TRS $\mathcal{R}$

$$\mathsf{h}(\mathsf{f},\mathsf{a},\mathsf{a}) \to \mathsf{h}(\mathsf{g},\mathsf{a},\mathsf{a}) \qquad \mathsf{a} \to \mathsf{b} \qquad \mathsf{h}(x,\mathsf{b},y) \to \mathsf{h}(x,y,y)$$

$$\mathsf{h}(\mathsf{g},\mathsf{a},\mathsf{a}) \to \mathsf{h}(\mathsf{f},\mathsf{a},\mathsf{a}) \qquad \mathsf{h}(x,y,\mathsf{b}) \to \mathsf{h}(x,y,y)$$

Remark

linearity cannot be weakened to left–linearity

Example

- left–linear TRS $\mathcal{R}$

$$\mathsf{h}(\mathsf{f},\mathsf{a},\mathsf{a}) \to \mathsf{h}(\mathsf{g},\mathsf{a},\mathsf{a}) \qquad \mathsf{a} \to \mathsf{b} \qquad \mathsf{h}(x,\mathsf{b},y) \to \mathsf{h}(x,y,y)$$
$$\mathsf{h}(\mathsf{g},\mathsf{a},\mathsf{a}) \to \mathsf{h}(\mathsf{f},\mathsf{a},\mathsf{a}) \qquad \mathsf{h}(x,y,\mathsf{b}) \to \mathsf{h}(x,y,y)$$

- $\mathcal{R}$ is strongly closed

Remark

linearity cannot be weakened to left–linearity

Example

- left–linear TRS $\mathcal{R}$

$$\mathsf{h}(\mathsf{f},\mathsf{a},\mathsf{a}) \to \mathsf{h}(\mathsf{g},\mathsf{a},\mathsf{a}) \qquad \mathsf{a} \to \mathsf{b} \qquad \mathsf{h}(x,\mathsf{b},y) \to \mathsf{h}(x,y,y)$$
$$\mathsf{h}(\mathsf{g},\mathsf{a},\mathsf{a}) \to \mathsf{h}(\mathsf{f},\mathsf{a},\mathsf{a}) \qquad \qquad \mathsf{h}(x,y,\mathsf{b}) \to \mathsf{h}(x,y,y)$$

- $\mathcal{R}$ is strongly closed
- $\mathcal{R}$ is not confluent

Remark

linearity cannot be weakened to left–linearity

Example

- left–linear TRS $\mathcal{R}$

$$\mathsf{h}(\mathsf{f},\mathsf{a},\mathsf{a}) \to \mathsf{h}(\mathsf{g},\mathsf{a},\mathsf{a}) \qquad \mathsf{a} \to \mathsf{b} \qquad \mathsf{h}(x,\mathsf{b},y) \to \mathsf{h}(x,y,y)$$

$$\mathsf{h}(\mathsf{g},\mathsf{a},\mathsf{a}) \to \mathsf{h}(\mathsf{f},\mathsf{a},\mathsf{a}) \qquad \mathsf{h}(x,y,\mathsf{b}) \to \mathsf{h}(x,y,y)$$

- $\mathcal{R}$ is strongly closed
- $\mathcal{R}$ is not confluent

$$\mathsf{h}(\mathsf{f},\mathsf{b},\mathsf{b}) \;{}^{*}\!\leftarrow \mathsf{h}(\mathsf{f},\mathsf{a},\mathsf{a}) \to \mathsf{h}(\mathsf{g},\mathsf{a},\mathsf{a}) \to^{*} \mathsf{h}(\mathsf{g},\mathsf{b},\mathsf{b})$$

Remark

linearity cannot be weakened to left–linearity

Example

- left–linear TRS $\mathcal{R}$

$$\mathsf{h}(\mathsf{f},\mathsf{a},\mathsf{a}) \to \mathsf{h}(\mathsf{g},\mathsf{a},\mathsf{a}) \qquad \mathsf{a} \to \mathsf{b} \qquad \mathsf{h}(x,\mathsf{b},y) \to \mathsf{h}(x,y,y)$$
$$\mathsf{h}(\mathsf{g},\mathsf{a},\mathsf{a}) \to \mathsf{h}(\mathsf{f},\mathsf{a},\mathsf{a}) \qquad\qquad \mathsf{h}(x,y,\mathsf{b}) \to \mathsf{h}(x,y,y)$$

- $\mathcal{R}$ is strongly closed
- $\mathcal{R}$ is not confluent

$$\mathsf{h}(\mathsf{f},\mathsf{b},\mathsf{b}) \;{}^{*}\!\!\leftarrow \mathsf{h}(\mathsf{f},\mathsf{a},\mathsf{a}) \to \mathsf{h}(\mathsf{g},\mathsf{a},\mathsf{a}) \to^{*} \mathsf{h}(\mathsf{g},\mathsf{b},\mathsf{b}) \qquad \mathsf{h}(\mathsf{f},\mathsf{b},\mathsf{b}) \not\downarrow \mathsf{h}(\mathsf{g},\mathsf{b},\mathsf{b})$$

Definition

TRS is parallel closed if $t \mathrel{+\!\!\!\mid\!\!\!\!\rightarrow} u$ for every critical pair $t \approx u$

Definition

TRS is parallel closed if $t \mathrel{+\!\!\!\!\!\!\longrightarrow} u$ for every critical pair $t \approx u$

Theorem

left−linear parallel closed TRSs are confluent

Definition

TRS is parallel closed if $t \twoheadrightarrow\!\!\!\!\!+\; u$ for every critical pair $t \approx u$

Theorem

left–linear parallel closed TRSs are confluent

Example

- left–linear TRS $\mathcal{R}$

$$x + y \to y + x \qquad (x + y) * z \to (x * z) + (y * z) \qquad (y + x) * z \to (x * z) + (y * z)$$

Definition

TRS is parallel closed if $t \twoheadrightarrow\!\!\!\!\!+\ u$ for every critical pair $t \approx u$

Theorem

left–linear parallel closed TRSs are confluent

Example

- left–linear TRS $\mathcal{R}$

$$x + y \to y + x \qquad (x + y) * z \to (x * z) + (y * z) \qquad (y + x) * z \to (x * z) + (y * z)$$

- 4 critical pairs

$$(y + x) * z \approx (x * z) + (y * z) \qquad (y * z) + (x * z) \approx (x * z) + (y * z)$$
$$(x + y) * z \approx (x * z) + (y * z) \qquad (x * z) + (y * z) \approx (y * z) + (x * z)$$

Definition

TRS is parallel closed if $t \twoheadrightarrow\!\!\!\!\!\!\!+\;\; u$ for every critical pair $t \approx u$

Theorem

left–linear parallel closed TRSs are confluent

Example

- left–linear TRS $\mathcal{R}$

$$x + y \to y + x \qquad (x + y) * z \to (x * z) + (y * z) \qquad (y + x) * z \to (x * z) + (y * z)$$

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$$(y + x) * z \approx (x * z) + (y * z) \qquad (y * z) + (x * z) \approx (x * z) + (y * z)$$
$$(x + y) * z \approx (x * z) + (y * z) \qquad (x * z) + (y * z) \approx (y * z) + (x * z)$$

- $\mathcal{R}$ is parallel closed

Definition

TRS is parallel closed if $t \twoheadrightarrow\!\!\!\!\!+\;\; u$ for every critical pair $t \approx u$

Theorem

left–linear parallel closed TRSs are confluent

Example

- left–linear TRS $\mathcal{R}$

$$x + y \to y + x \qquad (x + y) * z \to (x * z) + (y * z) \qquad (y + x) * z \to (x * z) + (y * z)$$

- 4 critical pairs

$$(y + x) * z \approx (x * z) + (y * z) \qquad (y * z) + (x * z) \approx (x * z) + (y * z)$$
$$(x + y) * z \approx (x * z) + (y * z) \qquad (x * z) + (y * z) \approx (y * z) + (x * z)$$

- $\mathcal{R}$ is parallel closed $\Longrightarrow$ $\mathcal{R}$ is confluent

Long–Standing Open Problem

is every left–linear TRS such that $t \mathrel{\leftarrow\!\!\shortparallel\!\!-} u$ for every critical pair $t \approx u$ confluent ?

Long–Standing Open Problem

is every left–linear TRS such that $t \leftarrow\!\!\!\!\!+\!\!\!\!+\!\!\!- u$ for every critical pair $t \approx u$ confluent ?

Definition

TRS is development closed if $t \multimap\!\!\!\rightarrow u$ for every critical pair $t \approx u$

Long–Standing Open Problem

is every left–linear TRS such that $t \mathrel{\leftarrow\!\!\!\!\!\!\parallel}\; u$ for every critical pair $t \approx u$ confluent ?

Definition

TRS is development closed if $t \mathrel{-\!\!\!\circ\!\!\!\rightarrow} u$ for every critical pair $t \approx u$

Theorem

left–linear development closed TRSs are confluent

Long–Standing Open Problem

is every left–linear TRS such that $t \mathrel{\leftarrow\!\!\!\!\!\parallel\!\!-} u$ for every critical pair $t \approx u$ confluent ?

Definition

TRS is development closed if $t \mathrel{-\!\!\!\circ\!\!\!\rightarrow} u$ for every critical pair $t \approx u$

Theorem

left–linear development closed TRSs are confluent

Remark

formalized proof employs proof terms (lecture 10)

Example

- left–linear TRS $\mathcal{R}$

$$\mathsf{b} \to \mathsf{c} \qquad \mathsf{f}(\mathsf{g}(\mathsf{a})) \to \mathsf{f}(\mathsf{c}) \qquad \mathsf{g}(x) \to \mathsf{h}(x, \mathsf{i}(x)) \qquad \mathsf{h}(\mathsf{a}, x) \to x \qquad \mathsf{i}(x) \to \mathsf{c}$$

Example

- left–linear TRS $\mathcal{R}$

$$\mathsf{b} \to \mathsf{c} \qquad \mathsf{f}(\mathsf{g}(\mathsf{a})) \to \mathsf{f}(\mathsf{c}) \qquad \mathsf{g}(x) \to \mathsf{h}(x, \mathsf{i}(x)) \qquad \mathsf{h}(\mathsf{a}, x) \to x \qquad \mathsf{i}(x) \to \mathsf{c}$$

- one critical pair

$$\mathsf{f}(\mathsf{h}(\mathsf{a}, \mathsf{i}(\mathsf{a}))) \approx \mathsf{f}(\mathsf{c})$$

Example

- left–linear TRS $\mathcal{R}$

$$\mathsf{b} \to \mathsf{c} \qquad \mathsf{f}(\mathsf{g}(\mathsf{a})) \to \mathsf{f}(\mathsf{c}) \qquad \mathsf{g}(x) \to \mathsf{h}(x, \mathsf{i}(x)) \qquad \mathsf{h}(\mathsf{a}, x) \to x \qquad \mathsf{i}(x) \to \mathsf{c}$$

- one critical pair

$$\mathsf{f}(\mathsf{h}(\mathsf{a}, \mathsf{i}(\mathsf{a}))) \approx \mathsf{f}(\mathsf{c})$$

- $\mathcal{R}$ is development closed

Example

- left–linear TRS $\mathcal{R}$

$$\mathsf{b} \to \mathsf{c} \qquad \mathsf{f}(\mathsf{g}(\mathsf{a})) \to \mathsf{f}(\mathsf{c}) \qquad \mathsf{g}(x) \to \mathsf{h}(x, \mathsf{i}(x)) \qquad \mathsf{h}(\mathsf{a}, x) \to x \qquad \mathsf{i}(x) \to \mathsf{c}$$

- one critical pair

$$\mathsf{f}(\mathsf{h}(\mathsf{a}, \mathsf{i}(\mathsf{a}))) \approx \mathsf{f}(\mathsf{c})$$

- $\mathcal{R}$ is development closed $\Longrightarrow$ $\mathcal{R}$ is confluent

Confluence Tools

ACP FORT Hakusan CONFident CSI

Example

- left–linear TRS $\mathcal{R}$

$$\mathsf{b} \to \mathsf{c} \qquad \mathsf{f}(\mathsf{g}(\mathsf{a})) \to \mathsf{f}(\mathsf{c}) \qquad \mathsf{g}(x) \to \mathsf{h}(x, \mathsf{i}(x)) \qquad \mathsf{h}(\mathsf{a}, x) \to x \qquad \mathsf{i}(x) \to \mathsf{c}$$

- one critical pair

$$\mathsf{f}(\mathsf{h}(\mathsf{a}, \mathsf{i}(\mathsf{a}))) \approx \mathsf{f}(\mathsf{c})$$

- $\mathcal{R}$ is development closed $\implies$ $\mathcal{R}$ is confluent

Confluence Tools

CSI

Example

- left–linear TRS $\mathcal{R}$

$$\mathsf{b} \to \mathsf{c} \qquad \mathsf{f}(\mathsf{g}(\mathsf{a})) \to \mathsf{f}(\mathsf{c}) \qquad \mathsf{g}(x) \to \mathsf{h}(x, \mathsf{i}(x)) \qquad \mathsf{h}(\mathsf{a}, x) \to x \qquad \mathsf{i}(x) \to \mathsf{c}$$

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Confluence Tools

ACP FORT Hakusan CONFident CSI

Confluence Competion

AriWeb

`https://project-coco.uibk.ac.at/`

`https://ari-web.uibk.ac.at/`

Outline

Homework Exercises for May 18

1. Exercise 6.3. (1)
2. Exercise 6.4. (2)
3. Exercise 6.12. (2)
4. Exercise 6.15. (2)
5. Exercise 6.14. ☆☆☆

Outline

Lecture Notes

- Section 3.2 (Definition 3.2.12 — Corollary 3.2.15)
- Section 6.1
- Section 6.3 (until Theorem 6.3.6)

Lecture Notes

- Section 3.2 (Definition 3.2.12 — Corollary 3.2.15)
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Important Concepts

- development closed
- maximal multi–step relation
- multi–step relation
- orthogonality
- parallel closed
- parallel moves lemma
- parallel rewriting
- strongly closed
- triangle property