Term Rewriting

Philipp Dablander and **Aart Middeldorp**

Outline

1. **Summary of Lecture 9**
2. **Proof Terms**
3. **Strategies**
4. **Normalization**
5. **Exercises**
6. **Further Reading**

Definition

orthogonal TRS is left–linear and lacks critical pairs

Definition

parallel rewriting $\twoheadrightarrow\!\!\!\!\!\!\parallel$ is inductively defined as follows:

① $x \mathrel{\|\!\!\!\to} x$ for all variables x

② $f(s_1, \ldots, s_n) \mathrel{\|\!\!\!\to} f(t_1, \ldots, t_n)$ if $s_i \mathrel{\|\!\!\!\to} t_i$ for all $1 \leqslant i \leqslant n$

③ $\ell\sigma \mathrel{\|\!\!\!\to} r\sigma$ if $\ell \to r \in \mathcal{R}$

Parallel Moves Lemma

$\mathrel{\leftarrow\!\!\!\|} \cdot \mathrel{\|\!\!\!\to} \subseteq \mathrel{\|\!\!\!\to} \cdot \mathrel{\leftarrow\!\!\!\|}$ for orthogonal TRSs $\Longrightarrow$ orthogonal TRSs are confluent

Definitions

multi–step relation $\multimap\!\!\!\rightarrow$ is inductively defined as follows:

① $x \multimap\!\!\!\rightarrow x$ for all variables x

② $f(s_1, \ldots, s_n) \multimap\!\!\!\rightarrow f(t_1, \ldots, t_n)$ if $s_i \multimap\!\!\!\rightarrow t_i$ for all $1 \leqslant i \leqslant n$

③ $\ell\sigma \multimap\!\!\!\rightarrow r\tau$ if $\ell \to r \in \mathcal{R}$ and $\underbrace{x\sigma \multimap\!\!\!\rightarrow x\tau \text{ for all variables } x}_{\sigma \multimap\!\!\!\rightarrow \tau}$

maximal multi–step relation $\bullet\!\!\!\rightarrow$ is inductively defined as follows:

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② $f(s_1, \ldots, s_n) \bullet\!\!\!\rightarrow f(t_1, \ldots, t_n)$ if $s_i \bullet\!\!\!\rightarrow t_i$ for all $1 \leqslant i \leqslant n$ and $f(s_1, \ldots, s_n)$ is no redex

③ $\ell\sigma \bullet\!\!\!\rightarrow r\tau$ if $\ell \to r \in \mathcal{R}$ and $\sigma \bullet\!\!\!\rightarrow \tau$

Lemma

$\to \;\subseteq\; \nrightarrow\!\!\!\!\!\shortparallel \;\subseteq\; \multimap\!\!\!\rightarrow \;\subseteq\; \to^*$ and $\bullet\!\!\!\rightarrow \;\subseteq\; \multimap\!\!\!\rightarrow$

Lemma (Triangle Property)

$\leftarrow\!\!\circ\!\!- \cdot \multimap\!\!\bullet\!\!\rightarrow \subseteq -\!\!\circ\!\!\rightarrow$ for orthogonal TRSs

Definitions

TRS

- $\cdots$ is strongly closed if $t \to^{=} \cdot {}^{*}\!\!\leftarrow u$ and $t \to^{*} \cdot {}^{=}\!\!\leftarrow u$
- $\cdots$ is parallel closed if $t \mathrel{+\!\!\!\!\!\!\;\twoheadrightarrow} u$
- $\cdots$ is development closed if $t \mathrel{-\!\!\circ\!\!\rightarrow} u$

for every critical pair $t \approx u$

Theorem

- linear strongly closed TRSs are confluent
- left–linear parallel closed TRSs are confluent
- left–linear development closed TRSs are confluent

Outline

Example

- left–linear TRS

$$\mathsf{h}(\mathsf{f}(x,\mathsf{g}(y))) \xrightarrow{\alpha} \mathsf{h}(\mathsf{f}(x,\mathsf{g}(x))) \qquad \mathsf{g}(\mathsf{a}) \xrightarrow{\gamma} \mathsf{g}(\mathsf{b})$$
$$\mathsf{f}(\mathsf{g}(x),y) \xrightarrow{\beta} \mathsf{f}(\mathsf{g}(x),\mathsf{g}(x)) \qquad \mathsf{b} \xrightarrow{\delta} \mathsf{a}$$

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rule symbol α associated to (left–linear) rewrite rule $\ell \to r$

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$$f(g(x),y) \xrightarrow{\beta} f(g(x),g(x)) \qquad b \xrightarrow{\delta} a$$

- term $s = h(f(h(f(g(a),g(a))),g(a)))$
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- $\mathsf{lhs}(\alpha)$ denotes ℓ
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- $\mathsf{var}(\alpha)$ denotes list $(x_1,\ldots,x_n)$ of variables appearing in ℓ in some fixed order
- arity of α is length of $\mathsf{var}(\alpha)$
- $\langle t_1,\ldots,t_n\rangle_\alpha$ denotes substitution $\{x_i \mapsto t_i \mid 1 \leqslant i \leqslant n\}$

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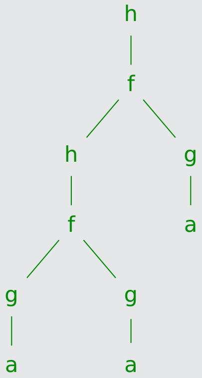

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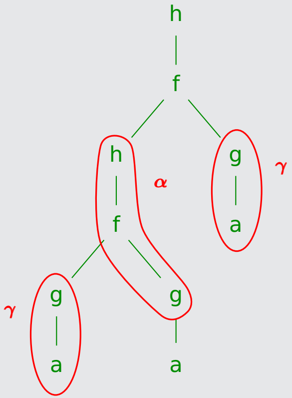

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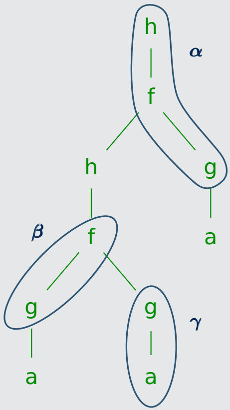

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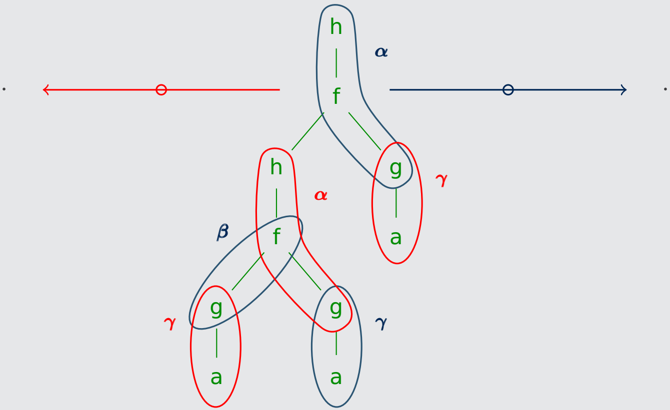

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$$
\begin{aligned}
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\mathsf{src}(\alpha(A_1, \dots, A_n)) &= \mathsf{lhs}(\alpha)\langle \mathsf{src}(A_1), \dots, \mathsf{src}(A_n)\rangle_{\alpha} \\
\mathsf{tgt}(f(A_1, \dots, A_n)) &= f(\mathsf{tgt}(A_1), \dots, \mathsf{tgt}(A_n)) \\
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\end{aligned}
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Remark

proof term A witnesses multi-step $\mathsf{src}(A) \twoheadrightarrow \mathsf{tgt}(A)$

Example (cont'd)

$$\mathsf{h}(\mathsf{f}(x,\mathsf{g}(y))) \xrightarrow{\alpha} \mathsf{h}(\mathsf{f}(x,\mathsf{g}(x))) \qquad \mathsf{g}(\mathsf{a}) \xrightarrow{\gamma} \mathsf{g}(\mathsf{b}) \qquad s = \mathsf{h}(\mathsf{f}(\mathsf{h}(\mathsf{f}(\mathsf{g}(\mathsf{a}),\mathsf{g}(\mathsf{a}))),\mathsf{g}(\mathsf{a})))$$

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- $\mathsf{tgt}(B) = \mathsf{h}(\mathsf{f}(\mathsf{tgt}(\mathsf{h}(\beta(\mathsf{a},\gamma))),\mathsf{g}(\mathsf{tgt}(\mathsf{h}(\beta(\mathsf{a},\gamma))))))$

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Example (cont'd)

$$\mathsf{h}(\mathsf{f}(x,\mathsf{g}(y))) \xrightarrow{\alpha} \mathsf{h}(\mathsf{f}(x,\mathsf{g}(x))) \quad \mathsf{g}(\mathsf{a}) \xrightarrow{\gamma} \mathsf{g}(\mathsf{b}) \quad s = \mathsf{h}(\mathsf{f}(\mathsf{h}(\mathsf{f}(\mathsf{g}(\mathsf{a}),\mathsf{g}(\mathsf{a}))),\mathsf{g}(\mathsf{a})))$$

$$\mathsf{f}(\mathsf{g}(x),y) \xrightarrow{\beta} \mathsf{f}(\mathsf{g}(x),\mathsf{g}(x)) \quad \mathsf{b} \xrightarrow{\delta} \mathsf{a} \quad A = \mathsf{h}(\mathsf{f}(\alpha(\gamma,\mathsf{a}),\gamma)) \quad B = \alpha(\mathsf{h}(\beta(\mathsf{a},\gamma)),\mathsf{a})$$

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Lemma

$s \multimap\!\!\to t \iff \mathrm{src}(A) = s$ and $\mathrm{tgt}(A) = t$ for some proof term A

Theorem

left−linear development closed TRSs are confluent

Remarks

- formalized proof employs proof terms

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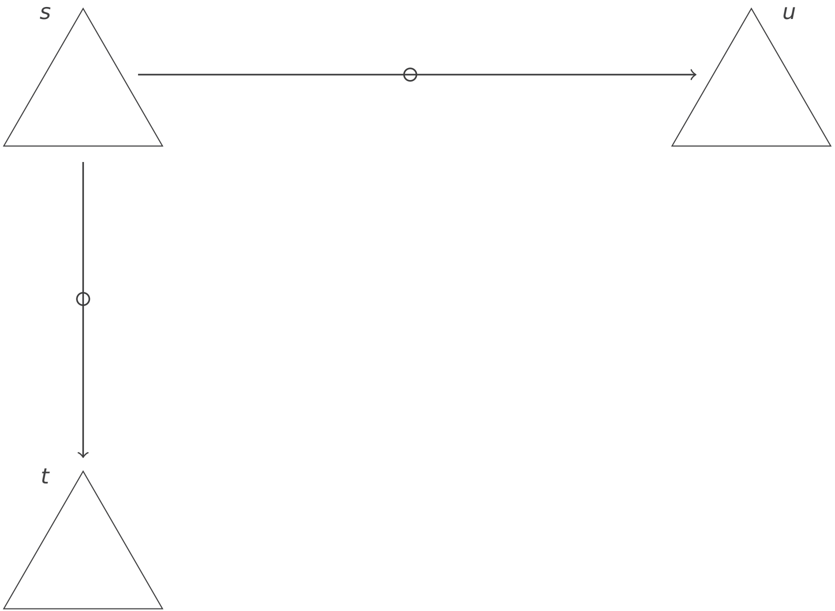
- result follows from diamond property of $\multimap$
- proof employs induction on amount of overlap between two multi−steps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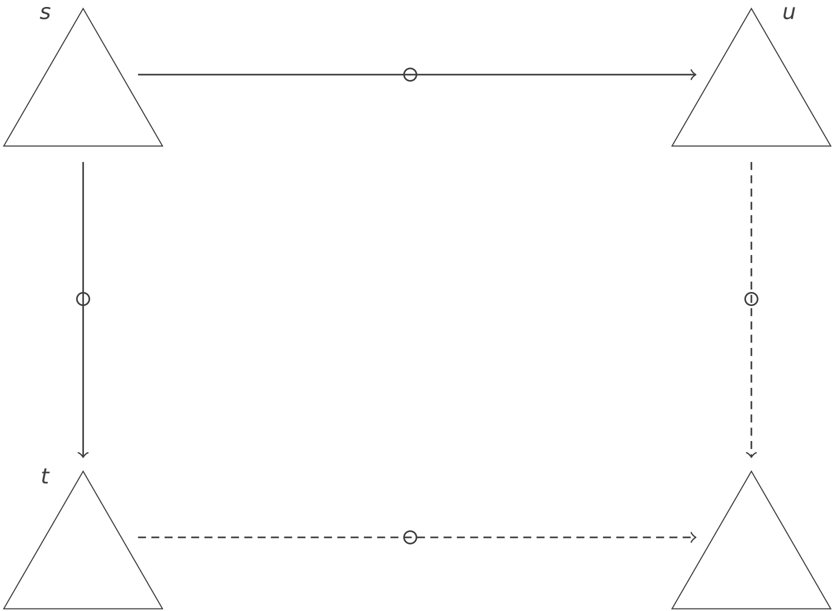
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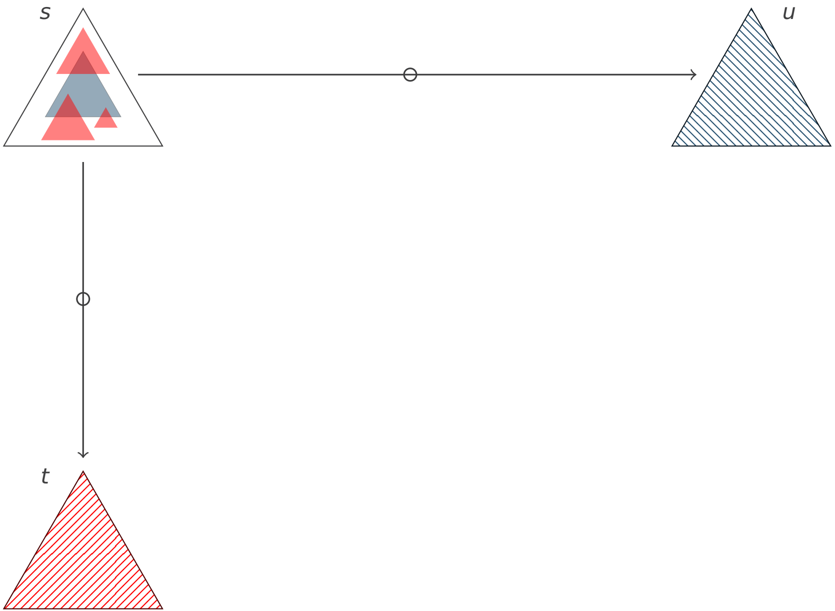
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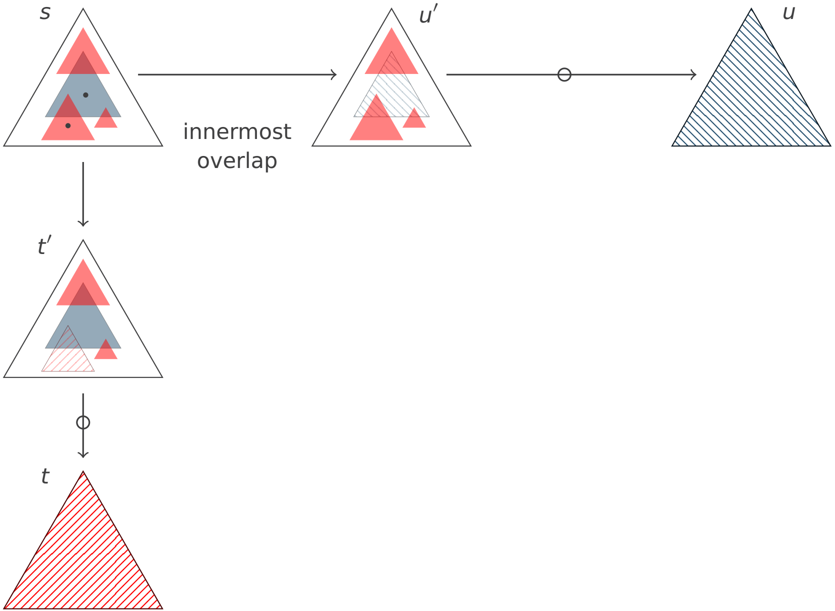
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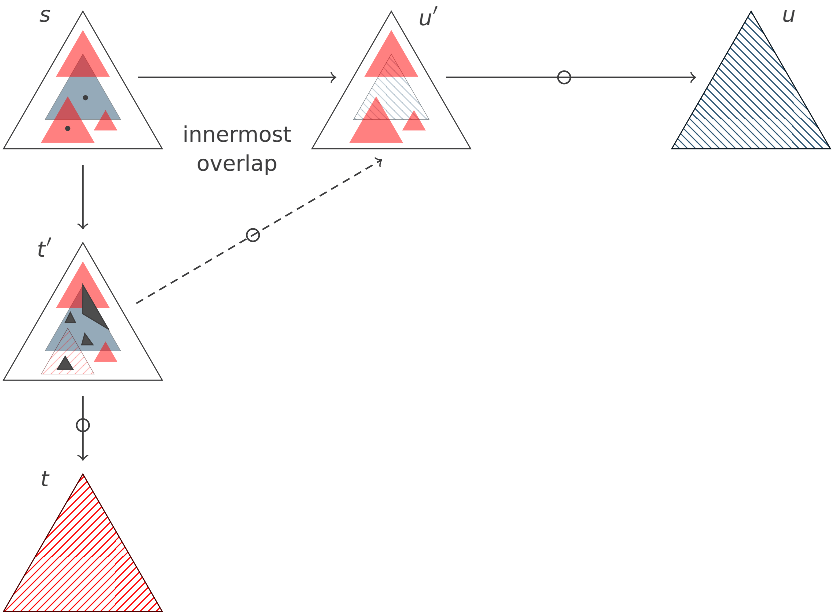

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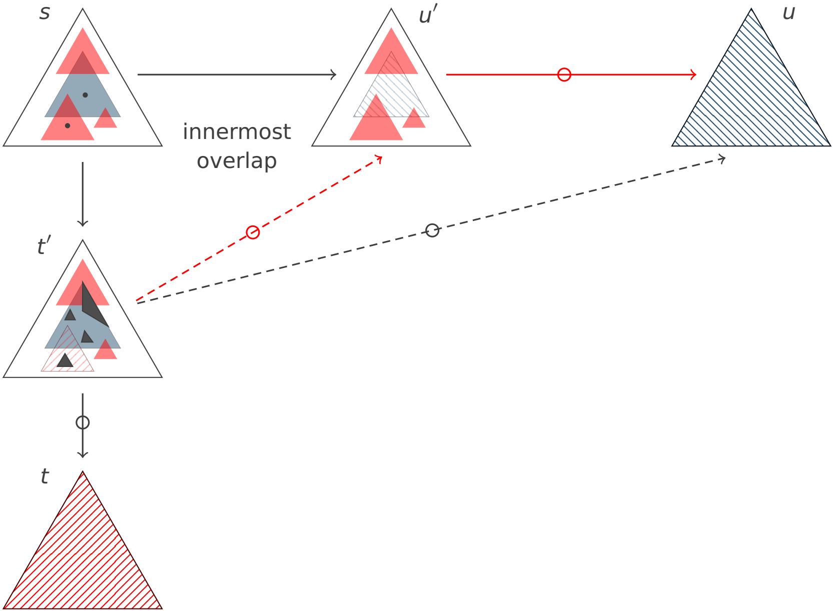
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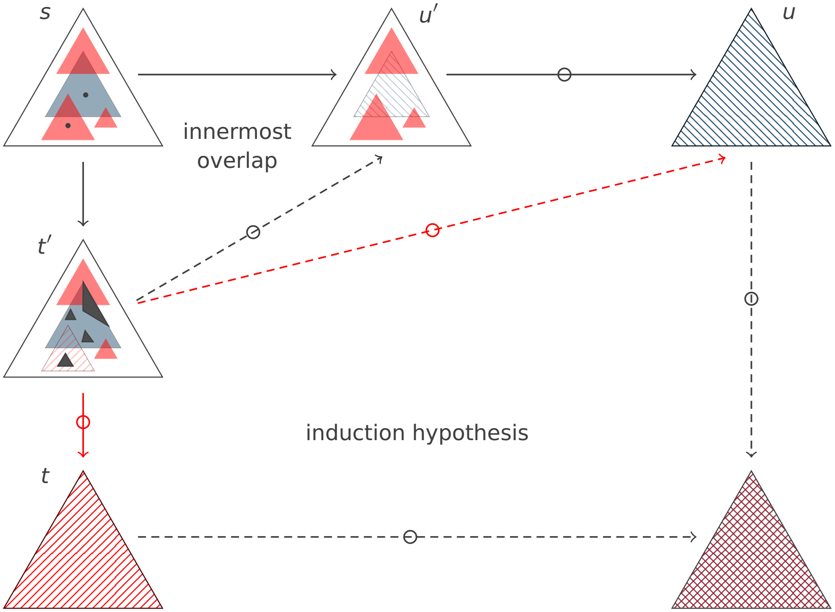
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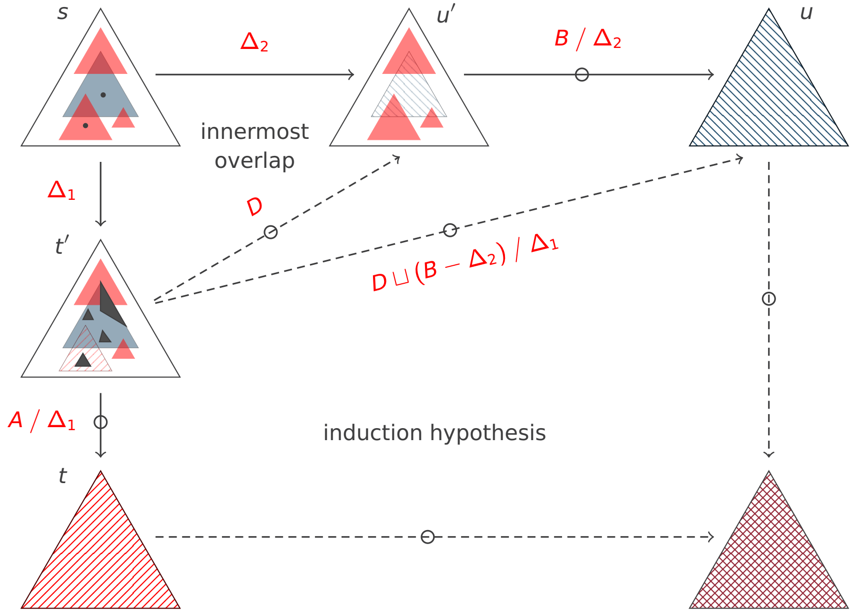
induction hypothesis



s
Δ2
u′
B / Δ2
u
innermost
overlap
Δ1
D
D ⊔ (B − Δ2) / Δ1
t′
A / Δ1
induction hypothesis
t

Outline

Definitions (Strategies)

- one–step rewrite strategy $\mathcal{S}$ for TRS $\mathcal{R}$ is relation $\xrightarrow{\mathcal{S}} \subseteq \to_{\mathcal{R}}$ such that $\mathsf{NF}(\mathcal{S}) = \mathsf{NF}(\mathcal{R})$

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Definitions (Strategies, cont'd)

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- (hyper–)normalizing strategy avoids non–terminating computations, if possible

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- perpetual strategy avoids terminating computations, if possible

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Remarks

- (hyper–)normalizing strategy avoids non–terminating computations, if possible
- perpetual strategy avoids terminating computations, if possible

Lemma

for terminating TRSs every strategy is hyper–normalizing and perpetual

Example

- rewrite rules

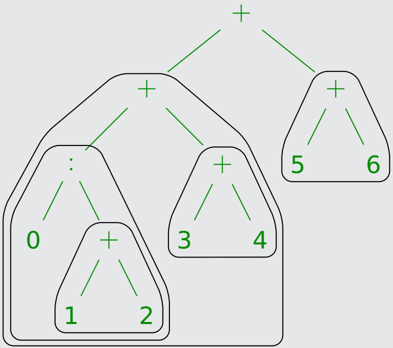
$0+0 \to 0$	$1+0 \to 1$	$\cdots$	$9+0 \to 9$	$0:x \to x$
$0+1 \to 1$	$1+1 \to 2$	$\cdots$	$9+1 \to 1:0$	$x+(y:z) \to y:(x+z)$
$0+2 \to 2$	$1+2 \to 3$	$\cdots$	$9+2 \to 1:1$	$(x:y)+z \to x:(y+z)$
$0+3 \to 3$	$1+3 \to 4$	$\cdots$	$9+3 \to 1:2$	$x:(y:z) \to (x+y):z$
$0+4 \to 4$	$1+4 \to 5$	$\cdots$	$9+4 \to 1:3$	
$0+5 \to 5$	$1+5 \to 6$	$\cdots$	$9+5 \to 1:4$	
$0+6 \to 6$	$1+6 \to 7$	$\cdots$	$9+6 \to 1:5$	
$0+7 \to 7$	$1+7 \to 8$	$\cdots$	$9+7 \to 1:6$	
$0+8 \to 8$	$1+8 \to 9$	$\cdots$	$9+8 \to 1:7$	
$0+9 \to 9$	$1+9 \to 1:0$	$\cdots$	$9+9 \to 1:8$	

- term $((0:(1+2))+(3+4))+(5+6)$

Example (cont'd)

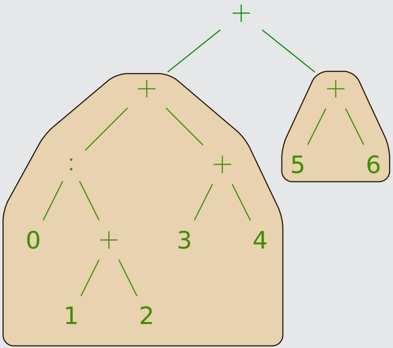
- term $\boxed{\boxed{0 : \boxed{1 + 2}} + \boxed{3 + 4}} + \boxed{5 + 6}$

- tree representation



Example (cont'd)

- term $0 : 1 + 2 + 3 + 4 + 5 + 6$
- tree representation



outermost redexes

Example (cont'd)

- term

$0 : 1 + 2 + 3 + 4 + 5 + 6$

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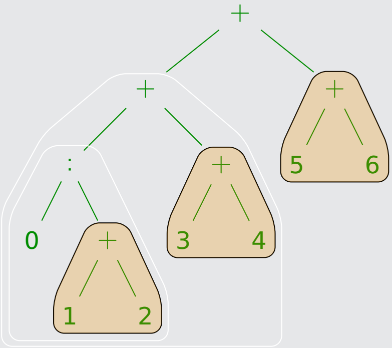

innermost redexes

Example (cont'd)

- term

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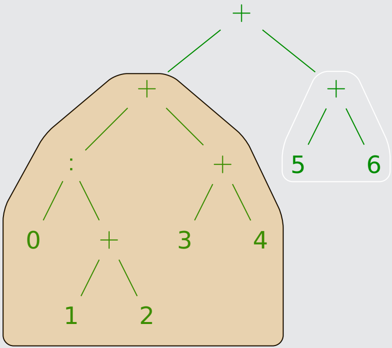
- tree representation



leftmost outermost strategy

Example (cont'd)

- term $0 : 1 + 2 + 3 + 4 + 5 + 6$

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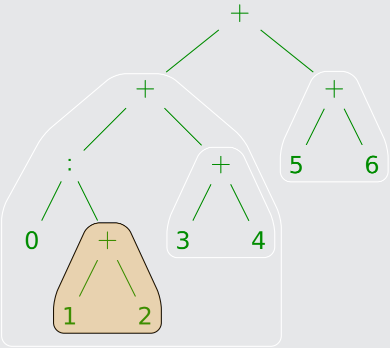

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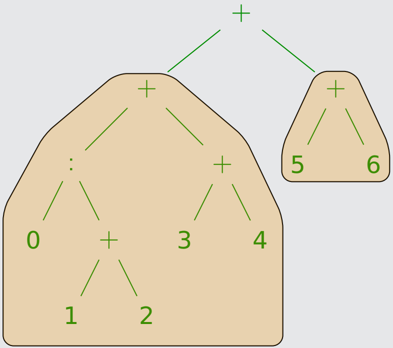
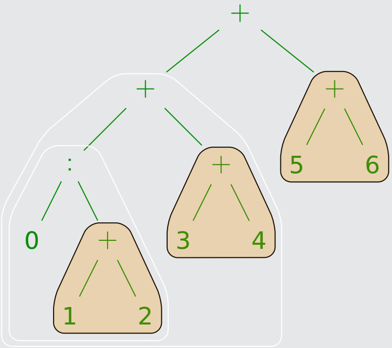


maximal outermost strategy

Example (cont'd)

- term $0 : (1 + 2) + (3 + 4) + (5 + 6)$
- tree representation



maximal innermost strategy

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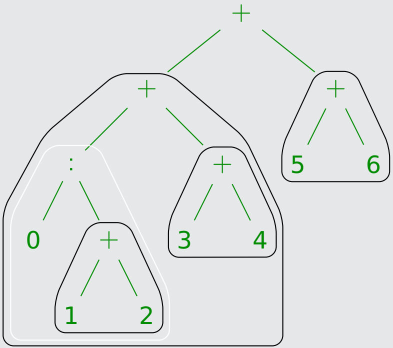

maximal strategy

Example (cont'd)

- leftmost outermost strategy

$$((0 : (1 + 2)) + (3 + 4)) + (5 + 6)$$

- leftmost innermost strategy

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Example (cont'd)

- leftmost outermost strategy

$$((0 : (1 + 2)) + (3 + 4)) + (5 + 6)$$
$$\to (0 : ((1 + 2) + (3 + 4))) + (5 + 6)$$

- leftmost innermost strategy

$$((0 : (1 + 2)) + (3 + 4)) + (5 + 6)$$
$$\to ((0 : 3) + (3 + 4)) + (5 + 6)$$

Example (cont'd)

- leftmost outermost strategy

$$((0 : (1+2)) + (3+4)) + (5+6)$$
$$\to (0 : ((1+2) + (3+4))) + (5+6) \to 0 : (((1+2) + (3+4)) + (5+6))$$

- leftmost innermost strategy

$$((0 : (1+2)) + (3+4)) + (5+6)$$
$$\to ((0 : 3) + (3+4)) + (5+6) \to (3 + (3+4)) + (5+6)$$

Example (cont'd)

- leftmost outermost strategy

$$((0 : (1 + 2)) + (3 + 4)) + (5 + 6)$$
$$\to (0 : ((1 + 2) + (3 + 4))) + (5 + 6) \to 0 : (((1 + 2) + (3 + 4)) + (5 + 6))$$
$$\to ((1 + 2) + (3 + 4)) + (5 + 6)$$

- leftmost innermost strategy

$$((0 : (1 + 2)) + (3 + 4)) + (5 + 6)$$
$$\to ((0 : 3) + (3 + 4)) + (5 + 6) \to (3 + (3 + 4)) + (5 + 6) \to (3 + 7) + (5 + 6)$$

Example (cont'd)

- leftmost outermost strategy

$$((0 : (1+2)) + (3+4)) + (5+6)$$
$$\to (0 : ((1+2) + (3+4))) + (5+6) \to 0 : (((1+2) + (3+4)) + (5+6))$$
$$\to ((1+2) + (3+4)) + (5+6) \to (3 + (3+4)) + (5+6)$$

- leftmost innermost strategy

$$((0 : (1+2)) + (3+4)) + (5+6)$$
$$\to ((0 : 3) + (3+4)) + (5+6) \to (3 + (3+4)) + (5+6) \to (3+7) + (5+6)$$
$$\to (1 : 0) + (5+6)$$

Example (cont'd)

- leftmost outermost strategy

$$((0:(1+2))+(3+4))+(5+6)$$
$$\to (0:((1+2)+(3+4)))+(5+6) \to 0:(((1+2)+(3+4))+(5+6))$$
$$\to ((1+2)+(3+4))+(5+6) \to (3+(3+4))+(5+6) \to (3+7)+(5+6)$$

- leftmost innermost strategy

$$((0:(1+2))+(3+4))+(5+6)$$
$$\to ((0:3)+(3+4))+(5+6) \to (3+(3+4))+(5+6) \to (3+7)+(5+6)$$
$$\to (1:0)+(5+6) \to (1:0)+(1:1)$$

Example (cont'd)

- leftmost outermost strategy

$$((0:(1+2))+(3+4))+(5+6)$$
$$\to (0:((1+2)+(3+4)))+(5+6) \to 0:(((1+2)+(3+4))+(5+6))$$
$$\to ((1+2)+(3+4))+(5+6) \to (3+(3+4))+(5+6) \to (3+7)+(5+6)$$
$$\to (1:0)+(5+6)$$

- leftmost innermost strategy

$$((0:(1+2))+(3+4))+(5+6)$$
$$\to ((0:3)+(3+4))+(5+6) \to (3+(3+4))+(5+6) \to (3+7)+(5+6)$$
$$\to (1:0)+(5+6) \to (1:0)+(1:1) \to 1:(0+(1:1))$$

Example (cont'd)

- leftmost outermost strategy

$$
\begin{aligned}
&((0 : (1+2)) + (3+4)) + (5+6) \\
&\quad \to (0 : ((1+2) + (3+4))) + (5+6) \to 0 : (((1+2) + (3+4)) + (5+6)) \\
&\quad \to ((1+2) + (3+4)) + (5+6) \to (3 + (3+4)) + (5+6) \to (3+7) + (5+6) \\
&\quad \to (1:0) + (5+6) \to 1 : (0 + (5+6))
\end{aligned}
$$

- leftmost innermost strategy

$$
\begin{aligned}
&((0 : (1+2)) + (3+4)) + (5+6) \\
&\quad \to ((0:3) + (3+4)) + (5+6) \to (3 + (3+4)) + (5+6) \to (3+7) + (5+6) \\
&\quad \to (1:0) + (5+6) \to (1:0) + (1:1) \to 1 : (0 + (1:1)) \to 1 : (1 : (0+1))
\end{aligned}
$$

Example (cont'd)

- leftmost outermost strategy

$$((0:(1+2))+(3+4))+(5+6)$$
$$\to (0:((1+2)+(3+4)))+(5+6) \to 0:(((1+2)+(3+4))+(5+6))$$
$$\to ((1+2)+(3+4))+(5+6) \to (3+(3+4))+(5+6) \to (3+7)+(5+6)$$
$$\to (1:0)+(5+6) \to 1:(0+(5+6)) \to 1:(0+(1:1))$$

- leftmost innermost strategy

$$((0:(1+2))+(3+4))+(5+6)$$
$$\to ((0:3)+(3+4))+(5+6) \to (3+(3+4))+(5+6) \to (3+7)+(5+6)$$
$$\to (1:0)+(5+6) \to (1:0)+(1:1) \to 1:(0+(1:1)) \to 1:(1:(0+1))$$
$$\to 1:(1:1)$$

Example (cont'd)

- leftmost outermost strategy

$$
\begin{aligned}
&((0 : (1+2)) + (3+4)) + (5+6) \\
&\quad \to (0 : ((1+2) + (3+4))) + (5+6) \to 0 : (((1+2) + (3+4)) + (5+6)) \\
&\quad \to ((1+2) + (3+4)) + (5+6) \to (3 + (3+4)) + (5+6) \to (3+7) + (5+6) \\
&\quad \to (1:0) + (5+6) \to 1 : (0 + (5+6)) \to 1 : (0 + (1:1)) \to 1 : (1 : (0+1))
\end{aligned}
$$

- leftmost innermost strategy

$$
\begin{aligned}
&((0 : (1+2)) + (3+4)) + (5+6) \\
&\quad \to ((0:3) + (3+4)) + (5+6) \to (3 + (3+4)) + (5+6) \to (3+7) + (5+6) \\
&\quad \to (1:0) + (5+6) \to (1:0) + (1:1) \to 1 : (0 + (1:1)) \to 1 : (1 : (0+1)) \\
&\quad \to 1 : (1:1) \to (1+1) : 1
\end{aligned}
$$

Example (cont'd)

- leftmost outermost strategy

$$
\begin{aligned}
&((0 : (1+2)) + (3+4)) + (5+6)\\
&\quad \to (0 : ((1+2) + (3+4))) + (5+6) \to 0 : (((1+2) + (3+4)) + (5+6))\\
&\quad \to ((1+2) + (3+4)) + (5+6) \to (3 + (3+4)) + (5+6) \to (3+7) + (5+6)\\
&\quad \to (1 : 0) + (5+6) \to 1 : (0 + (5+6)) \to 1 : (0 + (1 : 1)) \to 1 : (1 : (0+1))\\
&\quad \to (1+1) : (0+1)
\end{aligned}
$$

- leftmost innermost strategy (10 redexes)

$$
\begin{aligned}
&((0 : (1+2)) + (3+4)) + (5+6)\\
&\quad \to ((0 : 3) + (3+4)) + (5+6) \to (3 + (3+4)) + (5+6) \to (3+7) + (5+6)\\
&\quad \to (1 : 0) + (5+6) \to (1 : 0) + (1 : 1) \to 1 : (0 + (1 : 1)) \to 1 : (1 : (0+1))\\
&\quad \to 1 : (1 : 1) \to (1+1) : 1 \to 2 : 1
\end{aligned}
$$

Example (cont'd)

- leftmost outermost strategy

$$((0:(1+2))+(3+4))+(5+6)$$
$$\to (0:((1+2)+(3+4)))+(5+6) \to 0:(((1+2)+(3+4))+(5+6))$$
$$\to ((1+2)+(3+4))+(5+6) \to (3+(3+4))+(5+6) \to (3+7)+(5+6)$$
$$\to (1:0)+(5+6) \to 1:(0+(5+6)) \to 1:(0+(1:1)) \to 1:(1:(0+1))$$
$$\to (1+1):(0+1) \to 2:(0+1)$$

- leftmost innermost strategy (10 redexes)

$$((0:(1+2))+(3+4))+(5+6)$$
$$\to ((0:3)+(3+4))+(5+6) \to (3+(3+4))+(5+6) \to (3+7)+(5+6)$$
$$\to (1:0)+(5+6) \to (1:0)+(1:1) \to 1:(0+(1:1)) \to 1:(1:(0+1))$$
$$\to 1:(1:1) \to (1+1):1 \to 2:1$$

Example (cont'd)

- leftmost outermost strategy (12 redexes)

$$((0 : (1 + 2)) + (3 + 4)) + (5 + 6)$$
$$\to (0 : ((1 + 2) + (3 + 4))) + (5 + 6) \to 0 : (((1 + 2) + (3 + 4)) + (5 + 6))$$
$$\to ((1 + 2) + (3 + 4)) + (5 + 6) \to (3 + (3 + 4)) + (5 + 6) \to (3 + 7) + (5 + 6)$$
$$\to (1 : 0) + (5 + 6) \to 1 : (0 + (5 + 6)) \to 1 : (0 + (1 : 1)) \to 1 : (1 : (0 + 1))$$
$$\to (1 + 1) : (0 + 1) \to 2 : (0 + 1) \to 2 : 1$$

- leftmost innermost strategy (10 redexes)

$$((0 : (1 + 2)) + (3 + 4)) + (5 + 6)$$
$$\to ((0 : 3) + (3 + 4)) + (5 + 6) \to (3 + (3 + 4)) + (5 + 6) \to (3 + 7) + (5 + 6)$$
$$\to (1 : 0) + (5 + 6) \to (1 : 0) + (1 : 1) \to 1 : (0 + (1 : 1)) \to 1 : (1 : (0 + 1))$$
$$\to 1 : (1 : 1) \to (1 + 1) : 1 \to 2 : 1$$

Example (cont'd)

- leftmost outermost strategy (12 redexes)

$$((0:(1+2))+(3+4))+(5+6)$$
$$\to (0:((1+2)+(3+4)))+(5+6) \to 0:(((1+2)+(3+4))+(5+6))$$
$$\to ((1+2)+(3+4))+(5+6) \to (3+(3+4))+(5+6) \to (3+7)+(5+6)$$
$$\to (1:0)+(5+6) \to 1:(0+(5+6)) \to 1:(0+(1:1)) \to 1:(1:(0+1))$$
$$\to (1+1):(0+1) \to 2:(0+1) \to 2:1$$

- leftmost innermost strategy (10 redexes)

$$((0:(1+2))+(3+4))+(5+6)$$
$$\to ((0:3)+(3+4))+(5+6) \to (3+(3+4))+(5+6) \to (3+7)+(5+6)$$
$$\to (1:0)+(5+6) \to (1:0)+(1:1) \to 1:(0+(1:1)) \to 1:(1:(0+1))$$
$$\to 1:(1:1) \to (1+1):1 \to 2:1$$

Example (cont'd)

- leftmost outermost strategy (12 redexes)

$$((0:(1+2))+(3+4))+(5+6)$$
$$\to (0:((1+2)+(3+4)))+(5+6) \to 0:(((1+2)+(3+4))+(5+6))$$
$$\to ((1+2)+(3+4))+(5+6) \to (3+(3+4))+(5+6) \to (3+7)+(5+6)$$
$$\to (1:0)+(5+6) \to 1:(0+(5+6)) \to 1:(0+(1:1)) \to 1:(1:(0+1))$$
$$\to (1+1):(0+1) \to 2:(0+1) \to 2:1$$

- leftmost innermost strategy (10 redexes)

$$((0:(1+2))+(3+4))+(5+6)$$
$$\to ((0:3)+(3+4))+(5+6) \to (3+(3+4))+(5+6) \to (3+7)+(5+6)$$
$$\to (1:0)+(5+6) \to (1:0)+(1:1) \to 1:((1:0)+1)$$

Example (cont'd)

- leftmost outermost strategy (12 redexes)

$$((0:(1+2))+(3+4))+(5+6)$$
$$\to (0:((1+2)+(3+4)))+(5+6) \to 0:(((1+2)+(3+4))+(5+6))$$
$$\to ((1+2)+(3+4))+(5+6) \to (3+(3+4))+(5+6) \to (3+7)+(5+6)$$
$$\to (1:0)+(5+6) \to 1:(0+(5+6)) \to 1:(0+(1:1)) \to 1:(1:(0+1))$$
$$\to (1+1):(0+1) \to 2:(0+1) \to 2:1$$

- leftmost innermost strategy (10 redexes)

$$((0:(1+2))+(3+4))+(5+6)$$
$$\to ((0:3)+(3+4))+(5+6) \to (3+(3+4))+(5+6) \to (3+7)+(5+6)$$
$$\to (1:0)+(5+6) \to (1:0)+(1:1) \to 1:((1:0)+1) \to 1:(1:(0+1))$$

Example (cont'd)

- leftmost outermost strategy (12 redexes)

$$((0:(1+2))+(3+4))+(5+6)$$
$$\to (0:((1+2)+(3+4)))+(5+6) \to 0:(((1+2)+(3+4))+(5+6))$$
$$\to ((1+2)+(3+4))+(5+6) \to (3+(3+4))+(5+6) \to (3+7)+(5+6)$$
$$\to (1:0)+(5+6) \to 1:(0+(5+6)) \to 1:(0+(1:1)) \to 1:(1:(0+1))$$
$$\to (1+1):(0+1) \to 2:(0+1) \to 2:1$$

- leftmost innermost strategy (10 redexes)

$$((0:(1+2))+(3+4))+(5+6)$$
$$\to ((0:3)+(3+4))+(5+6) \to (3+(3+4))+(5+6) \to (3+7)+(5+6)$$
$$\to (1:0)+(5+6) \to (1:0)+(1:1) \to 1:((1:0)+1) \to 1:(1:(0+1))$$
$$\to 1:(1:1)$$

Example (cont'd)

- leftmost outermost strategy (12 redexes)

$$
\begin{aligned}
&((0 : (1 + 2)) + (3 + 4)) + (5 + 6)\\
&\quad \to (0 : ((1 + 2) + (3 + 4))) + (5 + 6) \to 0 : (((1 + 2) + (3 + 4)) + (5 + 6))\\
&\quad \to ((1 + 2) + (3 + 4)) + (5 + 6) \to (3 + (3 + 4)) + (5 + 6) \to (3 + 7) + (5 + 6)\\
&\quad \to (1 : 0) + (5 + 6) \to 1 : (0 + (5 + 6)) \to 1 : (0 + (1 : 1)) \to 1 : (1 : (0 + 1))\\
&\quad \to (1 + 1) : (0 + 1) \to 2 : (0 + 1) \to 2 : 1
\end{aligned}
$$

- leftmost innermost strategy (10 redexes)

$$
\begin{aligned}
&((0 : (1 + 2)) + (3 + 4)) + (5 + 6)\\
&\quad \to ((0 : 3) + (3 + 4)) + (5 + 6) \to (3 + (3 + 4)) + (5 + 6) \to (3 + 7) + (5 + 6)\\
&\quad \to (1 : 0) + (5 + 6) \to (1 : 0) + (1 : 1) \to 1 : ((1 : 0) + 1) \to 1 : (1 : (0 + 1))\\
&\quad \to 1 : (1 : 1) \to (1 + 1) : 1
\end{aligned}
$$

Example (cont'd)

- leftmost outermost strategy (12 redexes)

$$
\begin{aligned}
&((0:(1+2))+(3+4))+(5+6) \\
&\to (0:((1+2)+(3+4)))+(5+6) \to 0:(((1+2)+(3+4))+(5+6)) \\
&\to ((1+2)+(3+4))+(5+6) \to (3+(3+4))+(5+6) \to (3+7)+(5+6) \\
&\to (1:0)+(5+6) \to 1:(0+(5+6)) \to 1:(0+(1:1)) \to 1:(1:(0+1)) \\
&\to (1+1):(0+1) \to 2:(0+1) \to 2:1
\end{aligned}
$$

- leftmost innermost strategy (10 redexes)

$$
\begin{aligned}
&((0:(1+2))+(3+4))+(5+6) \\
&\to ((0:3)+(3+4))+(5+6) \to (3+(3+4))+(5+6) \to (3+7)+(5+6) \\
&\to (1:0)+(5+6) \to (1:0)+(1:1) \to 1:((1:0)+1) \to 1:(1:(0+1)) \\
&\to 1:(1:1) \to (1+1):1 \to 2:1
\end{aligned}
$$

Example (cont'd)

- maximal outermost strategy

$$((0 : (1+2)) + (3+4)) + (5+6)$$

- maximal innermost strategy

$$((0 : (1+2)) + (3+4)) + (5+6)$$

Example (cont'd)

- maximal outermost strategy

$$((0 : (1 + 2)) + (3 + 4)) + (5 + 6)$$
$$\mathrel{+\!\!\!\!\!\!\!\mapsto} (0 : ((1 + 2) + (3 + 4))) + (1 : 1)$$

- maximal innermost strategy

$$((0 : (1 + 2)) + (3 + 4)) + (5 + 6)$$
$$\mathrel{+\!\!\!\!\!\!\!\mapsto} ((0 : 3) + 7) + (1 : 1)$$

Example (cont'd)

- maximal outermost strategy

$$((0 : (1 + 2)) + (3 + 4)) + (5 + 6)$$
$$\;\not\twoheadrightarrow\!\!\!\!\!\!\!\!\!\!\;\;\; \nrightarrow (0 : ((1 + 2) + (3 + 4))) + (1 : 1) \nrightarrow 0 : (((1 + 2) + (3 + 4)) + (1 : 1))$$

- maximal innermost strategy

$$((0 : (1 + 2)) + (3 + 4)) + (5 + 6)$$
$$\nrightarrow ((0 : 3) + 7) + (1 : 1) \nrightarrow (3 + 7) + (1 : 1)$$

Example (cont'd)

- maximal outermost strategy

$$((0:(1+2))+(3+4))+(5+6)$$
$$\xrightarrow{\parallel} (0:((1+2)+(3+4)))+(1:1) \xrightarrow{\parallel} 0:(((1+2)+(3+4))+(1:1))$$
$$\xrightarrow{\parallel} ((1+2)+(3+4))+(1:1)$$

- maximal innermost strategy

$$((0:(1+2))+(3+4))+(5+6)$$
$$\xrightarrow{\parallel} ((0:3)+7)+(1:1) \xrightarrow{\parallel} (3+7)+(1:1) \xrightarrow{\parallel} (1:0)+(1:1)$$

Example (cont'd)

- maximal outermost strategy

$$((0 : (1 + 2)) + (3 + 4)) + (5 + 6)$$
$$\mathrel{+\!\!\!\!\!\!\to} (0 : ((1 + 2) + (3 + 4))) + (1 : 1) \mathrel{+\!\!\!\!\!\!\to} 0 : (((1 + 2) + (3 + 4)) + (1 : 1))$$
$$\mathrel{+\!\!\!\!\!\!\to} ((1 + 2) + (3 + 4)) + (1 : 1) \mathrel{+\!\!\!\!\!\!\to} 1 : (((1 + 2) + (3 + 4)) + 1)$$

- maximal innermost strategy

$$((0 : (1 + 2)) + (3 + 4)) + (5 + 6)$$
$$\mathrel{+\!\!\!\!\!\!\to} ((0 : 3) + 7) + (1 : 1) \mathrel{+\!\!\!\!\!\!\to} (3 + 7) + (1 : 1) \mathrel{+\!\!\!\!\!\!\to} (1 : 0) + (1 : 1)$$
$$\mathrel{+\!\!\!\!\!\!\to} 1 : (0 + (1 : 1))$$

Example (cont'd)

- maximal outermost strategy

$$((0 : (1 + 2)) + (3 + 4)) + (5 + 6)$$
$$\mathrel{+\!\!\!\!\rightarrow} (0 : ((1 + 2) + (3 + 4))) + (1 : 1) \mathrel{+\!\!\!\!\rightarrow} 0 : (((1 + 2) + (3 + 4)) + (1 : 1))$$
$$\mathrel{+\!\!\!\!\rightarrow} ((1 + 2) + (3 + 4)) + (1 : 1) \mathrel{+\!\!\!\!\rightarrow} 1 : (((1 + 2) + (3 + 4)) + 1)$$
$$\mathrel{+\!\!\!\!\rightarrow} 1 : ((3 + 7) + 1)$$

- maximal innermost strategy

$$((0 : (1 + 2)) + (3 + 4)) + (5 + 6)$$
$$\mathrel{+\!\!\!\!\rightarrow} ((0 : 3) + 7) + (1 : 1) \mathrel{+\!\!\!\!\rightarrow} (3 + 7) + (1 : 1) \mathrel{+\!\!\!\!\rightarrow} (1 : 0) + (1 : 1)$$
$$\mathrel{+\!\!\!\!\rightarrow} 1 : (0 + (1 : 1)) \mathrel{+\!\!\!\!\rightarrow} 1 : (1 : (0 + 1))$$

Example (cont'd)

- maximal outermost strategy

$$((0:(1+2))+(3+4))+(5+6)$$
$$\mathrel{+\!\!\!\!\!\rightarrow} (0:((1+2)+(3+4)))+(1:1) \mathrel{+\!\!\!\!\!\rightarrow} 0:(((1+2)+(3+4))+(1:1))$$
$$\mathrel{+\!\!\!\!\!\rightarrow} ((1+2)+(3+4))+(1:1) \mathrel{+\!\!\!\!\!\rightarrow} 1:(((1+2)+(3+4))+1)$$
$$\mathrel{+\!\!\!\!\!\rightarrow} 1:((3+7)+1) \mathrel{+\!\!\!\!\!\rightarrow} 1:((1:0)+1)$$

- maximal innermost strategy

$$((0:(1+2))+(3+4))+(5+6)$$
$$\mathrel{+\!\!\!\!\!\rightarrow} ((0:3)+7)+(1:1) \mathrel{+\!\!\!\!\!\rightarrow} (3+7)+(1:1) \mathrel{+\!\!\!\!\!\rightarrow} (1:0)+(1:1)$$
$$\mathrel{+\!\!\!\!\!\rightarrow} 1:(0+(1:1)) \mathrel{+\!\!\!\!\!\rightarrow} 1:(1:(0+1)) \mathrel{+\!\!\!\!\!\rightarrow} 1:(1:1)$$

Example (cont'd)

- maximal outermost strategy

$$((0:(1+2))+(3+4))+(5+6)$$
$$\xrightarrow{\#} (0:((1+2)+(3+4)))+(1:1) \xrightarrow{\#} 0:(((1+2)+(3+4))+(1:1))$$
$$\xrightarrow{\#} ((1+2)+(3+4))+(1:1) \xrightarrow{\#} 1:(((1+2)+(3+4))+1)$$
$$\xrightarrow{\#} 1:((3+7)+1) \xrightarrow{\#} 1:((1:0)+1) \xrightarrow{\#} 1:(1:(0+1))$$

- maximal innermost strategy

$$((0:(1+2))+(3+4))+(5+6)$$
$$\xrightarrow{\#} ((0:3)+7)+(1:1) \xrightarrow{\#} (3+7)+(1:1) \xrightarrow{\#} (1:0)+(1:1)$$
$$\xrightarrow{\#} 1:(0+(1:1)) \xrightarrow{\#} 1:(1:(0+1)) \xrightarrow{\#} 1:(1:1) \xrightarrow{\#} (1+1):1$$

Example (cont'd)

- maximal outermost strategy

$$
\begin{aligned}
&((0 : (1 + 2)) + (3 + 4)) + (5 + 6) \\
&\quad \mathrel{-\!\!\|\!\!\!\to} (0 : ((1 + 2) + (3 + 4))) + (1 : 1) \mathrel{-\!\!\|\!\!\!\to} 0 : (((1 + 2) + (3 + 4)) + (1 : 1)) \\
&\quad \mathrel{-\!\!\|\!\!\!\to} ((1 + 2) + (3 + 4)) + (1 : 1) \mathrel{-\!\!\|\!\!\!\to} 1 : (((1 + 2) + (3 + 4)) + 1) \\
&\quad \mathrel{-\!\!\|\!\!\!\to} 1 : ((3 + 7) + 1) \mathrel{-\!\!\|\!\!\!\to} 1 : ((1 : 0) + 1) \mathrel{-\!\!\|\!\!\!\to} 1 : (1 : (0 + 1)) \\
&\quad \mathrel{-\!\!\|\!\!\!\to} (1 + 1) : (0 + 1)
\end{aligned}
$$

- maximal innermost strategy (10 redexes in 8 steps)

$$
\begin{aligned}
&((0 : (1 + 2)) + (3 + 4)) + (5 + 6) \\
&\quad \mathrel{-\!\!\|\!\!\!\to} ((0 : 3) + 7) + (1 : 1) \mathrel{-\!\!\|\!\!\!\to} (3 + 7) + (1 : 1) \mathrel{-\!\!\|\!\!\!\to} (1 : 0) + (1 : 1) \\
&\quad \mathrel{-\!\!\|\!\!\!\to} 1 : (0 + (1 : 1)) \mathrel{-\!\!\|\!\!\!\to} 1 : (1 : (0 + 1)) \mathrel{-\!\!\|\!\!\!\to} 1 : (1 : 1) \mathrel{-\!\!\|\!\!\!\to} (1 + 1) : 1 \mathrel{-\!\!\|\!\!\!\to} 2 : 1
\end{aligned}
$$

Example (cont'd)

- maximal outermost strategy (12 redexes in 9 steps)

$$
\begin{aligned}
&((0:(1+2))+(3+4))+(5+6)\\
&\quad \twoheadrightarrow\!\!\!\!\!\!\parallel\;\; (0:((1+2)+(3+4)))+(1:1) \twoheadrightarrow\!\!\!\!\!\!\parallel\;\; 0:(((1+2)+(3+4))+(1:1))\\
&\quad \twoheadrightarrow\!\!\!\!\!\!\parallel\;\; ((1+2)+(3+4))+(1:1) \twoheadrightarrow\!\!\!\!\!\!\parallel\;\; 1:(((1+2)+(3+4))+1)\\
&\quad \twoheadrightarrow\!\!\!\!\!\!\parallel\;\; 1:((3+7)+1) \twoheadrightarrow\!\!\!\!\!\!\parallel\;\; 1:((1:0)+1) \twoheadrightarrow\!\!\!\!\!\!\parallel\;\; 1:(1:(0+1))\\
&\quad \twoheadrightarrow\!\!\!\!\!\!\parallel\;\; (1+1):(0+1) \twoheadrightarrow\!\!\!\!\!\!\parallel\;\; 2:1
\end{aligned}
$$

- maximal innermost strategy (10 redexes in 8 steps)

$$
\begin{aligned}
&((0:(1+2))+(3+4))+(5+6)\\
&\quad \twoheadrightarrow\!\!\!\!\!\!\parallel\;\; ((0:3)+7)+(1:1) \twoheadrightarrow\!\!\!\!\!\!\parallel\;\; (3+7)+(1:1) \twoheadrightarrow\!\!\!\!\!\!\parallel\;\; (1:0)+(1:1)\\
&\quad \twoheadrightarrow\!\!\!\!\!\!\parallel\;\; 1:(0+(1:1)) \twoheadrightarrow\!\!\!\!\!\!\parallel\;\; 1:(1:(0+1)) \twoheadrightarrow\!\!\!\!\!\!\parallel\;\; 1:(1:1) \twoheadrightarrow\!\!\!\!\!\!\parallel\;\; (1+1):1 \twoheadrightarrow\!\!\!\!\!\!\parallel\;\; 2:1
\end{aligned}
$$

Definition (Maximal Multi–Step Rewriting)

maximal multi–step relation $\multimap\!\!\!\rightarrow$ is inductively defined as follows:

① $x \multimap\!\!\!\rightarrow x$ for all variables x

② $f(s_1, \ldots, s_n) \multimap\!\!\!\rightarrow f(t_1, \ldots, t_n)$ if $s_i \multimap\!\!\!\rightarrow t_i$ for all $1 \leqslant i \leqslant n$ and $f(s_1, \ldots, s_n)$ is no redex

③ $\ell\sigma \multimap\!\!\!\rightarrow r\tau$ if $\ell \rightarrow r \in \mathcal{R}$ and $\sigma \multimap\!\!\!\rightarrow \tau$

Definition (Maximal Multi–Step Rewriting)

maximal multi–step relation $\mathrel{-\!\!\bullet\!\!\!\to}$ is inductively defined as follows:

① $x \mathrel{-\!\!\bullet\!\!\!\to} x$ for all variables x

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③ $\ell\sigma \mathrel{-\!\!\bullet\!\!\!\to} r\tau$ if $\ell \to r \in \mathcal{R}$ and $\sigma \mathrel{-\!\!\bullet\!\!\!\to} \tau$

Definition (Maximal Strategy)

maximal strategy performs maximal multi–step for every reducible term

Example (cont'd)

- maximal strategy

$$((0 : \underline{(1 + 2)}) + \underline{(3 + 4)}) + \underline{(5 + 6)}$$

Example (cont'd)

- maximal strategy

$$((0 : (\underline{1+2})) + (\underline{3+4})) + (\underline{5+6})$$

$$\xrightarrow{\bullet} (0 : (3+7)) + (1 : 1)$$

Example (cont'd)

- maximal strategy

$$
\overline{((0 : \underline{(1+2)}) + \underline{(3+4)})} + \underline{(5+6)}
$$

$$
\xrightarrow{\bullet} \overline{\underline{(0 : \overline{(3+7)})} + (1:1)} \xrightarrow{\bullet} 1 : (\underline{(1:0) + 1})
$$

Example (cont'd)

- maximal strategy

$$\overline{((0 : \underline{(1+2)}) + \underline{(3+4)}}) + \underline{(5+6)}$$

$$\twoheadrightarrow \overline{\underline{(0 : (\overline{3+7}))} + (1:1)} \twoheadrightarrow 1 : (\underline{(1:0) + 1}) \twoheadrightarrow \overline{1 : (1 : (\underline{0+1}))}$$

Example (cont'd)

- maximal strategy

$$(\overline{(0 : (\underline{1+2})) + (\underline{3+4})}) + (\underline{5+6})$$

$$\mathrel{-\!\!\bullet\!\!\!\to} \overline{(\underline{0 : (\overline{3+7})}) + (1:1)} \mathrel{-\!\!\bullet\!\!\!\to} 1 : (\underline{(1:0)+1}) \mathrel{-\!\!\bullet\!\!\!\to} \overline{1 : (1 : (\underline{0+1}))}$$

$$\mathrel{-\!\!\bullet\!\!\!\to} (\underline{1+1}) : 1$$

Example (cont'd)

- maximal strategy (11 redexes in 5 steps)

$$\overline{((0 : (\underline{1+2})) + (\underline{3+4}))} + (\underline{5+6})$$

$$\mathrel{\multimap\!\!\!\to} \overline{(\underline{0 : (\overline{3+7})}) + (1:1)} \mathrel{\multimap\!\!\!\to} 1 : (\underline{(1:0)+1}) \mathrel{\multimap\!\!\!\to} \overline{1 : (1 : (\underline{0+1}))}$$

$$\mathrel{\multimap\!\!\!\to} (\underline{1+1}) : 1 \mathrel{\multimap\!\!\!\to} 2 : 1$$

Example (cont'd)

- maximal strategy (11 redexes in 5 steps)

$$t = (\overline{(0 : \underline{(1+2)}) + \underline{(3+4)}}) + \underline{(5+6)}$$

$$\mathrel{\bullet\!\!\to} \overline{(\underline{0 : (\overline{3+7})}) + (1:1)} \mathrel{\bullet\!\!\to} 1 : (\underline{(1:0) + 1}) \mathrel{\bullet\!\!\to} \overline{1 : (1 : (\underline{0+1}))}$$

$$\mathrel{\bullet\!\!\to} (\underline{1+1}) : 1 \mathrel{\bullet\!\!\to} 2 : 1$$

- maximal strategy (12 redexes in 6 steps)

$$t \mathrel{\bullet\!\!\to} \overline{(0 : (\underline{3+7})) + (1:1)} \mathrel{\bullet\!\!\to} \overline{0 : (\underline{(1:0) + (1:1)})} \mathrel{\bullet\!\!\to} 1 : (\underline{0 + (1:1)})$$

$$\mathrel{\bullet\!\!\to} \overline{1 : (1 : (\underline{0+1}))} \mathrel{\bullet\!\!\to} \underline{(1+1)} : 1 \mathrel{\bullet\!\!\to} 2 : 1$$

Example (cont'd)

- maximal strategy (11 redexes in 5 steps)

$$t = ((0 : (1 + 2)) + (3 + 4)) + (5 + 6)$$
$$\twoheadrightarrow (0 : (3 + 7)) + (1 : 1) \twoheadrightarrow 1 : ((1 : 0) + 1) \twoheadrightarrow 1 : (1 : (0 + 1))$$
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$$t \twoheadrightarrow (0 : (3 + 7)) + (1 : 1) \twoheadrightarrow 0 : ((1 : 0) + (1 : 1)) \twoheadrightarrow 1 : (0 + (1 : 1))$$
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Outline

1. **Summary of Lecture 9**
2. **Proof Terms**
3. **Strategies**
4. **Normalization**

   Maximal Strategy   Innermost Strategies   Leftmost Outermost Strategy

5. **Exercises**
6. **Further Reading**

Remark

all strategies defined so−far are deterministic for orthogonal TRSs

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Theorem

for orthogonal TRSs

- maximal and maximal outermost strategies are hyper−normalizing

Remark

all strategies defined so−far are deterministic for orthogonal TRSs

Theorem

for orthogonal TRSs

- maximal and maximal outermost strategies are hyper−normalizing
- innermost strategies are perpetual

Outline

Definition (Cofinality)

rewrite strategy $\mathcal{S}$ is cofinal for TRS $\mathcal{R}$ if for every maximal sequence

$$s = s_0 \xrightarrow{\mathcal{S}} s_1 \xrightarrow{\mathcal{S}} s_2 \xrightarrow{\mathcal{S}} \cdots$$

and every $s \to_{\mathcal{R}}^* t$ there exists $k \geqslant 0$ such that $t \to_{\mathcal{R}}^* s_k$

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cofinal strategies are normalizing

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- consider maximal sequence $s = s_0 \xrightarrow{\mathcal{S}} s_1 \xrightarrow{\mathcal{S}} s_2 \xrightarrow{\mathcal{S}} \cdots$ and let $s \to_{\mathcal{R}}^! t$

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- consider maximal sequence $s = s_0 \xrightarrow{\mathcal{S}} s_1 \xrightarrow{\mathcal{S}} s_2 \xrightarrow{\mathcal{S}} \cdots$ and let $s \to_{\mathcal{R}}^! t$
- cofinality $\implies$ $t \to_{\mathcal{R}}^* s_k$ for some $k \geqslant 0$
- $t = s_k$

Theorem

maximal strategy is cofinal for orthogonal TRSs

Proof

- assume $s \to^* t$

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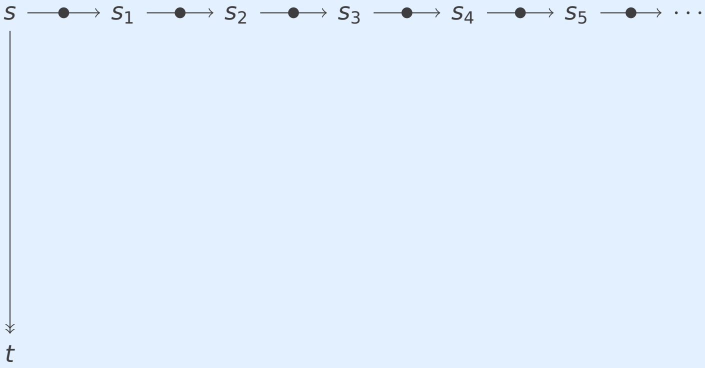

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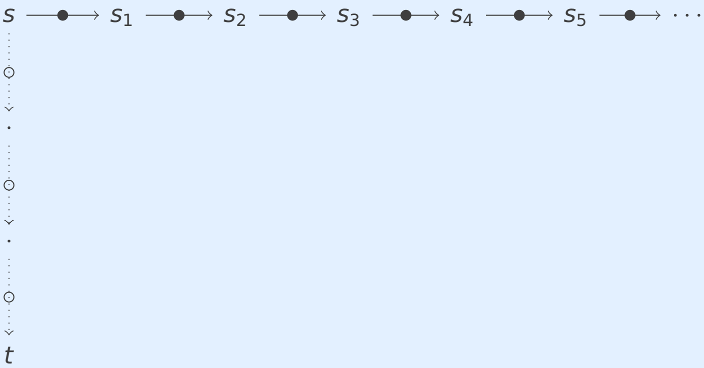

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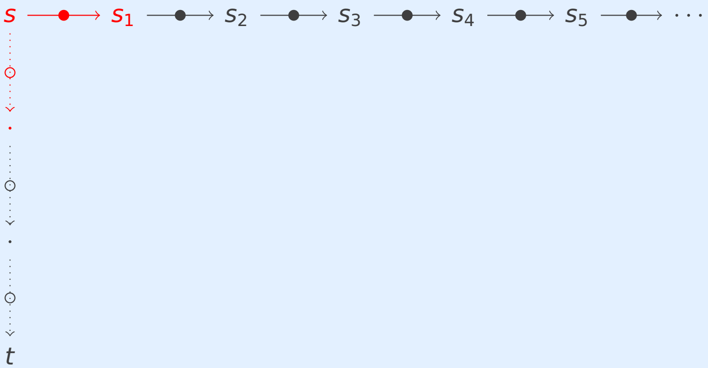

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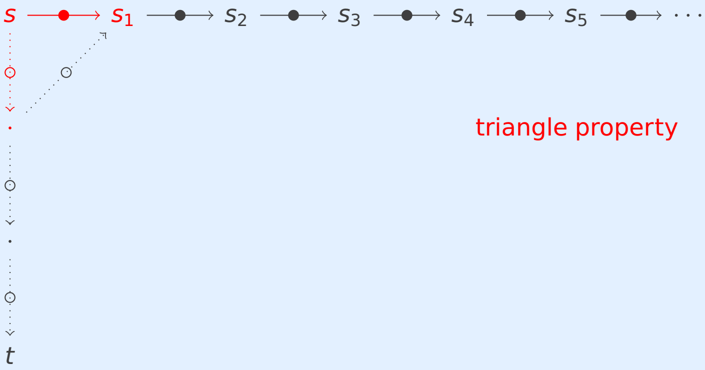

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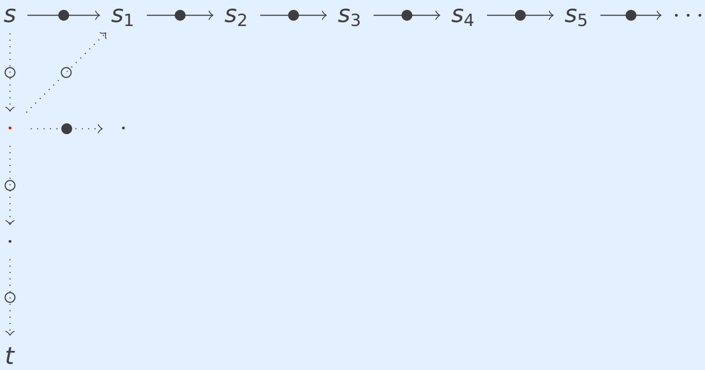

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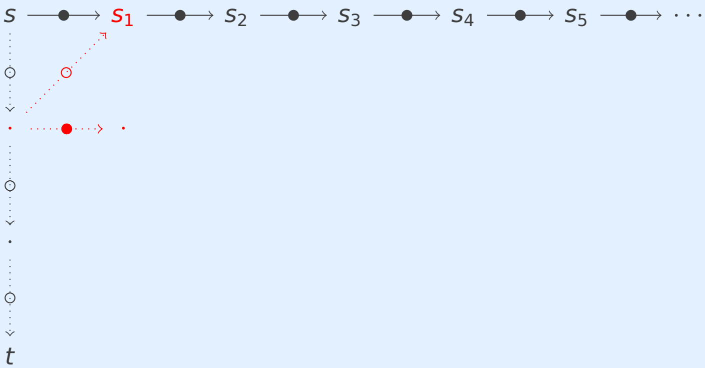

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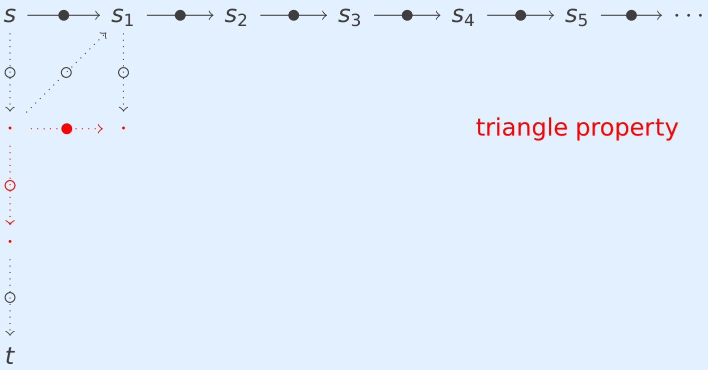

triangle property

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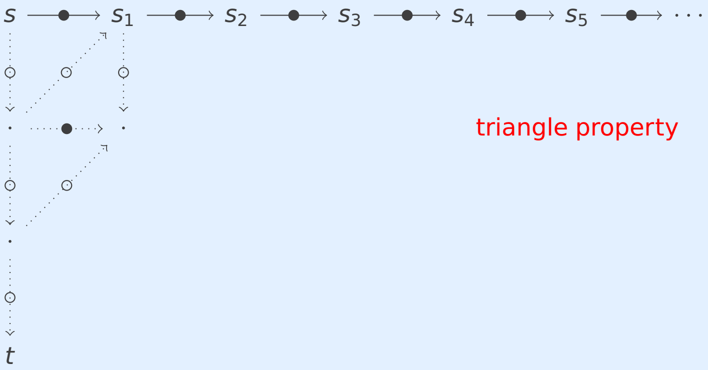

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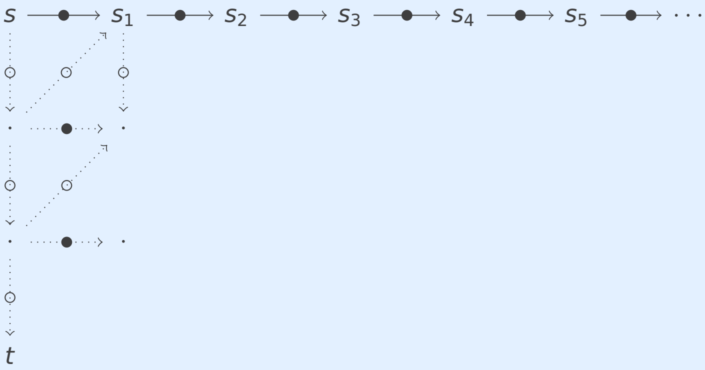

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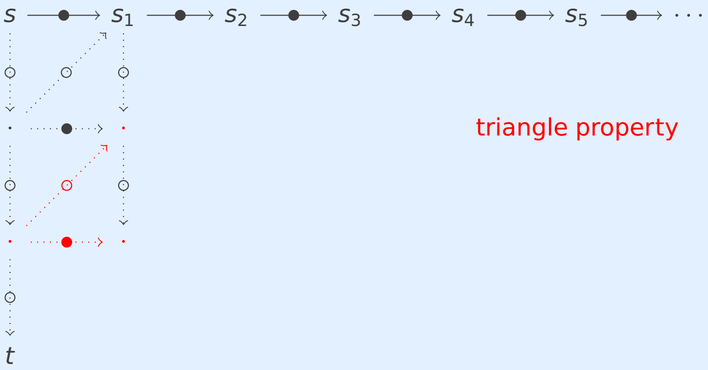

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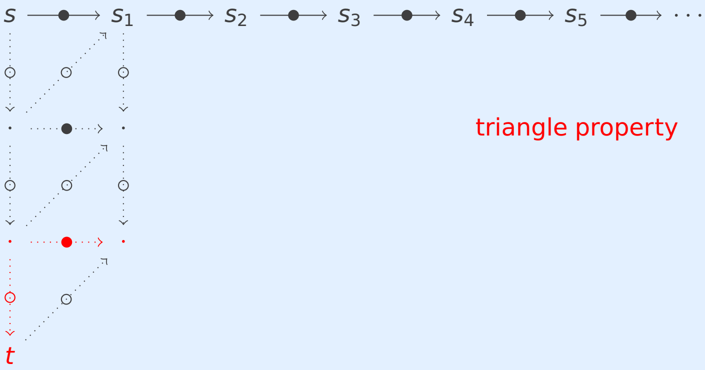

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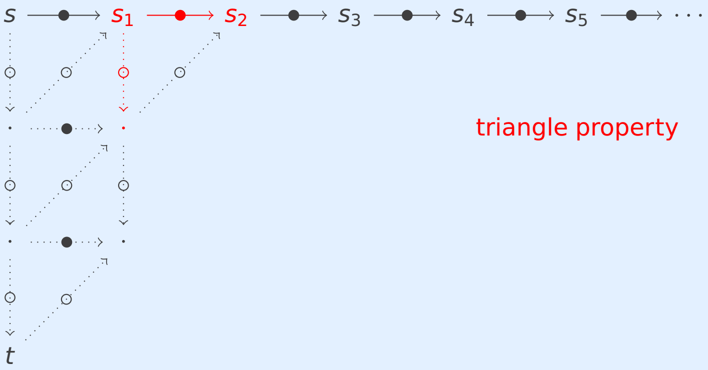

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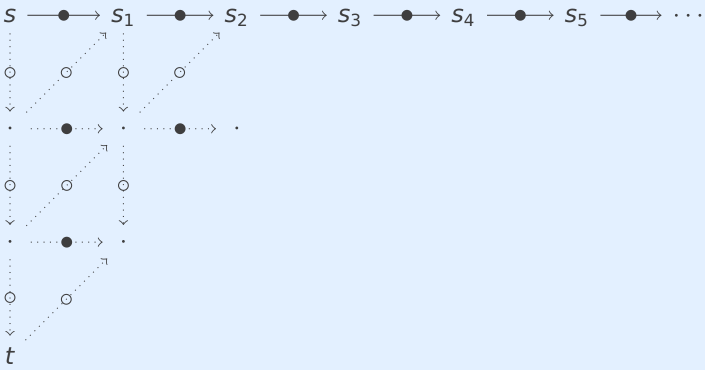

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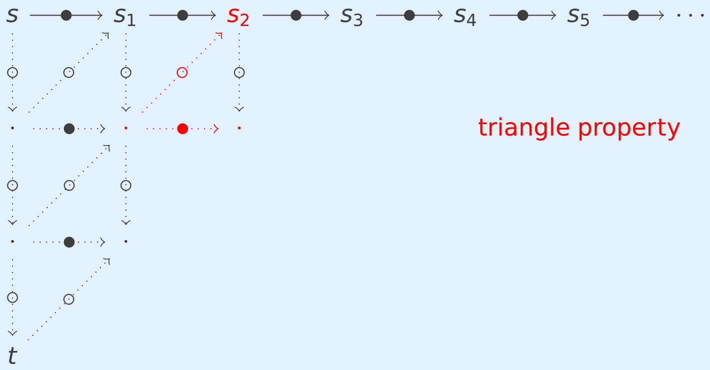

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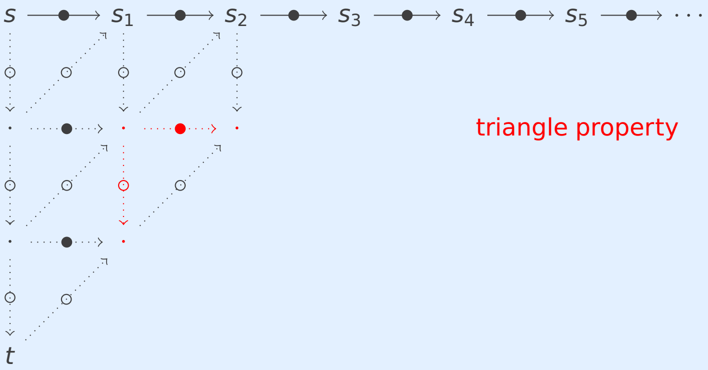

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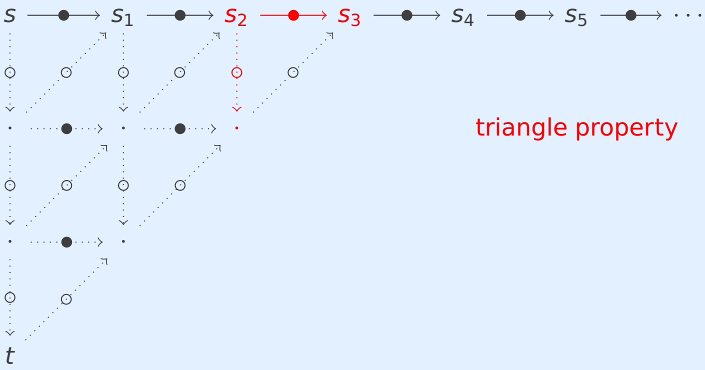

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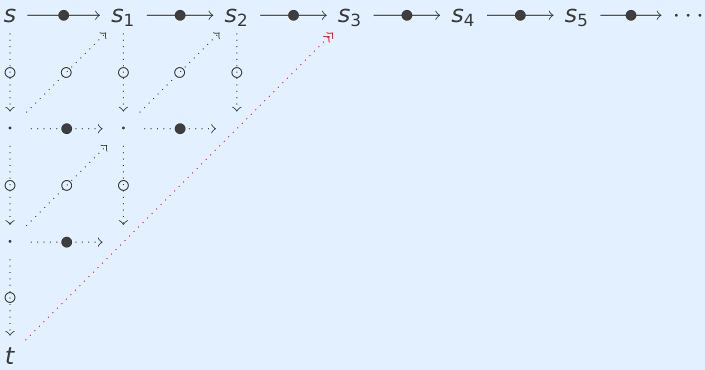

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Theorem

every innermost strategy is perpetual for orthogonal TRSs

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Proof Outline

① innermost rewriting has random descent

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Notation

$a \xleftrightarrow[\langle l, r \rangle]{*} b$ if conversion $a \leftrightarrow^* b$ has l left $(\leftarrow)$ steps and r right $(\rightarrow)$ steps

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Definition (Random Descent)

ARS $\mathcal{A} = \langle A, \rightarrow \rangle$ has random descent if $a \rightarrow^{r-l} b$ whenever $a \xleftrightarrow[\langle l,r \rangle]{*} b$ with $b \in \mathrm{NF}(\mathcal{A})$

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if ARS $\mathcal{A}$ has random descent and $a \leftrightarrow^* b$ with $b \in \mathrm{NF}(\mathcal{A})$ then

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2. all rewrite sequences from a to b have same length

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skip proofs

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suppose $a \xleftrightarrow[\langle l, r \rangle]{*} b \in \mathrm{NF}(\mathcal{A})$ $\qquad r - l \geqslant 0$ by random descent

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  $a \to^{r-l} a_{r-l}$

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- suppose a is non-terminating: $a = a_0 \to a_1 \to a_2 \to \cdots$

  $a \to^{r-l} a_{r-l}$ and thus $a_{r-l} \xleftrightarrow[\langle r,r\rangle]{*} b$

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Proof

suppose $a \xleftrightarrow[\langle l,r \rangle]{*} b \in \mathrm{NF}(\mathcal{A})$

- a is terminating and confluent

Theorem

if ARS $\mathcal{A}$ has random descent and $a \leftrightarrow^* b$ with $b \in \mathsf{NF}(\mathcal{A})$ then

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Proof

suppose $a \xleftrightarrow[\langle l, r \rangle]{*} b \in \mathsf{NF}(\mathcal{A})$

- a is terminating and confluent:

  if $c \ {}^*\!\!\leftarrow a \rightarrow^* d$ then $c \leftrightarrow^* b$ and $d \leftrightarrow^* b$

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  $c \downarrow d$ by random descent

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Proof

suppose $a \xleftrightarrow[\langle l, r \rangle]{*} b \in \mathsf{NF}(\mathcal{A})$

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- suppose $a \to^m b$ and $a \to^n b$

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  $b \xleftrightarrow[\langle m, n \rangle]{*} b$

  $b \to^{n-m} b$ by random descent and thus $m = n$

Theorem

innermost rewriting has random descent for orthogonal TRSs

skip proof

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skip proof

Proof

$\overset{\mathsf{i}}{\leftarrow} \cdot \overset{\mathsf{i}}{\rightarrow} \subseteq \overset{\mathsf{i}}{\rightarrow} \cdot \overset{\mathsf{i}}{\leftarrow} \cup =$

Theorem

innermost rewriting has random descent for orthogonal TRSs

▸ skip proof

Proof

$\overset{\mathsf{i}}{\leftarrow} \cdot \overset{\mathsf{i}}{\rightarrow} \subseteq \overset{\mathsf{i}}{\rightarrow} \cdot \overset{\mathsf{i}}{\leftarrow} \cup =$:

- let $s \overset{\mathsf{i}}{\rightarrow} t_1$ by contracting innermost redex Δ_1 at position p_1

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- if p_1 and p_2 are parallel then $t_1 \overset{\mathsf{i}}{\rightarrow} \cdot \overset{\mathsf{i}}{\leftarrow} t_2$

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Proof Outline

① innermost rewriting has random descent

② innermost normalizing terms are innermost terminating

③ innermost terminating terms are terminating

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$s \xrightarrow{\text{i}}^! t$

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$s \xrightarrow{\mathsf{i}}{}^{!}\, t \quad \Longrightarrow \quad s$ is complete with respect to $\xrightarrow{\mathsf{i}}$

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Theorem

innermost termination and termination coincide for (terms in) orthogonal TRSs

Outline

1. **Summary of Lecture 9**
2. **Proof Terms**
3. **Strategies**
4. **Normalization**

   Maximal Strategy   Innermost Strategies   Leftmost Outermost Strategy

5. **Exercises**
6. **Further Reading**

Remark

leftmost outermost strategy is not normalizing for all orthogonal TRSs

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Example

$$\mathsf{a} \to \mathsf{b} \qquad \mathsf{c} \to \mathsf{c} \qquad \mathsf{f}(x, \mathsf{b}) \to \mathsf{b}$$

- leftmost outermost $\mathsf{f}(\mathsf{c}, \mathsf{a})$
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Example

$$a \to b \qquad c \to c \qquad f(x, b) \to b$$

- leftmost outermost $f(c, a) \to f(c, a)$
- leftmost innermost $f(c, a) \to f(c, a)$
- maximal outermost $f(c, a) \twoheadrightarrow\!\!\!\!\!\!\!\shortparallel\;\; f(c, b)$
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- maximal $f(c, a) \multimap\!\!\!\!\rightarrow f(c, b)$

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Remark

important result: Combinatory Logic is left-normal

Outline

Homework Exercises for June 1

1. Exercise 6.7. ①
2. Exercise 6.8(a,b). ②
3. Exercise 7.1. ②
4. Exercise 7.8. ①
5. Exercise 7.9. ①
6. Exercise 7.6. ☆☆☆

Outline

1. **Summary of Lecture 9**
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Lecture Notes

- Section 1.5
- Section 6.2
- Section 7.1
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Important Concepts

- cofinal
- hyper−normalization
- innermost normalization
- innermost rewriting
- innermost termination
- left−normality
- leftmost innermost strategy
- leftmost outermost strategy
- maximal innermost strategy
- maximal outermost strategy
- maximal strategy
- normalization
- outermost rewriting
- perpetual
- proof term
- random descent
- relative rewriting
- rewrite strategy
- $\mathrm{src}(A)$
- $\mathrm{tgt}(A)$