



Term Rewriting

Philipp Dablander and **Aart Middeldorp**

Outline

- 1. Summary of Lecture 8**
- 2. Orthogonality**
- 3. Parallel Rewriting**
- 4. Multi-Step Rewriting**
- 5. Critical Pair Conditions**
- 6. Exercises**
- 7. Further Reading**

Definitions

- ▶ **weight function** (w, w_0) consists of mapping $w: \mathcal{F} \rightarrow \mathbb{N}$ and constant $w_0 > 0$ such that $w(c) \geq w_0$ for all constants $c \in \mathcal{F}$
- ▶ **weight** of term t : $w(t) = w_0$ if $t \in \mathcal{V}$ and $w(t) = w(f) + \sum_{i=1}^n w(t_i)$ if $t = f(t_1, \dots, t_n)$
- ▶ weight function (w, w_0) is **admissible** for precedence $>$ if $f > g$ for all $g \in \mathcal{F} \setminus \{f\}$, whenever f is unary function symbol in \mathcal{F} with $w(f) = 0$
- ▶ binary relation $>_{\text{kbo}}$ on terms (**Knuth-Bendix order**): $s >_{\text{kbo}} t$ if $|s|_x \geq |t|_x$ for all $x \in \mathcal{V}$ and either $w(s) > w(t)$ or both $w(s) = w(t)$ and either
 - ① $s = f^n(t)$ for some $n > 0$ and $t \in \mathcal{V}$
 - ② $s = f(s_1, \dots, s_n)$ and $t = f(t_1, \dots, t_n)$ and for some $1 \leq i \leq n$
 - a $s_j = t_j$ for all $1 \leq j < i$
 - b $s_i >_{\text{kbo}} t_i$
 - ③ $s = f(s_1, \dots, s_n)$ and $t = g(t_1, \dots, t_m)$ and $f > g$

Theorem

- ▶ $>_{\text{kbo}}$ is **reduction order** if $>$ is well-founded and (w, w_0) is admissible for $>$
- ▶ if $> \subseteq \sqsupset$ then $>_{\text{kbo}} \subseteq \sqsupset_{\text{kbo}}$ (**incrementality**)
- ▶ if $>$ is total then $>_{\text{kbo}}$ is **total on ground terms**
- ▶ following problem is **decidable**:

instance: finite TRS \mathcal{R}

question: \exists weight function (w, w_0) such that (w, w_0) is admissible for $>$?
 \exists precedence $>$ and $\mathcal{R} \subseteq >_{\text{kbo}}$

Definitions

- ▶ TRS \mathcal{R} is **reduced** if $r \in \text{NF}(\mathcal{R})$ and $l \in \text{NF}(\mathcal{R} \setminus \{l \rightarrow r\})$ for all $l \rightarrow r \in \mathcal{R}$
- ▶ reduced complete TRS is **canonical**

Definitions

- ▶ TRSs \mathcal{R} and \mathcal{S} are **conversion equivalent** if $\leftrightarrow_{\mathcal{R}}^* = \leftrightarrow_{\mathcal{S}}^*$
- ▶ TRSs \mathcal{R} and \mathcal{S} are **normalization equivalent** if $\rightarrow_{\mathcal{R}}^! = \rightarrow_{\mathcal{S}}^!$

Theorem

conversion equivalent canonical TRSs that are compatible with same reduction order are unique up to literal similarity

Definitions

- ▶ $\dot{\mathcal{R}} = \{l \rightarrow r \downarrow_{\mathcal{R}} \mid l \rightarrow r \in \mathcal{R}\}$
- ▶ $\ddot{\mathcal{R}} = \{l \rightarrow r \in \dot{\mathcal{R}} \mid l \in \text{NF}(\dot{\mathcal{R}} \setminus \{l \rightarrow r\})\}$

Theorem

if \mathcal{R} is complete TRS then $\ddot{\mathcal{R}}$ is (normalization) equivalent canonical TRS

Definition

set of equations \mathcal{E} set of rewrite rules \mathcal{R} reduction order $>$

inference system **KB** consists of eight rules

delete
$$\frac{\mathcal{E} \uplus \{s \approx s\}, \mathcal{R}}{\mathcal{E}, \mathcal{R}}$$

deduce
$$\frac{\mathcal{E}, \mathcal{R}}{\mathcal{E} \cup \{s \approx t\}, \mathcal{R}} \quad \text{if } s \mathcal{R} \leftarrow \cdot \rightarrow_{\mathcal{R}} t$$

compose
$$\frac{\mathcal{E}, \mathcal{R} \uplus \{s \rightarrow t\}}{\mathcal{E}, \mathcal{R} \cup \{s \rightarrow u\}} \quad \text{if } t \rightarrow_{\mathcal{R}} u$$

collapse
$$\frac{\mathcal{E}, \mathcal{R} \uplus \{t \rightarrow s\}}{\mathcal{E} \cup \{u \approx s\}, \mathcal{R}} \quad \text{if } t \rightarrow_{\mathcal{R}} u$$

orient
$$\frac{\mathcal{E} \uplus \{s \approx t\}, \mathcal{R}}{\mathcal{E}, \mathcal{R} \cup \{s \rightarrow t\}} \quad \text{if } s > t$$

simplify
$$\frac{\mathcal{E} \uplus \{s \approx t\}, \mathcal{R}}{\mathcal{E} \cup \{s \approx u\}, \mathcal{R}} \quad \text{if } t \rightarrow_{\mathcal{R}} u$$

$$\frac{\mathcal{E} \uplus \{t \approx s\}, \mathcal{R}}{\mathcal{E}, \mathcal{R} \cup \{s \rightarrow t\}}$$

$$\frac{\mathcal{E} \uplus \{t \approx s\}, \mathcal{R}}{\mathcal{E} \cup \{u \approx s\}, \mathcal{R}}$$

Definition

run for ES \mathcal{E} is finite sequence

$$\mathcal{E}_0, \mathcal{R}_0 \vdash_{\text{KB}} \mathcal{E}_1, \mathcal{R}_1 \vdash_{\text{KB}} \cdots \vdash_{\text{KB}} \mathcal{E}_n, \mathcal{R}_n$$

such that $\mathcal{E}_0 = \mathcal{E}$ and $\mathcal{R}_0 = \emptyset$

▶ run **fails** if $\mathcal{E}_n \neq \emptyset$

▶ run is **fair** if $\text{PCP}(\mathcal{R}_n) \subseteq \downarrow_{\mathcal{R}_n} \cup \bigcup_{i=0}^n \leftrightarrow_{\mathcal{E}_i}$

Theorem

\mathcal{R}_n is complete presentation of \mathcal{E}_0 for every **fair non-failing** run $(\mathcal{E}_0, \mathcal{R}_0) \vdash_{\text{KB}} \cdots \vdash_{\text{KB}} (\mathcal{E}_n, \mathcal{R}_n)$

Outline

1. Summary of Lecture 8
- 2. Orthogonality**
3. Parallel Rewriting
4. Multi-Step Rewriting
5. Critical Pair Conditions
6. Exercises
7. Further Reading

Example

TRS \mathcal{R} modeling **Sieve of Eratosthenes** for generating list of prime numbers

$\text{primes} \rightarrow \text{sieve}(\text{from}(\text{s}(\text{s}(0))))$

$\text{sieve}(0 : y) \rightarrow \text{sieve}(y)$

$\text{from}(x) \rightarrow x : \text{from}(\text{s}(x))$

$\text{sieve}(\text{s}(x) : y) \rightarrow \text{s}(x) : \text{sieve}(\text{filter}(x, y, x))$

$\text{head}(x : y) \rightarrow x$

$\text{filter}(0, y : z, w) \rightarrow 0 : \text{filter}(w, z, w)$

$\text{tail}(x : y) \rightarrow y$

$\text{filter}(\text{s}(x), y : z, w) \rightarrow y : \text{filter}(x, z, w)$

- ▶ \mathcal{R} is confluent but not terminating:

$\text{from}(0) \rightarrow 0 : \text{from}(\text{s}(0)) \rightarrow 0 : (\text{s}(0) : \text{from}(\text{s}(\text{s}(0)))) \rightarrow \dots$

- ▶ how to prove confluence of \mathcal{R} ? **orthogonality** (lectures 9, 12)
- ▶ \exists non-terminating terms with (unique) normal form

$\text{head}(\text{tail}(\text{tail}(\text{primes}))) \rightarrow^! \text{s}(\text{s}(\text{s}(\text{s}(0))))$

- ▶ how to compute normal forms in \mathcal{R} ? **strategy** (lectures 10, 11)

Confluence Methods

critical pair closing systems

decreasing diagrams

development closed critical pairs

discrimination pairs

joinable critical pairs for terminating systems

orthogonality

parallel closed critical pairs

parallel critical pairs

redundant rules

rule labeling

simultaneous critical pairs

source labeling

strongly closed critical pairs

tree automata

weak orthogonality

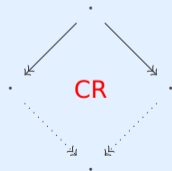
Z property

...

Confluence

every two co-initial rewrite sequences can be joined

- ▶ ... yields uniqueness of normal forms
- ▶ ... is decidable for terminating TRSs
- ▶ ... what about non-terminating TRSs ?



Examples (Non-Confluence)

- ▶ no confluence because of critical pairs

$$a \rightarrow b$$

$$a \rightarrow c$$

$$b \leftarrow a \rightarrow c$$

- ▶ no critical pairs but no confluence

$$f(x, x) \rightarrow a$$

$$g(x) \rightarrow f(x, g(x))$$

$$c \rightarrow g(c)$$

$$a \leftarrow^* c \rightarrow g(c) \rightarrow^* g(a)$$

- ▶ no critical pairs but no confluence

$$f(x, x) \rightarrow a$$

$$f(x, g(x)) \rightarrow b$$

$$c \rightarrow g(c)$$

$$a \leftarrow f(c, c) \rightarrow f(c, g(c)) \rightarrow b$$

Confluence via Critical Pairs

control interference of rewrite rules

(notation: $s \leftarrow \times \rightarrow t$ if $s \approx t$ is critical pair)

- ▶ critical pair lemma (lecture 6):

$$\text{WCR} \iff \leftarrow \times \rightarrow \subseteq \downarrow$$

- ▶ combine with Newman's Lemma (lecture 2):

$$\text{SN} \ \& \ \leftarrow \times \rightarrow \subseteq \downarrow \implies \text{CR}$$

- ▶ observe from preceding examples:

$$\leftarrow \times \rightarrow = \emptyset \not\implies \text{CR}$$

Confluence via Orthogonality

forbid interference of rewrite rules

- ▶ **no critical pairs**
- ▶ **no equality checks**

Definitions (Linearity)

- ▶ term t is **linear** if each variable in $\text{Var}(t)$ occurs exactly once in t
- ▶ rewrite rule $\ell \rightarrow r$ is **left-linear** if ℓ is linear
- ▶ TRS is left-linear if all rewrite rules are left-linear
- ▶ rewrite rule $\ell \rightarrow r$ is **right-linear** if r is linear
- ▶ TRS is right-linear if all rewrite rules are right-linear
- ▶ rewrite rule $\ell \rightarrow r$ is **linear** if ℓ and r is linear
- ▶ TRS is linear if all rewrite rules are linear

Examples

- ▶ $g(x) \rightarrow f(x, g(x))$ left-linear but not right-linear
- ▶ $f(x, x) \rightarrow a$ right-linear but not left-linear

Definition (Orthogonality)

orthogonal TRS is left-linear and lacks critical pairs

Examples

$$I \cdot x \rightarrow x$$

$$(K \cdot x) \cdot y \rightarrow x$$

$$((S \cdot x) \cdot y) \cdot z \rightarrow (x \cdot z) \cdot (y \cdot z)$$

$$\text{ack}(0, y) \rightarrow s(y)$$

$$\text{ack}(s(x), 0) \rightarrow \text{ack}(x, s(0))$$

$$\text{ack}(s(x), s(y)) \rightarrow \text{ack}(x, \text{ack}(s(x), y))$$

Theorem

orthogonal TRSs are **confluent**

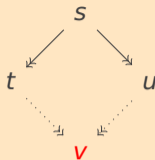
Outline

1. Summary of Lecture 8
2. Orthogonality
- 3. Parallel Rewriting**
4. Multi-Step Rewriting
5. Critical Pair Conditions
6. Exercises
7. Further Reading

Theorem

orthogonal TRSs are **confluent**

$\forall s, t, u$



$\exists v$

Observation

for orthogonal TRSs there is canonical way to compute common reduct v

Definition (Parallel Rewriting)

parallel rewriting \twoheadrightarrow is inductively defined as follows:

- ① $x \twoheadrightarrow x$ for all variables x
- ② $f(s_1, \dots, s_n) \twoheadrightarrow f(t_1, \dots, t_n)$ if $s_i \twoheadrightarrow t_i$ for all $1 \leq i \leq n$
- ③ $l\sigma \twoheadrightarrow r\sigma$ if $l \rightarrow r \in \mathcal{R}$

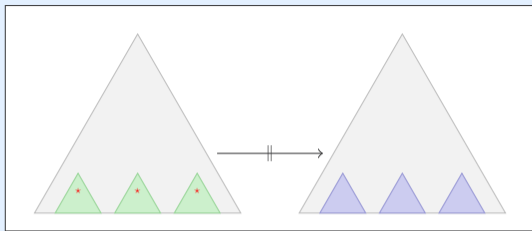
$$\frac{x \in \mathcal{V}}{x \twoheadrightarrow x}$$

$$\frac{s \rightarrow t}{s \twoheadrightarrow t}$$

$$\frac{s_1 \twoheadrightarrow t_1 \quad \dots \quad s_n \twoheadrightarrow t_n}{f(s_1, \dots, s_n) \twoheadrightarrow f(t_1, \dots, t_n)}$$

Lemma

$s \twoheadrightarrow t \iff \begin{array}{l} \exists \text{ context } C \text{ with } n \geq 0 \text{ holes} \\ \exists \text{ terms } s_1, \dots, s_n, t_1, \dots, t_n \end{array} \text{ such that } \begin{array}{l} s = C[s_1, \dots, s_n] \\ t = C[t_1, \dots, t_n] \\ s_i \rightarrow t_i \text{ for all } 1 \leq i \leq n \end{array}$



Example

► TRS

$$0 + y \rightarrow y$$

$$0 \times y \rightarrow 0$$

$$s(x) + y \rightarrow s(x + y)$$

$$s(x) \times y \rightarrow (x \times y) + y$$

► rewrite sequences

$$s(0 \times 0) + s(0) \times (0 + s(0)) \dashrightarrow s(0) + (0 \times (0 + s(0)) + (0 + s(0)))$$

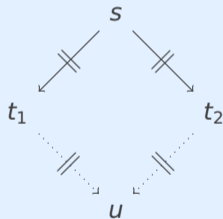
$$s(0 \times 0) + s(0) \times (0 + s(0)) \not\rightarrow s(0 \times 0) + ((0 \times s(0)) + (0 + s(0)))$$

$$s(0 \times 0) + s(0) \times (0 + s(0)) \dashrightarrow s(0 \times 0) + s(0) \times (0 + s(0))$$

Lemma

$\rightarrow \subseteq \Rrightarrow \subseteq \rightarrow^*$

Parallel Moves Lemma

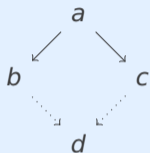


Definition (Diamond Property)

▶ **diamond property** \diamond

▶ $\leftarrow \cdot \rightarrow \subseteq \rightarrow \cdot \leftarrow$

▶ $\forall a, b, c$



$\exists d$

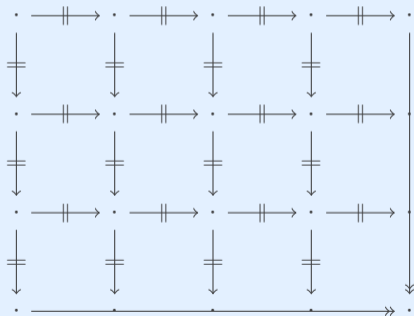
Lemma

ARS $\langle A, \rightarrow \rangle$ is confluent if $\rightarrow \subseteq \twoheadrightarrow \subseteq \rightarrow^*$ for some relation \twoheadrightarrow on A with diamond property

Corollary

orthogonal TRSs are confluent

Proof



Outline

1. Summary of Lecture 8
2. Orthogonality
3. Parallel Rewriting
- 4. Multi-Step Rewriting**
5. Critical Pair Conditions
6. Exercises
7. Further Reading

Definition (Multi-Step Rewriting)

multi-step relation \twoheadrightarrow is inductively defined as follows:

- ① $x \twoheadrightarrow x$ for all variables x
- ② $f(s_1, \dots, s_n) \twoheadrightarrow f(t_1, \dots, t_n)$ if $s_i \twoheadrightarrow t_i$ for all $1 \leq i \leq n$
- ③ $l\sigma \twoheadrightarrow r\tau$ if $l \rightarrow r \in \mathcal{R}$ and $\underbrace{x\sigma \twoheadrightarrow x\tau}_{\sigma \twoheadrightarrow \tau}$ for all variables x

Example

► TRS

$$0 + y \rightarrow y$$

$$0 \times y \rightarrow 0$$

$$s(x) + y \rightarrow s(x + y)$$

$$s(x) \times y \rightarrow (x \times y) + y$$

► rewrite sequences

$$s(0 \times 0) + s(0) \times (0 + s(0)) \not\rightarrow s(0) + (0 \times (0 + s(0))) + s(0)$$

$$s(0 \times 0) + s(0) \times (0 + s(0)) \twoheadrightarrow s(0) + (0 \times s(0)) + s(0)$$

- ① $x \rightarrow x$ for all variables x
- ② $f(s_1, \dots, s_n) \rightarrow f(t_1, \dots, t_n)$ if $s_i \rightarrow t_i$ for all $1 \leq i \leq n$
- ③ $l\sigma \rightarrow r\tau$ if $l \rightarrow r \in \mathcal{R}$ and $\sigma \rightarrow \tau$

Example

TRS

$$0 + y \rightarrow y \qquad 0 \times y \rightarrow 0$$

$$s(x) + y \rightarrow s(x + y) \qquad s(x) \times y \rightarrow (x \times y) + y$$

$$\begin{array}{c}
 \textcircled{3} \frac{0 \times y \rightarrow 0 \quad \frac{}{0 \rightarrow 0} \textcircled{2}}{0 \times 0 \rightarrow 0} \qquad \frac{s(x) \times y \rightarrow (x \times y) + y \quad \frac{}{0 \rightarrow 0} \textcircled{2}}{s(0) \times (0 + s(0)) \rightarrow (0 \times s(0)) + s(0)} \textcircled{3} \\
 \textcircled{2} \frac{s(0 \times 0) \rightarrow s(0)}{s(0 \times 0) + s(0) \times (0 + s(0)) \rightarrow s(0) + ((0 \times s(0)) + s(0))} \textcircled{2}
 \end{array}$$

Lemma

$$\rightarrow \subseteq \twoheadrightarrow \subseteq \rightarrow^* \subseteq \rightarrow^*$$

Definition (Maximal Multi-Step Rewriting)

maximal multi-step relation \twoheadrightarrow is inductively defined as follows:

- ① $x \twoheadrightarrow x$ for all variables x
- ② $f(s_1, \dots, s_n) \twoheadrightarrow f(t_1, \dots, t_n)$ if $s_i \twoheadrightarrow t_i$ for all $1 \leq i \leq n$ and $f(s_1, \dots, s_n)$ is no redex
- ③ $l\sigma \twoheadrightarrow r\tau$ if $l \rightarrow r \in \mathcal{R}$ and $\sigma \twoheadrightarrow \tau$

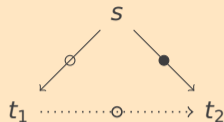
Remark

\twoheadrightarrow is deterministic for orthogonal TRSs

Lemma

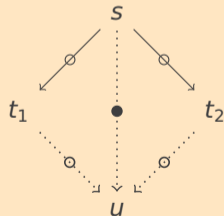
for orthogonal TRSs

triangle property



Corollary

for orthogonal TRSs



Outline

1. Summary of Lecture 8
2. Orthogonality
3. Parallel Rewriting
4. Multi-Step Rewriting
- 5. Critical Pair Conditions**
6. Exercises
7. Further Reading

Definition

TRS is **strongly closed** if $t \rightarrow^= \cdot * \leftarrow u$ and $t \rightarrow^* \cdot = \leftarrow u$ for every critical pair $t \approx u$

Theorem

linear strongly closed TRSs are confluent

Example 1

► TRS \mathcal{R}

$$f(f(x, y), z) \rightarrow f(x, f(y, z))$$

$$f(x, y) \rightarrow f(y, x)$$

► 4 critical pairs

$$f(f(x, f(y, z)), v) \approx f(f(x, y), f(z, v))$$

$$f(x, f(y, z)) \approx f(z, f(x, y))$$

$$f(z, f(x, y)) \approx f(x, f(y, z))$$

$$f(f(y, x), z) \approx f(x, f(y, z))$$

► \mathcal{R} is linear and strongly closed $\implies \mathcal{R}$ is confluent

Example 2

- ▶ linear TRS \mathcal{R}

$$(x + y) + z \rightarrow x + (y + z) \quad x + (y + z) \rightarrow (x + y) + z \quad x + y \rightarrow y + x$$

- ▶ 12 critical pairs

$$(x + (y + z)) + w \approx (x + y) + (z + w)$$

$$(y + x) + z \approx x + (y + z)$$

$$((x + y) + z) + w \approx x + ((y + z) + w)$$

$$z + (y + x) \approx x + (y + z)$$

$$((x + y) + z) + w \approx x + (y + (z + w))$$

$$x + (y + z) \approx z + (y + x)$$

$$x + (y + (z + w)) \approx ((x + y) + z) + w$$

$$x + (z + y) \approx (x + y) + z$$

$$x + (y + (z + w)) \approx (x + (y + z)) + w$$

$$(y + z) + x \approx (x + y) + z$$

$$x + ((y + z) + w) \approx (x + y) + (z + w)$$

$$(x + y) + z \approx (y + z) + x$$

- ▶ \mathcal{R} is strongly closed

$$(x + (y + z)) + w \leftrightarrow ((x + y) + z) + w \leftrightarrow (x + y) + (z + w)$$

Remark

linearity cannot be weakened to **left-linearity**

Example

- ▶ left-linear TRS \mathcal{R}

$$h(f, a, a) \rightarrow h(g, a, a)$$

$$a \rightarrow b$$

$$h(x, b, y) \rightarrow h(x, y, y)$$

$$h(g, a, a) \rightarrow h(f, a, a)$$

$$h(x, y, b) \rightarrow h(x, y, y)$$

- ▶ \mathcal{R} is strongly closed
- ▶ \mathcal{R} is not confluent

$$h(f, b, b) \overset{*}{\leftarrow} h(f, a, a) \rightarrow h(g, a, a) \overset{*}{\rightarrow} h(g, b, b)$$

$$h(f, b, b) \not\rightarrow h(g, b, b)$$

Definition

TRS is **parallel closed** if $t \dashrightarrow u$ for every critical pair $t \approx u$

Theorem

left-linear parallel closed TRSs are confluent

Example

▶ left-linear TRS \mathcal{R}

$$x + y \rightarrow y + x \quad (x + y) * z \rightarrow (x * z) + (y * z) \quad (y + x) * z \rightarrow (x * z) + (y * z)$$

▶ 4 critical pairs

$$(y + x) * z \approx (x * z) + (y * z) \quad (y * z) + (x * z) \approx (x * z) + (y * z)$$

$$(x + y) * z \approx (x * z) + (y * z) \quad (x * z) + (y * z) \approx (y * z) + (x * z)$$

▶ \mathcal{R} is parallel closed $\implies \mathcal{R}$ is confluent

Long-Standing Open Problem

is every left-linear TRS such that $t \leftarrow\!\! \leftarrow u$ for every critical pair $t \approx u$ confluent ?

Definition

TRS is **development closed** if $t \twoheadrightarrow u$ for every critical pair $t \approx u$

Theorem

left-linear development closed TRSs are confluent

Remark

formalized proof employs **proof terms** (lecture 10)

Example

- ▶ left-linear TRS \mathcal{R}

$$b \rightarrow c \quad f(g(a)) \rightarrow f(c) \quad g(x) \rightarrow h(x, i(x)) \quad h(a, x) \rightarrow x \quad i(x) \rightarrow c$$

- ▶ one critical pair

$$f(h(a, i(a))) \approx f(c)$$

- ▶ \mathcal{R} is development closed $\implies \mathcal{R}$ is confluent

Confluence Tools

ACP

FORT

Hakusan

CONFident

CSI

Confluence Competition

AriWeb

<https://project-coco.uibk.ac.at/>

<https://ari-web.uibk.ac.at/>

Outline

1. Summary of Lecture 8
2. Orthogonality
3. Parallel Rewriting
4. Multi-Step Rewriting
5. Critical Pair Conditions
- 6. Exercises**
7. Further Reading

Homework Exercises for May 18

① Exercise 6.3.

1

② Exercise 6.4.

2

③ Exercise 6.12.

2

④ Exercise 6.15.

2

⑤ Exercise 6.14.

★ ★ ★

Outline

1. Summary of Lecture 8
2. Orthogonality
3. Parallel Rewriting
4. Multi-Step Rewriting
5. Critical Pair Conditions
6. Exercises
- 7. Further Reading**

Lecture Notes

- ▶ Section 3.2 (Definition 3.2.12 — Corollary 3.2.15)
- ▶ Section 6.1
- ▶ Section 6.3 (until Theorem 6.3.6)

Important Concepts

- ▶ development closed
- ▶ maximal multi-step relation
- ▶ multi-step relation
- ▶ orthogonality
- ▶ parallel closed
- ▶ parallel moves lemma
- ▶ parallel rewriting
- ▶ strongly closed
- ▶ triangle property