universität innsbruck

26S lecture 9

Term Rewriting

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Outline

Definitions

- weight function (w, w_0) consists of mapping $w \colon \mathcal{F} \to \mathbb{N}$ and constant $w_0 > 0$ such that $w(c) \geqslant w_0$ for all constants $c \in \mathcal{F}$
- weight of term t: $w(t) = w_0$ if $t \in \mathcal{V}$ and $w(t) = w(f) + \sum_{i=1}^{n} w(t_i)$ if $t = f(t_1, \dots, t_n)$
- weight function (w, w_0) is admissible for precedence $>$ if $f > g$ for all $g \in \mathcal{F} \setminus \{f\}$, whenever f is unary function symbol in $\mathcal{F}$ with $w(f) = 0$
- binary relation $>_{\text{kbo}}$ on terms (Knuth–Bendix order): $s >_{\text{kbo}} t$ if $|s|_x \geqslant |t|_x$ for all $x \in \mathcal{V}$ and either $w(s) > w(t)$ or both $w(s) = w(t)$ and either
  - ① $s = f^n(t)$ for some $n > 0$ and $t \in \mathcal{V}$
  - ② $s = f(s_1, \dots, s_n)$ and $t = f(t_1, \dots, t_n)$ and for some $1 \leqslant i \leqslant n$
    - **a** $s_j = t_j$ for all $1 \leqslant j < i$
    - **b** $s_i >_{\text{kbo}} t_i$
  - ③ $s = f(s_1, \dots, s_n)$ and $t = g(t_1, \dots, t_m)$ and $f > g$

Theorem

- $>_{\text{kbo}}$ is reduction order if $>$ is well–founded and (w, w_0) is admissible for $>$
- if $> \subseteq \sqsupset$ then $>_{\text{kbo}} \subseteq \sqsupset_{\text{kbo}}$ (incrementality)
- if $>$ is total then $>_{\text{kbo}}$ is total on ground terms
- following problem is decidable:

  instance: finite TRS $\mathcal{R}$

  question: $\exists$ weight function (w, w_0) $\exists$ precedence $>$ such that (w, w_0) is admissible for $>$ and $\mathcal{R} \subseteq >_{\text{kbo}}$?

Definitions

- TRS $\mathcal{R}$ is reduced if $r \in \mathsf{NF}(\mathcal{R})$ and $\ell \in \mathsf{NF}(\mathcal{R} \setminus \{\ell \to r\})$ for all $\ell \to r \in \mathcal{R}$
- reduced complete TRS is canonical

Definitions

- TRSs $\mathcal{R}$ and $\mathcal{S}$ are conversion equivalent if $\leftrightarrow^*_\mathcal{R} = \leftrightarrow^*_\mathcal{S}$
- TRSs $\mathcal{R}$ and $\mathcal{S}$ are normalization equivalent if $\to^!_\mathcal{R} = \to^!_\mathcal{S}$

Theorem

conversion equivalent canonical TRSs that are compatible with same reduction order are unique up to literal similarity

Definitions

- $\dot{\mathcal{R}} = \{\ell \to r{\downarrow_\mathcal{R}} \mid \ell \to r \in \mathcal{R}\}$
- $\ddot{\mathcal{R}} = \{\ell \to r \in \dot{\mathcal{R}} \mid \ell \in \mathsf{NF}(\dot{\mathcal{R}} \setminus \{\ell \to r\})\}$

Theorem

if $\mathcal{R}$ is complete TRS then $\ddot{\mathcal{R}}$ is (normalization) equivalent canonical TRS

Definition

set of equations $\mathcal{E}$ set of rewrite rules $\mathcal{R}$ reduction order $>$

inference system KB consists of eight rules

delete $\dfrac{\mathcal{E} \uplus \{s \approx s\}, \mathcal{R}}{\mathcal{E}, \mathcal{R}}$

deduce $\dfrac{\mathcal{E}, \mathcal{R}}{\mathcal{E} \cup \{s \approx t\}, \mathcal{R}}$ if $s \;{}_\mathcal{R}\!\leftarrow \cdot \to_\mathcal{R} t$

compose $\dfrac{\mathcal{E}, \mathcal{R} \uplus \{s \to t\}}{\mathcal{E}, \mathcal{R} \cup \{s \to u\}}$ if $t \to_\mathcal{R} u$

collapse $\dfrac{\mathcal{E}, \mathcal{R} \uplus \{t \to s\}}{\mathcal{E} \cup \{u \approx s\}, \mathcal{R}}$ if $t \to_\mathcal{R} u$

orient $\dfrac{\mathcal{E} \uplus \{s \approx t\}, \mathcal{R}}{\mathcal{E}, \mathcal{R} \cup \{s \to t\}}$ $\dfrac{\mathcal{E} \uplus \{t \approx s\}, \mathcal{R}}{\mathcal{E}, \mathcal{R} \cup \{s \to t\}}$ if $s > t$

simplify $\dfrac{\mathcal{E} \uplus \{s \approx t\}, \mathcal{R}}{\mathcal{E} \cup \{s \approx u\}, \mathcal{R}}$ $\dfrac{\mathcal{E} \uplus \{t \approx s\}, \mathcal{R}}{\mathcal{E} \cup \{u \approx s\}, \mathcal{R}}$ if $t \to_\mathcal{R} u$

Definition

run for ES $\mathcal{E}$ is finite sequence

$$\mathcal{E}_0, \mathcal{R}_0 \vdash_{\mathsf{KB}} \mathcal{E}_1, \mathcal{R}_1 \vdash_{\mathsf{KB}} \cdots \vdash_{\mathsf{KB}} \mathcal{E}_n, \mathcal{R}_n$$

such that $\mathcal{E}_0 = \mathcal{E}$ and $\mathcal{R}_0 = \varnothing$

- run fails if $\mathcal{E}_n \neq \varnothing$
- run is fair if $\mathsf{PCP}(\mathcal{R}_n) \subseteq {\downarrow_{\mathcal{R}_n}} \cup \bigcup_{i=0}^{n} \leftrightarrow_{\mathcal{E}_i}$

Theorem

$\mathcal{R}_n$ is complete presentation of $\mathcal{E}_0$ for every fair non–failing run $(\mathcal{E}_0, \mathcal{R}_0) \vdash_{\mathsf{KB}} \cdots \vdash_{\mathsf{KB}} (\mathcal{E}_n, \mathcal{R}_n)$

Outline

Example

TRS $\mathcal{R}$ modeling Sieve of Eratosthenes for generating list of prime numbers

$$\begin{aligned} \mathsf{primes} &\to \mathsf{sieve}(\mathsf{from}(\mathsf{s}(\mathsf{s}(0)))) & \mathsf{sieve}(0:y) &\to \mathsf{sieve}(y) \\ \mathsf{from}(x) &\to x:\mathsf{from}(\mathsf{s}(x)) & \mathsf{sieve}(\mathsf{s}(x):y) &\to \mathsf{s}(x):\mathsf{sieve}(\mathsf{filter}(x,y,x)) \\ \mathsf{head}(x:y) &\to x & \mathsf{filter}(0,y:z,w) &\to 0:\mathsf{filter}(w,z,w) \\ \mathsf{tail}(x:y) &\to y & \mathsf{filter}(\mathsf{s}(x),y:z,w) &\to y:\mathsf{filter}(x,z,w) \end{aligned}$$

- $\mathcal{R}$ is confluent but not terminating:

$$\mathsf{from}(0) \to 0:\mathsf{from}(\mathsf{s}(0)) \to 0:(\mathsf{s}(0):\mathsf{from}(\mathsf{s}(\mathsf{s}(0)))) \to \cdots$$

- how to prove confluence of $\mathcal{R}$? orthogonality (lectures 9, 12)
- $\exists$ non–terminating terms with (unique) normal form

$$\mathsf{head}(\mathsf{tail}(\mathsf{tail}(\mathsf{primes}))) \to^! \mathsf{s}(\mathsf{s}(\mathsf{s}(\mathsf{s}(\mathsf{s}(0)))))$$

- how to compute normal forms in $\mathcal{R}$? strategy (lectures 10, 11)

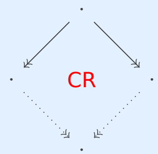
Confluence Methods

critical pair closing systems · decreasing diagrams · development closed critical pairs · discrimination pairs · joinable critical pairs for terminating systems · orthogonality · parallel closed critical pairs · parallel critical pairs · redundant rules · rule labeling · simultaneous critical pairs · source labeling · strongly closed critical pairs · tree automata · weak orthogonality · Z property · $\cdots$

Confluence

every two co–initial rewrite sequences can be joined

- $\cdots$ yields uniqueness of normal forms
- $\cdots$ is decidable for terminating TRSs
- $\cdots$ what about non–terminating TRSs ?



Examples (Non–Confluence)

- no confluence because of critical pairs

$$\begin{aligned} \mathsf{a} &\to \mathsf{b} \\ \mathsf{a} &\to \mathsf{c} \end{aligned} \qquad \mathsf{b} \leftarrow \mathsf{a} \to \mathsf{c}$$

- no critical pairs but no confluence

$$\begin{aligned} \mathsf{f}(x,x) &\to \mathsf{a} \\ \mathsf{g}(x) &\to \mathsf{f}(x,\mathsf{g}(x)) \\ \mathsf{c} &\to \mathsf{g}(\mathsf{c}) \end{aligned} \qquad \mathsf{a} \; {}^*\!\!\leftarrow \mathsf{c} \to \mathsf{g}(\mathsf{c}) \to^* \mathsf{g}(\mathsf{a})$$

- no critical pairs but no confluence

$$\begin{aligned} \mathsf{f}(x,x) &\to \mathsf{a} \\ \mathsf{f}(x,\mathsf{g}(x)) &\to \mathsf{b} \\ \mathsf{c} &\to \mathsf{g}(\mathsf{c}) \end{aligned} \qquad \mathsf{a} \leftarrow \mathsf{f}(\mathsf{c},\mathsf{c}) \to \mathsf{f}(\mathsf{c},\mathsf{g}(\mathsf{c})) \to \mathsf{b}$$

Confluence via Critical Pairs

control interference of rewrite rules (notation: $s \leftarrow\!\rtimes\!\to t$ if $s \approx t$ is critical pair)

- critical pair lemma (lecture 6):

  $\mathrm{WCR} \iff {\leftarrow\!\rtimes\!\to} \subseteq {\downarrow}$

- combine with Newman's Lemma (lecture 2):

  SN & ${\leftarrow\!\rtimes\!\to} \subseteq {\downarrow} \implies \mathrm{CR}$

- observe from preceding examples:

  ${\leftarrow\!\rtimes\!\to} = \varnothing \;\not\Longrightarrow\; \mathrm{CR}$

Confluence via Orthogonality

forbid interference of rewrite rules

- no critical pairs
- no equality checks

Definitions (Linearity)

- term t is linear if each variable in $\mathcal{V}\mathrm{ar}(t)$ occurs exactly once in t
- rewrite rule $\ell \to r$ is left–linear if ℓ is linear
- TRS is left–linear if all rewrite rules are left–linear
- rewrite rule $\ell \to r$ is right–linear if r is linear
- TRS is right–linear if all rewrite rules are right–linear
- rewrite rule $\ell \to r$ is linear if ℓ and r is linear
- TRS is linear if all rewrite rules are linear

Examples

- $\mathsf{g}(x) \to \mathsf{f}(x, \mathsf{g}(x))$ left–linear but not right–linear
- $\mathsf{f}(x, x) \to \mathsf{a}$ right–linear but not left–linear

Definition (Orthogonality)

orthogonal TRS is left–linear and lacks critical pairs

Examples

$$\mathsf{I} \cdot x \to x \qquad \mathsf{ack}(0, y) \to \mathsf{s}(y)$$

$$(\mathsf{K} \cdot x) \cdot y \to x \qquad \mathsf{ack}(\mathsf{s}(x), 0) \to \mathsf{ack}(x, \mathsf{s}(0))$$

$$((\mathsf{S} \cdot x) \cdot y) \cdot z \to (x \cdot z) \cdot (y \cdot z) \qquad \mathsf{ack}(\mathsf{s}(x), \mathsf{s}(y)) \to \mathsf{ack}(x, \mathsf{ack}(\mathsf{s}(x), y))$$

Theorem

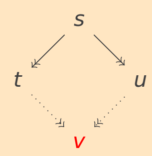
orthogonal TRSs are confluent

Outline

Theorem

orthogonal TRSs are confluent

$\forall\, s, t, u$



$\exists\, v$

Observation

for orthogonal TRSs there is canonical way to compute common reduct v

Definition (Parallel Rewriting)

parallel rewriting $\mathrel{-\!\!\|\!\!\rightarrow}$ is inductively defined as follows:

① $x \mathrel{-\!\!\|\!\!\rightarrow} x$ for all variables x

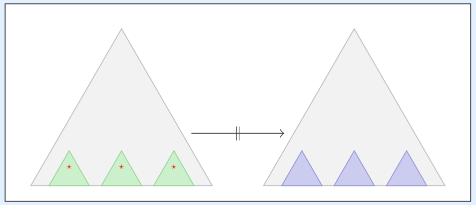
② $f(s_1, \ldots, s_n) \mathrel{-\!\!\|\!\!\rightarrow} f(t_1, \ldots, t_n)$ if $s_i \mathrel{-\!\!\|\!\!\rightarrow} t_i$ for all $1 \leqslant i \leqslant n$

③ $\ell\sigma \mathrel{-\!\!\|\!\!\rightarrow} r\sigma$ if $\ell \to r \in \mathcal{R}$

$$\frac{x \in \mathcal{V}}{x \mathrel{-\!\!\|\!\!\rightarrow} x} \qquad \frac{s \to t}{s \mathrel{-\!\!\|\!\!\rightarrow} t} \qquad \frac{s_1 \mathrel{-\!\!\|\!\!\rightarrow} t_1 \quad \cdots \quad s_n \mathrel{-\!\!\|\!\!\rightarrow} t_n}{f(s_1, \ldots, s_n) \mathrel{-\!\!\|\!\!\rightarrow} f(t_1, \ldots, t_n)}$$

Lemma

$s \mathrel{-\!\!\|\!\!\rightarrow} t \iff$ $\exists$ context C with $n \geqslant 0$ holes, $\exists$ terms $s_1, \ldots, s_n, t_1, \ldots, t_n$ such that $s = C[s_1, \ldots, s_n]$, $t = C[t_1, \ldots, t_n]$, $s_i \to t_i$ for all $1 \leqslant i \leqslant n$



Example

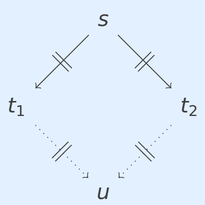
- TRS
  $0 + y \to y \qquad 0 \times y \to 0$
  $\mathsf{s}(x) + y \to \mathsf{s}(x + y) \qquad \mathsf{s}(x) \times y \to (x \times y) + y$
- rewrite sequences

$$\mathsf{s}(0 \times 0) + \mathsf{s}(0) \times (0 + \mathsf{s}(0)) \mathrel{-\!\!\|\!\!\rightarrow} \mathsf{s}(0) + (0 \times (0 + \mathsf{s}(0)) + (0 + \mathsf{s}(0)))$$
$$\mathsf{s}(0 \times 0) + \mathsf{s}(0) \times (0 + \mathsf{s}(0)) \not\mathrel{-\!\!\|\!\!\rightarrow} \mathsf{s}(0 \times 0) + ((0 \times \mathsf{s}(0)) + (0 + \mathsf{s}(0)))$$
$$\mathsf{s}(0 \times 0) + \mathsf{s}(0) \times (0 + \mathsf{s}(0)) \mathrel{-\!\!\|\!\!\rightarrow} \mathsf{s}(0 \times 0) + \mathsf{s}(0) \times (0 + \mathsf{s}(0))$$

Lemma

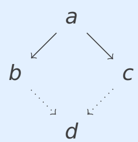
$\to \;\subseteq\; \mathrel{-\!\!\|\!\!\rightarrow} \;\subseteq\; \to^*$

Parallel Moves Lemma



Definition (Diamond Property)

- diamond property $\diamond$
  - $\leftarrow \cdot \to \;\subseteq\; \to \cdot \leftarrow$
  - $\forall\, a, b, c$ $\exists\, d$



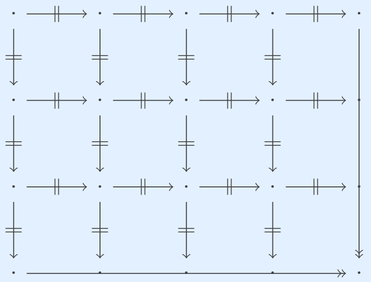
Lemma

ARS $\langle A, \to \rangle$ is confluent if $\to \;\subseteq\; \rightarrowtail \;\subseteq\; \to^*$ for some relation $\rightarrowtail$ on A with diamond property

Corollary

orthogonal TRSs are confluent

Proof



Outline

Definition (Multi–Step Rewriting)

multi–step relation $\mathrel{\circ\!\!\!\rightarrow}$ is inductively defined as follows:

① $x \mathrel{\circ\!\!\!\rightarrow} x$ for all variables x

② $f(s_1, \dots, s_n) \mathrel{\circ\!\!\!\rightarrow} f(t_1, \dots, t_n)$ if $s_i \mathrel{\circ\!\!\!\rightarrow} t_i$ for all $1 \leqslant i \leqslant n$

③ $\ell\sigma \mathrel{\circ\!\!\!\rightarrow} r\tau$ if $\ell \to r \in \mathcal{R}$ and $\underbrace{x\sigma \mathrel{\circ\!\!\!\rightarrow} x\tau \text{ for all variables } x}_{\sigma \mathrel{\circ\!\!\!\rightarrow} \tau}$

Example

- TRS
  $$0 + y \to y \qquad 0 \times y \to 0$$
  $$\mathsf{s}(x) + y \to \mathsf{s}(x + y) \qquad \mathsf{s}(x) \times y \to (x \times y) + y$$
- rewrite sequences
  $$\mathsf{s}(0 \times 0) + \mathsf{s}(0) \times (0 + \mathsf{s}(0)) \not\mathrel{\circ\!\!\!\rightarrow} \mathsf{s}(0) + (0 \times (0 + \mathsf{s}(0)) + \mathsf{s}(0))$$
  $$\mathsf{s}(0 \times 0) + \mathsf{s}(0) \times (0 + \mathsf{s}(0)) \mathrel{\circ\!\!\!\rightarrow} \mathsf{s}(0) + (0 \times \mathsf{s}(0) + \mathsf{s}(0))$$

① $x \mathrel{\circ\!\!\!\rightarrow} x$ for all variables x

② $f(s_1, \dots, s_n) \mathrel{\circ\!\!\!\rightarrow} f(t_1, \dots, t_n)$ if $s_i \mathrel{\circ\!\!\!\rightarrow} t_i$ for all $1 \leqslant i \leqslant n$

③ $\ell\sigma \mathrel{\circ\!\!\!\rightarrow} r\tau$ if $\ell \to r \in \mathcal{R}$ and $\sigma \mathrel{\circ\!\!\!\rightarrow} \tau$

Example

TRS
$$0 + y \to y \qquad 0 \times y \to 0$$
$$\mathsf{s}(x) + y \to \mathsf{s}(x + y) \qquad \mathsf{s}(x) \times y \to (x \times y) + y$$

$$\dfrac{\dfrac{\dfrac{0 \times y \to 0 \qquad \dfrac{}{0 \mathrel{\circ\!\!\!\rightarrow} 0}\,②}{0 \times 0 \mathrel{\circ\!\!\!\rightarrow} 0}\,③}{\mathsf{s}(0 \times 0) \mathrel{\circ\!\!\!\rightarrow} \mathsf{s}(0)}\,② \qquad \dfrac{\mathsf{s}(x) \times y \to (x \times y) + y \qquad \dfrac{}{0 \mathrel{\circ\!\!\!\rightarrow} 0}\,② \qquad \dfrac{0 + y \to y \qquad \dfrac{\dfrac{}{0 \mathrel{\circ\!\!\!\rightarrow} 0}\,②}{\mathsf{s}(0) \mathrel{\circ\!\!\!\rightarrow} \mathsf{s}(0)}\,②}{0 + \mathsf{s}(0) \mathrel{\circ\!\!\!\rightarrow} \mathsf{s}(0)}\,③}{\mathsf{s}(0) \times (0 + \mathsf{s}(0)) \mathrel{\circ\!\!\!\rightarrow} (0 \times \mathsf{s}(0)) + \mathsf{s}(0)}\,③}{\mathsf{s}(0 \times 0) + \mathsf{s}(0) \times (0 + \mathsf{s}(0)) \mathrel{\circ\!\!\!\rightarrow} \mathsf{s}(0) + ((0 \times \mathsf{s}(0)) + \mathsf{s}(0))}\,②$$

Lemma

$\to \;\subseteq\; \mathrel{+\!\!\!\!\to} \;\subseteq\; \mathrel{\circ\!\!\!\!\to} \;\subseteq\; \to^*$

Definition (Maximal Multi–Step Rewriting)

maximal multi–step relation $\mathrel{\bullet\!\!\!\!\to}$ is inductively defined as follows:

① $x \mathrel{\bullet\!\!\!\!\to} x$ for all variables x

② $f(s_1, \ldots, s_n) \mathrel{\bullet\!\!\!\!\to} f(t_1, \ldots, t_n)$ if $s_i \mathrel{\bullet\!\!\!\!\to} t_i$ for all $1 \leqslant i \leqslant n$ and $f(s_1, \ldots, s_n)$ is no redex

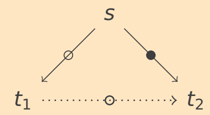
③ $\ell\sigma \mathrel{\bullet\!\!\!\!\to} r\tau$ if $\ell \to r \in \mathcal{R}$ and $\sigma \mathrel{\bullet\!\!\!\!\to} \tau$

Remark

$\mathrel{\bullet\!\!\!\!\to}$ is deterministic for orthogonal TRSs

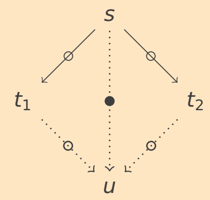
Lemma

for orthogonal TRSs triangle property



Corollary

for orthogonal TRSs



Outline

Definition

TRS is strongly closed if $t \to^{=} \cdot \;^{*}\!\!\leftarrow u$ and $t \to^{*} \cdot \;^{=}\!\!\leftarrow u$ for every critical pair $t \approx u$

Theorem

linear strongly closed TRSs are confluent

Example ❶

- TRS $\mathcal{R}$

$$\mathsf{f}(\mathsf{f}(x,y),z) \to \mathsf{f}(x,\mathsf{f}(y,z)) \qquad \mathsf{f}(x,y) \to \mathsf{f}(y,x)$$

- 4 critical pairs

$$\mathsf{f}(\mathsf{f}(x,\mathsf{f}(y,z)),v) \approx \mathsf{f}(\mathsf{f}(x,y),\mathsf{f}(z,v)) \qquad \mathsf{f}(x,\mathsf{f}(y,z)) \approx \mathsf{f}(z,\mathsf{f}(x,y))$$
$$\mathsf{f}(z,\mathsf{f}(x,y)) \approx \mathsf{f}(x,\mathsf{f}(y,z)) \qquad \mathsf{f}(\mathsf{f}(y,x),z) \approx \mathsf{f}(x,\mathsf{f}(y,z))$$

- $\mathcal{R}$ is linear and strongly closed $\Longrightarrow$ $\mathcal{R}$ is confluent

Example ❷

- linear TRS $\mathcal{R}$

$$(x+y)+z \to x+(y+z) \qquad x+(y+z) \to (x+y)+z \qquad x+y \to y+x$$

- 12 critical pairs

$$\begin{array}{ll}
(x+(y+z))+w \approx (x+y)+(z+w) & (y+x)+z \approx x+(y+z) \\
((x+y)+z)+w \approx x+((y+z)+w) & z+(y+x) \approx x+(y+z) \\
((x+y)+z)+w \approx x+(y+(z+w)) & x+(y+z) \approx z+(y+x) \\
x+(y+(z+w)) \approx ((x+y)+z)+w & x+(z+y) \approx (x+y)+z \\
x+(y+(z+w)) \approx (x+(y+z))+w & (y+z)+x \approx (x+y)+z \\
x+((y+z)+w) \approx (x+y)+(z+w) & (x+y)+z \approx (y+z)+x
\end{array}$$

- $\mathcal{R}$ is strongly closed

$$(x+(y+z))+w \leftrightarrows ((x+y)+z)+w \leftrightarrows (x+y)+(z+w)$$

Remark

linearity cannot be weakened to left–linearity

Example

- left–linear TRS $\mathcal{R}$

$$\begin{array}{lll}
\mathsf{h(f,a,a)} \to \mathsf{h(g,a,a)} & \mathsf{a} \to \mathsf{b} & \mathsf{h}(x,\mathsf{b},y) \to \mathsf{h}(x,y,y) \\
\mathsf{h(g,a,a)} \to \mathsf{h(f,a,a)} & & \mathsf{h}(x,y,\mathsf{b}) \to \mathsf{h}(x,y,y)
\end{array}$$

- $\mathcal{R}$ is strongly closed
- $\mathcal{R}$ is not confluent

$$\mathsf{h(f,b,b)} \;{}^*\!\leftarrow \mathsf{h(f,a,a)} \to \mathsf{h(g,a,a)} \to^* \mathsf{h(g,b,b)} \qquad \mathsf{h(f,b,b)} \not\downarrow \mathsf{h(g,b,b)}$$

Definition

TRS is parallel closed if $t \mathrel{+\!\!\!\!\!\rightarrow} u$ for every critical pair $t \approx u$

Theorem

left–linear parallel closed TRSs are confluent

Example

- left–linear TRS $\mathcal{R}$

$$x+y \to y+x \qquad (x+y)*z \to (x*z)+(y*z) \qquad (y+x)*z \to (x*z)+(y*z)$$

- 4 critical pairs

$$\begin{array}{ll}
(y+x)*z \approx (x*z)+(y*z) & (y*z)+(x*z) \approx (x*z)+(y*z) \\
(x+y)*z \approx (x*z)+(y*z) & (x*z)+(y*z) \approx (y*z)+(x*z)
\end{array}$$

- $\mathcal{R}$ is parallel closed $\implies$ $\mathcal{R}$ is confluent

Long–Standing Open Problem

is every left–linear TRS such that $t \mathrel{\leftarrow\!\!\!\!\!+\!\!\!\!\!\rightarrow} u$ for every critical pair $t \approx u$ confluent ?

Definition

TRS is development closed if $t \mathrel{\circ\!\!\!\!\!\rightarrow} u$ for every critical pair $t \approx u$

Theorem

left–linear development closed TRSs are confluent

Remark

formalized proof employs proof terms (lecture 10)

Example

- left-linear TRS $\mathcal{R}$

$$\mathsf{b} \to \mathsf{c} \qquad \mathsf{f}(\mathsf{g}(\mathsf{a})) \to \mathsf{f}(\mathsf{c}) \qquad \mathsf{g}(x) \to \mathsf{h}(x, \mathsf{i}(x)) \qquad \mathsf{h}(\mathsf{a}, x) \to x \qquad \mathsf{i}(x) \to \mathsf{c}$$

- one critical pair

$$\mathsf{f}(\mathsf{h}(\mathsf{a}, \mathsf{i}(\mathsf{a}))) \approx \mathsf{f}(\mathsf{c})$$

- $\mathcal{R}$ is development closed $\Longrightarrow$ $\mathcal{R}$ is confluent

Confluence Tools

ACP FORT Hakusan CONFident CSI

Confluence Competion **AriWeb**

`https://project-coco.uibk.ac.at/` `https://ari-web.uibk.ac.at/`

Outline

Homework Exercises for May 18

① Exercise 6.3. 1
② Exercise 6.4. 2
③ Exercise 6.12. 2
④ Exercise 6.15. 2
⑤ Exercise 6.14. ☆☆☆

Outline

Lecture Notes

- Section 3.2 (Definition 3.2.12 — Corollary 3.2.15)
- Section 6.1
- Section 6.3 (until Theorem 6.3.6)

Important Concepts

- development closed
- maximal multi–step relation
- multi–step relation
- orthogonality
- parallel closed
- parallel moves lemma
- parallel rewriting
- strongly closed
- triangle property