universität innsbruck

26S lecture 10

Term Rewriting

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Outline

1. **Summary of Lecture 9**
2. **Proof Terms**
3. **Strategies**
4. **Normalization**
5. **Exercises**
6. **Further Reading**

Definition

orthogonal TRS is left−linear and lacks critical pairs

Definition

parallel rewriting $\twoheadrightarrow$ is inductively defined as follows:

1. $x \twoheadrightarrow x$ for all variables x
2. $f(s_1, \ldots, s_n) \twoheadrightarrow f(t_1, \ldots, t_n)$ if $s_i \twoheadrightarrow t_i$ for all $1 \leqslant i \leqslant n$
3. $\ell\sigma \twoheadrightarrow r\sigma$ if $\ell \to r \in \mathcal{R}$

Parallel Moves Lemma

$\twoheadleftarrow \cdot \twoheadrightarrow \subseteq \twoheadrightarrow \cdot \twoheadleftarrow$ for orthogonal TRSs $\Longrightarrow$ orthogonal TRSs are confluent

Definitions

multi–step relation $\twoheadrightarrow_{\circ}$ is inductively defined as follows:

① $x \twoheadrightarrow_{\circ} x$ for all variables x

② $f(s_1, \ldots, s_n) \twoheadrightarrow_{\circ} f(t_1, \ldots, t_n)$ if $s_i \twoheadrightarrow_{\circ} t_i$ for all $1 \leqslant i \leqslant n$

③ $\ell\sigma \twoheadrightarrow_{\circ} r\tau$ if $\ell \to r \in \mathcal{R}$ and $\underbrace{x\sigma \twoheadrightarrow_{\circ} x\tau \text{ for all variables } x}_{\sigma \twoheadrightarrow_{\circ} \tau}$

maximal multi–step relation $\twoheadrightarrow_{\bullet}$ is inductively defined as follows:

① $x \twoheadrightarrow_{\bullet} x$ for all variables x

② $f(s_1, \ldots, s_n) \twoheadrightarrow_{\bullet} f(t_1, \ldots, t_n)$ if $s_i \twoheadrightarrow_{\bullet} t_i$ for all $1 \leqslant i \leqslant n$ and $f(s_1, \ldots, s_n)$ is no redex

③ $\ell\sigma \twoheadrightarrow_{\bullet} r\tau$ if $\ell \to r \in \mathcal{R}$ and $\sigma \twoheadrightarrow_{\bullet} \tau$

Lemma

$\to \; \subseteq \; \twoheadrightarrow_{\parallel} \; \subseteq \; \twoheadrightarrow_{\circ} \; \subseteq \; \to^*$ and $\twoheadrightarrow_{\bullet} \; \subseteq \; \twoheadrightarrow_{\circ}$

Lemma (Triangle Property)

$\leftarrow\!\!\circ\!\!- \cdot -\!\!\bullet\!\!\rightarrow \subseteq -\!\!\circ\!\!\rightarrow$ for orthogonal TRSs

Definitions

TRS

- $\cdots$ is strongly closed if $t \rightarrow^{=} \cdot {}^{*}\!\!\leftarrow u$ and $t \rightarrow^{*} \cdot {}^{=}\!\!\leftarrow u$
- $\cdots$ is parallel closed if $t \mathrel{+\!\!\!\!\!\!\rightarrow} u$
- $\cdots$ is development closed if $t -\!\!\circ\!\!\rightarrow u$

for every critical pair $t \approx u$

Theorem

- linear strongly closed TRSs are confluent
- left–linear parallel closed TRSs are confluent
- left–linear development closed TRSs are confluent

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Example

- left–linear TRS $\quad \mathsf{h}(\mathsf{f}(x,\mathsf{g}(y))) \xrightarrow{\alpha} \mathsf{h}(\mathsf{f}(x,\mathsf{g}(x))) \qquad \mathsf{g}(\mathsf{a}) \xrightarrow{\gamma} \mathsf{g}(\mathsf{b})$

  $\mathsf{f}(\mathsf{g}(x),y) \xrightarrow{\beta} \mathsf{f}(\mathsf{g}(x),\mathsf{g}(x)) \qquad \mathsf{b} \xrightarrow{\delta} \mathsf{a}$
- term $s = \mathsf{h}(\mathsf{f}(\mathsf{h}(\mathsf{f}(\mathsf{g}(\mathsf{a}),\mathsf{g}(\mathsf{a}))),\mathsf{g}(\mathsf{a})))$
- proof terms $A = \mathsf{h}(\mathsf{f}(\alpha(\gamma,\mathsf{a}),\gamma)$ and $B = \alpha(\mathsf{h}(\beta(\mathsf{a},\gamma)),\mathsf{a})$

Definitions

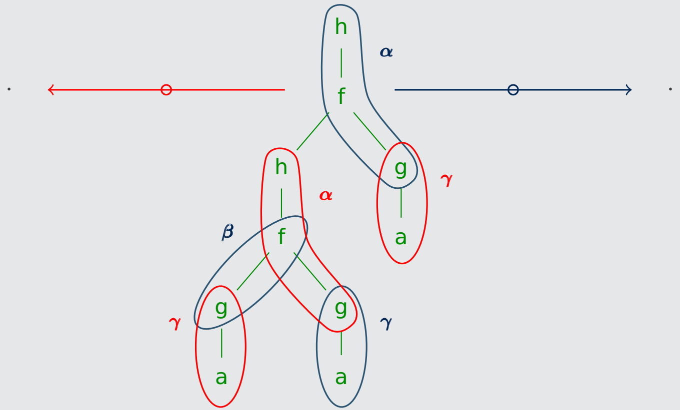
rule symbol α associated to (left–linear) rewrite rule $\ell \to r$

- $\mathsf{lhs}(\alpha)$ denotes ℓ
- $\mathsf{rhs}(\alpha)$ denotes r
- $\mathsf{var}(\alpha)$ denotes list $(x_1, \dots, x_n)$ of variables appearing in ℓ in some fixed order
- arity of α is length of $\mathsf{var}(\alpha)$
- $\langle t_1, \dots, t_n \rangle_\alpha$ denotes substitution $\{x_i \mapsto t_i \mid 1 \leqslant i \leqslant n\}$

Example

$$\mathsf{h}(\mathsf{f}(x,\mathsf{g}(y))) \xrightarrow{\alpha} \mathsf{h}(\mathsf{f}(x,\mathsf{g}(x))) \quad \mathsf{g}(\mathsf{a}) \xrightarrow{\gamma} \mathsf{g}(\mathsf{b}) \quad s = \mathsf{h}(\mathsf{f}(\mathsf{h}(\mathsf{f}(\mathsf{g}(\mathsf{a}),\mathsf{g}(\mathsf{a}))),\mathsf{g}(\mathsf{a})))$$

$$\mathsf{f}(\mathsf{g}(x),y) \xrightarrow{\beta} \mathsf{f}(\mathsf{g}(x),\mathsf{g}(x)) \quad \mathsf{b} \xrightarrow{\delta} \mathsf{a} \quad A = \mathsf{h}(\mathsf{f}(\alpha(\gamma,\mathsf{a}),\gamma)) \quad B = \alpha(\mathsf{h}(\beta(\mathsf{a},\gamma)),\mathsf{a})$$



Definitions

- proof terms are terms built from function symbols, variables, rule symbols
- source $\mathsf{src}(A)$ and target $\mathsf{tgt}(A)$ of proof term A

$$\mathsf{src}(x) = \mathsf{tgt}(x) = x$$

$$\mathsf{src}(f(A_1, \ldots, A_n)) = f(\mathsf{src}(A_1), \ldots, \mathsf{src}(A_n))$$

$$\mathsf{src}(\alpha(A_1, \ldots, A_n)) = \mathsf{lhs}(\alpha)\langle \mathsf{src}(A_1), \ldots, \mathsf{src}(A_n)\rangle_{\alpha}$$

$$\mathsf{tgt}(f(A_1, \ldots, A_n)) = f(\mathsf{tgt}(A_1), \ldots, \mathsf{tgt}(A_n))$$

$$\mathsf{tgt}(\alpha(A_1, \ldots, A_n)) = \mathsf{rhs}(\alpha)\langle \mathsf{tgt}(A_1), \ldots, \mathsf{tgt}(A_n)\rangle_{\alpha}$$

- proof terms A and B are co–initial if $\mathsf{src}(A) = \mathsf{src}(B)$

Remark

proof term A witnesses multi–step $\mathsf{src}(A) \twoheadrightarrow\!\!\!\!\circ\ \mathsf{tgt}(A)$

Example (cont'd)

$$\mathsf{h}(\mathsf{f}(x,\mathsf{g}(y))) \xrightarrow{\alpha} \mathsf{h}(\mathsf{f}(x,\mathsf{g}(x))) \quad \mathsf{g}(\mathsf{a}) \xrightarrow{\gamma} \mathsf{g}(\mathsf{b}) \quad s = \mathsf{h}(\mathsf{f}(\mathsf{h}(\mathsf{f}(\mathsf{g}(\mathsf{a}),\mathsf{g}(\mathsf{a}))),\mathsf{g}(\mathsf{a})))$$

$$\mathsf{f}(\mathsf{g}(x),y) \xrightarrow{\beta} \mathsf{f}(\mathsf{g}(x),\mathsf{g}(x)) \quad \mathsf{b} \xrightarrow{\delta} \mathsf{a} \quad A = \mathsf{h}(\mathsf{f}(\alpha(\gamma,\mathsf{a}),\gamma)) \quad B = \alpha(\mathsf{h}(\beta(\mathsf{a},\gamma)),\mathsf{a})$$

- $\mathsf{src}(A) = \mathsf{h}(\mathsf{src}(\mathsf{f}(\alpha(\gamma,\mathsf{a}),\gamma))) = \mathsf{h}(\mathsf{f}(\mathsf{src}(\alpha(\gamma,\mathsf{a})),\mathsf{src}(\gamma)))$
  $= \mathsf{h}(\mathsf{f}(\mathsf{h}(\mathsf{f}(\mathsf{src}(\gamma),\mathsf{g}(\mathsf{src}(\mathsf{a})),\mathsf{g}(\mathsf{a})))) = \mathsf{h}(\mathsf{f}(\mathsf{h}(\mathsf{f}(\mathsf{g}(\mathsf{a}),\mathsf{g}(\mathsf{a}))),\mathsf{g}(\mathsf{a})))$
- $\mathsf{tgt}(B) = \mathsf{h}(\mathsf{f}(\mathsf{tgt}(\mathsf{h}(\beta(\mathsf{a},\gamma))),\mathsf{g}(\mathsf{tgt}(\mathsf{h}(\beta(\mathsf{a},\gamma)))))) = \mathsf{h}(\mathsf{f}(\mathsf{h}(\mathsf{tgt}(\beta(\mathsf{a},\gamma))),\mathsf{g}(\mathsf{h}(\mathsf{tgt}(\beta(\mathsf{a},\gamma))))))$
  $= \mathsf{h}(\mathsf{f}(\mathsf{h}(\mathsf{f}(\mathsf{g}(\mathsf{tgt}(\mathsf{a})),\mathsf{g}(\mathsf{tgt}(\mathsf{a})))),\mathsf{g}(\mathsf{h}(\mathsf{f}(\mathsf{g}(\mathsf{tgt}(\mathsf{a})),\mathsf{g}(\mathsf{tgt}(\mathsf{a})))))))$
  $= \mathsf{h}(\mathsf{f}(\mathsf{h}(\mathsf{f}(\mathsf{g}(\mathsf{a}),\mathsf{g}(\mathsf{a}))),\mathsf{g}(\mathsf{h}(\mathsf{f}(\mathsf{g}(\mathsf{a}),\mathsf{g}(\mathsf{a}))))))$

Lemma

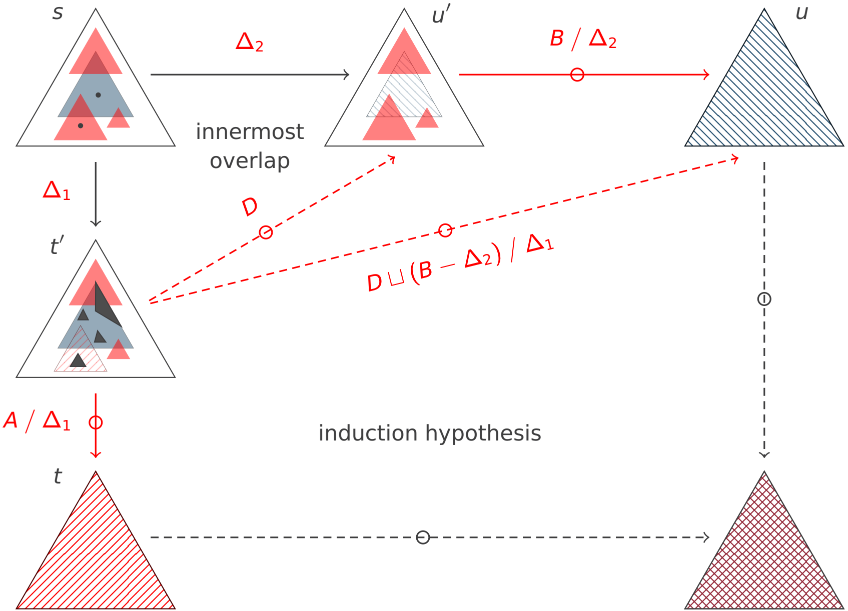
$s \mathrel{\rightarrow\!\!\!\circ\!\!\rightarrow} t \iff \mathsf{src}(A) = s$ and $\mathsf{tgt}(A) = t$ for some proof term A

Theorem

left–linear development closed TRSs are confluent

Remarks

- formalized proof employs proof terms
- result follows from diamond property of $\twoheadrightarrow\!\!\!\!\!\circ$ (multi-step)
- proof employs induction on amount of overlap between two multi–steps


s
Δ_2
u'
$B \,/\, \Delta_2$
u
innermost
overlap
Δ_1
D
t'
$D \sqcup (B - \Delta_2) \,/\, \Delta_1$
$A \,/\, \Delta_1$
induction hypothesis
t

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Definitions (Strategies)

- one–step rewrite strategy $\mathcal{S}$ for TRS $\mathcal{R}$ is relation $\xrightarrow{\mathcal{S}} \subseteq \to_{\mathcal{R}}$ such that $\mathsf{NF}(\mathcal{S}) = \mathsf{NF}(\mathcal{R})$
- many–step rewrite strategy $\mathcal{S}$ is relation $\xrightarrow{\mathcal{S}} \subseteq \to_{\mathcal{R}}^{+}$ such that $\mathsf{NF}(\mathcal{S}) = \mathsf{NF}(\mathcal{R})$
- rewrite strategy $\mathcal{S}$ is deterministic if $\xleftarrow{\mathcal{S}} \cdot \xrightarrow{\mathcal{S}} \subseteq =$
- rewrite strategy $\mathcal{S}$ normalizes term t if all $\mathcal{S}$–rewrite sequences starting from t are finite
- rewrite strategy $\mathcal{S}$ is normalizing if it normalizes every normalizing term:

$$\forall\, t \;\big(\mathsf{WN}_{\mathcal{R}}(t) \implies \mathsf{SN}_{\mathcal{S}}(t)\big)$$

- relative rewriting: $\to_{\mathcal{S}/\mathcal{R}} = \to_{\mathcal{R}}^{*} \cdot \to_{\mathcal{S}} \cdot \to_{\mathcal{R}}^{*}$
- rewrite strategy $\mathcal{S}$ is hyper–normalizing if every $\mathcal{R}$–normalizing term is $\mathcal{S}/\mathcal{R}$–terminating

Definitions (Strategies, cont'd)

- rewrite strategy $\mathcal{S}$ is perpetual if every maximal $\mathcal{S}$–rewrite sequence starting from any non–terminating term is infinite:

$$\forall\, t\; \big(\mathsf{WN}_{\mathcal{S}}(t) \implies \mathsf{SN}_{\mathcal{R}}(t)\big)$$

Remarks

- (hyper–)normalizing strategy avoids non–terminating computations, if possible
- perpetual strategy avoids terminating computations, if possible

Lemma

for terminating TRSs every strategy is hyper–normalizing and perpetual

Example

- rewrite rules

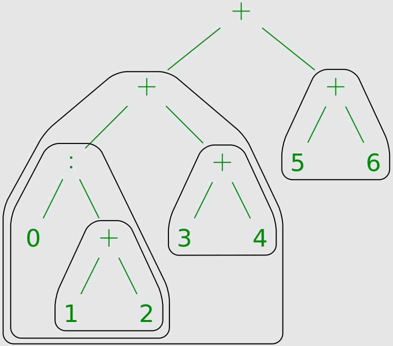
$$
\begin{array}{lllll}
0+0 \to 0 & 1+0 \to 1 & \cdots & 9+0 \to 9 & 0 : x \to x \\
0+1 \to 1 & 1+1 \to 2 & \cdots & 9+1 \to 1:0 & x + (y:z) \to y:(x+z) \\
0+2 \to 2 & 1+2 \to 3 & \cdots & 9+2 \to 1:1 & (x:y)+z \to x:(y+z) \\
0+3 \to 3 & 1+3 \to 4 & \cdots & 9+3 \to 1:2 & x:(y:z) \to (x+y):z \\
0+4 \to 4 & 1+4 \to 5 & \cdots & 9+4 \to 1:3 & \\
0+5 \to 5 & 1+5 \to 6 & \cdots & 9+5 \to 1:4 & \\
0+6 \to 6 & 1+6 \to 7 & \cdots & 9+6 \to 1:5 & \\
0+7 \to 7 & 1+7 \to 8 & \cdots & 9+7 \to 1:6 & \\
0+8 \to 8 & 1+8 \to 9 & \cdots & 9+8 \to 1:7 & \\
0+9 \to 9 & 1+9 \to 1:0 & \cdots & 9+9 \to 1:8 &
\end{array}
$$

- term $((0:(1+2))+(3+4))+(5+6)$

Example (cont'd)

- term $\boxed{\boxed{0 : \boxed{1 + 2}} + \boxed{3 + 4}} + \boxed{5 + 6}$

- tree representation



maximal / leftmost outermost / innermost strategies

Example (cont'd)

- leftmost outermost strategy (12 redexes)

$$((0 : (1 + 2)) + (3 + 4)) + (5 + 6)$$
$$\to (0 : ((1 + 2) + (3 + 4))) + (5 + 6) \to 0 : (((1 + 2) + (3 + 4)) + (5 + 6))$$
$$\to ((1 + 2) + (3 + 4)) + (5 + 6) \to (3 + (3 + 4)) + (5 + 6) \to (3 + 7) + (5 + 6)$$
$$\to (1 : 0) + (5 + 6) \to 1 : (0 + (5 + 6)) \to 1 : (0 + (1 : 1)) \to 1 : (1 : (0 + 1))$$
$$\to (1 + 1) : (0 + 1) \to 2 : (0 + 1) \to 2 : 1$$

- leftmost innermost strategy (10 redexes)

$$((0 : (1 + 2)) + (3 + 4)) + (5 + 6)$$
$$\to ((0 : 3) + (3 + 4)) + (5 + 6) \to (3 + (3 + 4)) + (5 + 6) \to (3 + 7) + (5 + 6)$$
$$\to (1 : 0) + (5 + 6) \to (1 : 0) + (1 : 1) \to 1 : ((1 : 0) + 1) \to 1 : (1 : (0 + 1))$$
$$\to 1 : (1 : 1) \to (1 + 1) : 1 \to 2 : 1$$

Example (cont'd)

- maximal outermost strategy (12 redexes in 9 steps)

$$((0:(1+2))+(3+4))+(5+6)$$
$$\twoheadrightarrow\!\!\!\!\!\!\shortparallel\; (0:((1+2)+(3+4)))+(1:1) \twoheadrightarrow\!\!\!\!\!\!\shortparallel\; 0:(((1+2)+(3+4))+(1:1))$$
$$\twoheadrightarrow\!\!\!\!\!\!\shortparallel\; ((1+2)+(3+4))+(1:1) \twoheadrightarrow\!\!\!\!\!\!\shortparallel\; 1:(((1+2)+(3+4))+1)$$
$$\twoheadrightarrow\!\!\!\!\!\!\shortparallel\; 1:((3+7)+1) \twoheadrightarrow\!\!\!\!\!\!\shortparallel\; 1:((1:0)+1) \twoheadrightarrow\!\!\!\!\!\!\shortparallel\; 1:(1:(0+1))$$
$$\twoheadrightarrow\!\!\!\!\!\!\shortparallel\; (1+1):(0+1) \twoheadrightarrow\!\!\!\!\!\!\shortparallel\; 2:1$$

- maximal innermost strategy (10 redexes in 8 steps)

$$((0:(1+2))+(3+4))+(5+6)$$
$$\twoheadrightarrow\!\!\!\!\!\!\shortparallel\; ((0:3)+7)+(1:1) \twoheadrightarrow\!\!\!\!\!\!\shortparallel\; (3+7)+(1:1) \twoheadrightarrow\!\!\!\!\!\!\shortparallel\; (1:0)+(1:1)$$
$$\twoheadrightarrow\!\!\!\!\!\!\shortparallel\; 1:(0+(1:1)) \twoheadrightarrow\!\!\!\!\!\!\shortparallel\; 1:(1:(0+1)) \twoheadrightarrow\!\!\!\!\!\!\shortparallel\; 1:(1:1) \twoheadrightarrow\!\!\!\!\!\!\shortparallel\; (1+1):1 \twoheadrightarrow\!\!\!\!\!\!\shortparallel\; 2:1$$

Definition (Maximal Multi–Step Rewriting)

maximal multi–step relation $\multimap\!\!\!\to$ is inductively defined as follows:

① $x \multimap\!\!\!\to x$ for all variables x

② $f(s_1, \dots, s_n) \multimap\!\!\!\to f(t_1, \dots, t_n)$ if $s_i \multimap\!\!\!\to t_i$ for all $1 \leqslant i \leqslant n$ and $f(s_1, \dots, s_n)$ is no redex

③ $\ell\sigma \multimap\!\!\!\to r\tau$ if $\ell \to r \in \mathcal{R}$ and $\sigma \multimap\!\!\!\to \tau$

Definition (Maximal Strategy)

maximal strategy performs maximal multi–step for every reducible term

Example (cont'd)

- maximal strategy (11 redexes in 5 steps)

$$t = ((0 : (1 + 2)) + (3 + 4)) + (5 + 6)$$
$$\mathrel{\bullet\!\!\!\to} (0 : (3 + 7)) + (1 : 1) \mathrel{\bullet\!\!\!\to} 1 : ((1 : 0) + 1) \mathrel{\bullet\!\!\!\to} 1 : (1 : (0 + 1))$$
$$\mathrel{\bullet\!\!\!\to} (1 + 1) : 1 \mathrel{\bullet\!\!\!\to} 2 : 1$$

- maximal strategy (12 redexes in 6 steps)

$$t \mathrel{\bullet\!\!\!\to} (0 : (3 + 7)) + (1 : 1) \mathrel{\bullet\!\!\!\to} 0 : ((1 : 0) + (1 : 1)) \mathrel{\bullet\!\!\!\to} 1 : (0 + (1 : 1))$$
$$\mathrel{\bullet\!\!\!\to} 1 : (1 : (0 + 1)) \mathrel{\bullet\!\!\!\to} (1 + 1) : 1 \mathrel{\bullet\!\!\!\to} 2 : 1$$

- maximal strategy (12 redexes in 6 steps)

$$t \mathrel{\bullet\!\!\!\to} (0 : (3 + 7)) + (1 : 1) \mathrel{\bullet\!\!\!\to} 0 : ((1 : 0) + (1 : 1)) \mathrel{\bullet\!\!\!\to} 1 : ((1 : 0) + 1)$$
$$\mathrel{\bullet\!\!\!\to} 1 : (1 : (0 + 1)) \mathrel{\bullet\!\!\!\to} (1 + 1) : 1 \mathrel{\bullet\!\!\!\to} 2 : 1$$

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   Maximal Strategy Innermost Strategies Leftmost Outermost Strategy
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6. **Further Reading**

Remark

all strategies defined so−far are deterministic for orthogonal TRSs

Theorem

for orthogonal TRSs

- maximal and maximal outermost strategies are hyper−normalizing
- innermost strategies are perpetual

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Definition (Cofinality)

rewrite strategy $\mathcal{S}$ is cofinal for TRS $\mathcal{R}$ if for every maximal sequence

$$s = s_0 \xrightarrow{\mathcal{S}} s_1 \xrightarrow{\mathcal{S}} s_2 \xrightarrow{\mathcal{S}} \cdots$$

and every $s \to_{\mathcal{R}}^* t$ there exists $k \geqslant 0$ such that $t \to_{\mathcal{R}}^* s_k$

Lemma

cofinal strategies are normalizing

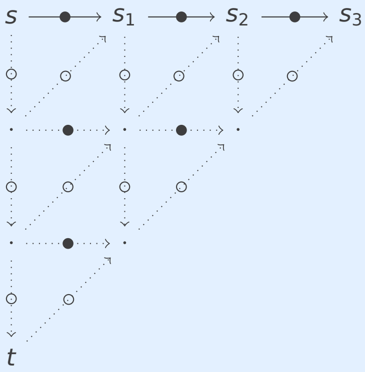
Proof

- let $\mathcal{S}$ be cofinal strategy for TRS $\mathcal{R}$
- consider maximal sequence $s = s_0 \xrightarrow{\mathcal{S}} s_1 \xrightarrow{\mathcal{S}} s_2 \xrightarrow{\mathcal{S}} \cdots$ and let $s \to_{\mathcal{R}}^! t$
- cofinality $\implies$ $t \to_{\mathcal{R}}^* s_k$ for some $k \geqslant 0$
- $t = s_k$

Theorem

maximal strategy is cofinal for orthogonal TRSs

Proof



triangle property

- assume $s \to^* t$

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Theorem

every innermost strategy is perpetual for orthogonal TRSs

Proof Outline

① innermost rewriting has random descent

② innermost normalizing terms are innermost terminating

③ innermost terminating terms are terminating

Notation

$a \xleftrightarrow[\langle l,r \rangle]{*} b$ if conversion $a \leftrightarrow^* b$ has l left $(\leftarrow)$ steps and r right $(\rightarrow)$ steps

Definition (Random Descent)

ARS $\mathcal{A} = \langle A, \rightarrow \rangle$ has random descent if $a \rightarrow^{r-l} b$ whenever $a \xleftrightarrow[\langle l,r \rangle]{*} b$ with $b \in \mathrm{NF}(\mathcal{A})$

Theorem

if ARS $\mathcal{A}$ has random descent and $a \leftrightarrow^* b$ with $b \in \mathrm{NF}(\mathcal{A})$ then

1. a is complete (confluent and terminating)
2. all rewrite sequences from a to b have same length

skip proofs

Theorem

innermost rewriting has random descent for orthogonal TRSs

skip proof

Proof

$\xleftarrow{\text{i}} \cdot \xrightarrow{\text{i}} \subseteq \xrightarrow{\text{i}} \cdot \xleftarrow{\text{i}} \cup = :$

- let $s \xrightarrow{\text{i}} t_1$ by contracting innermost redex Δ_1 at position p_1

  let $s \xrightarrow{\text{i}} t_2$ by contracting innermost redex Δ_2 at position p_2
- if p_1 and p_2 are parallel then $t_1 \xrightarrow{\text{i}} \cdot \xleftarrow{\text{i}} t_2$
- $p_1 < p_2$ and $p_2 < p_1$ are impossible because p_1 and p_2 are positions of innermost redexes
- if $p_1 = p_2$ then $\Delta_1 = \Delta_2$ and thus $t_1 = t_2$ by orthogonality

$\xleftarrow{\text{i}} \cdot \xrightarrow{\text{i}} \subseteq \xrightarrow{\text{i}} \cdot \xleftarrow{\text{i}} \cup = \quad \Longrightarrow \quad \xrightarrow{\text{i}}$ has random descent

Proof Outline

① innermost rewriting has random descent

② innermost normalizing terms are innermost terminating

③ innermost terminating terms are terminating

Proof

$s \xrightarrow{\mathsf{i}}{}^! t \quad \Longrightarrow \quad s$ is complete with respect to $\xrightarrow{\mathsf{i}} \quad \Longrightarrow \quad s$ is innermost terminating

Theorem

innermost termination and termination coincide for (terms in) orthogonal TRSs

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Remark

leftmost outermost strategy is not normalizing for all orthogonal TRSs

Example

$$\mathsf{a} \to \mathsf{b} \qquad \mathsf{c} \to \mathsf{c} \qquad \mathsf{f}(x, \mathsf{b}) \to \mathsf{b}$$

- leftmost outermost $\quad \mathsf{f}(\mathsf{c}, \mathsf{a}) \to \mathsf{f}(\mathsf{c}, \mathsf{a}) \to \mathsf{f}(\mathsf{c}, \mathsf{a}) \to \cdots$
- leftmost innermost $\quad \mathsf{f}(\mathsf{c}, \mathsf{a}) \to \mathsf{f}(\mathsf{c}, \mathsf{a}) \to \mathsf{f}(\mathsf{c}, \mathsf{a}) \to \cdots$
- maximal outermost $\quad \mathsf{f}(\mathsf{c}, \mathsf{a}) \mathrel{+\!\!\!\!\!\!+\!\!\!\!\to} \mathsf{f}(\mathsf{c}, \mathsf{b}) \mathrel{+\!\!\!\!\!\!+\!\!\!\!\to} \mathsf{b}$
- maximal innermost $\quad \mathsf{f}(\mathsf{c}, \mathsf{a}) \mathrel{+\!\!\!\!\!\!+\!\!\!\!\to} \mathsf{f}(\mathsf{c}, \mathsf{b}) \mathrel{+\!\!\!\!\!\!+\!\!\!\!\to} \mathsf{f}(\mathsf{c}, \mathsf{b}) \mathrel{+\!\!\!\!\!\!+\!\!\!\!\to} \cdots$
- maximal $\quad \mathsf{f}(\mathsf{c}, \mathsf{a}) \mathrel{-\!\!\bullet\!\!\to} \mathsf{f}(\mathsf{c}, \mathsf{b}) \mathrel{-\!\!\bullet\!\!\to} \mathsf{b}$

Definitions (Left-Normality)

- term t is left-normal if variables do not precede (in prefix notation) function symbols
- TRS $\mathcal{R}$ is left-normal if all left-hand sides of rules in $\mathcal{R}$ are left-normal

Examples

- $x + (y : z) \to y : (x + z)$ not left-normal $+ \ x \ : \ y \ z$
- $(x : y) + z \to x : (y + z)$ left-normal $+ \ : \ x \ y \ z$

Theorem

leftmost outermost strategy is normalizing for orthogonal left-normal TRSs

Remark

important result: Combinatory Logic is left-normal

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Homework Exercises for June 1

1. Exercise 6.7. ①
2. Exercise 6.8(a,b). ②
3. Exercise 7.1. ②
4. Exercise 7.8. ①
5. Exercise 7.9. ①
6. Exercise 7.6. ☆☆☆

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Lecture Notes

- Section 1.5
- Section 6.2
- Section 7.1
- Section 7.2

Important Concepts

- cofinal
- hyper−normalization
- innermost normalization
- innermost rewriting
- innermost termination
- left−normality
- leftmost innermost strategy
- leftmost outermost strategy
- maximal innermost strategy
- maximal outermost strategy
- maximal strategy
- normalization
- outermost rewriting
- perpetual
- proof term
- random descent
- relative rewriting
- rewrite strategy
- src(A)
- tgt(A)