



Term Rewriting

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Definition

orthogonal TRS is left-linear and lacks critical pairs

Definition

parallel rewriting \twoheadrightarrow is inductively defined as follows:

- ① $x \twoheadrightarrow x$ for all variables x
- ② $f(s_1, \dots, s_n) \twoheadrightarrow f(t_1, \dots, t_n)$ if $s_i \twoheadrightarrow t_i$ for all $1 \leq i \leq n$
- ③ $l\sigma \twoheadrightarrow r\sigma$ if $l \rightarrow r \in \mathcal{R}$

Parallel Moves Lemma

$\leftarrow \twoheadrightarrow \cdot \twoheadrightarrow \subseteq \twoheadrightarrow \cdot \leftarrow$ for orthogonal TRSs \implies orthogonal TRSs are **confluent**

Outline

1. Summary of Lecture 9
2. Proof Terms
3. Strategies
4. Normalization
5. Exercises
6. Further Reading

Definitions

multi-step relation \twoheadrightarrow is inductively defined as follows:

- ① $x \twoheadrightarrow x$ for all variables x
- ② $f(s_1, \dots, s_n) \twoheadrightarrow f(t_1, \dots, t_n)$ if $s_i \twoheadrightarrow t_i$ for all $1 \leq i \leq n$
- ③ $l\sigma \twoheadrightarrow r\tau$ if $l \rightarrow r \in \mathcal{R}$ and $\underbrace{x\sigma \twoheadrightarrow x\tau}_{\sigma \twoheadrightarrow \tau}$ for all variables x

maximal multi-step relation \twoheadrightarrow is inductively defined as follows:

- ① $x \twoheadrightarrow x$ for all variables x
- ② $f(s_1, \dots, s_n) \twoheadrightarrow f(t_1, \dots, t_n)$ if $s_i \twoheadrightarrow t_i$ for all $1 \leq i \leq n$ and $f(s_1, \dots, s_n)$ is no redex
- ③ $l\sigma \twoheadrightarrow r\tau$ if $l \rightarrow r \in \mathcal{R}$ and $\sigma \twoheadrightarrow \tau$

Lemma

$\rightarrow \subseteq \twoheadrightarrow \subseteq \twoheadrightarrow \subseteq \rightarrow^*$ and $\twoheadrightarrow \subseteq \twoheadrightarrow$

Lemma (Triangle Property)

$\leftarrow \cdot \rightarrow \subseteq \rightarrow \cdot \rightarrow$ for orthogonal TRSs

Definitions

TRS

- ▶ ... is **strongly closed** if $t \rightarrow^* \cdot \leftarrow u$ and $t \rightarrow^* \cdot \leftarrow u$
- ▶ ... is **parallel closed** if $t \leftrightarrow u$
- ▶ ... is **development closed** if $t \rightarrow u$

for every critical pair $t \approx u$

Theorem

- ▶ **linear** strongly closed TRSs are confluent
- ▶ **left-linear** parallel closed TRSs are confluent
- ▶ **left-linear** development closed TRSs are confluent

Example

- ▶ left-linear TRS
$$\begin{array}{l} h(f(x, g(y))) \xrightarrow{\alpha} h(f(x, g(x))) \quad g(a) \xrightarrow{\gamma} g(b) \\ f(g(x), y) \xrightarrow{\beta} f(g(x), g(x)) \quad b \xrightarrow{\delta} a \end{array}$$

▶ term $s = h(f(h(f(g(a), g(a))), g(a)))$

▶ proof terms $A = h(f(\alpha(\gamma, a), \gamma))$ and $B = \alpha(h(\beta(a, \gamma)), a)$

Definitions

rule symbol α associated to (left-linear) rewrite rule $\ell \rightarrow r$

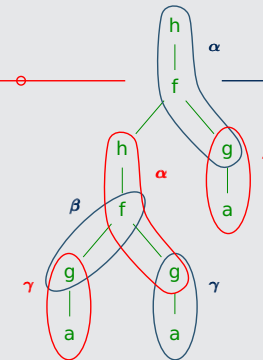
- ▶ **lhs**(α) denotes ℓ
- ▶ **rhs**(α) denotes r
- ▶ **var**(α) denotes list (x_1, \dots, x_n) of variables appearing in ℓ in some fixed order
- ▶ arity of α is length of **var**(α)
- ▶ $\langle t_1, \dots, t_n \rangle \alpha$ denotes substitution $\{x_i \mapsto t_i \mid 1 \leq i \leq n\}$

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Example

- ▶
$$\begin{array}{l} h(f(x, g(y))) \xrightarrow{\alpha} h(f(x, g(x))) \quad g(a) \xrightarrow{\gamma} g(b) \quad s = h(f(h(f(g(a), g(a))), g(a))) \\ f(g(x), y) \xrightarrow{\beta} f(g(x), g(x)) \quad b \xrightarrow{\delta} a \quad A = h(f(\alpha(\gamma, a), \gamma)) \quad B = \alpha(h(\beta(a, \gamma)), a) \end{array}$$



Definitions

- ▶ **proof terms** are terms built from function symbols, variables, rule symbols
- ▶ source $\text{src}(A)$ and target $\text{tgt}(A)$ of proof term A

$$\text{src}(x) = \text{tgt}(x) = x$$

$$\text{src}(f(A_1, \dots, A_n)) = f(\text{src}(A_1), \dots, \text{src}(A_n))$$

$$\text{src}(\alpha(A_1, \dots, A_n)) = \text{lhs}(\alpha)(\text{src}(A_1), \dots, \text{src}(A_n))_\alpha$$

$$\text{tgt}(f(A_1, \dots, A_n)) = f(\text{tgt}(A_1), \dots, \text{tgt}(A_n))$$

$$\text{tgt}(\alpha(A_1, \dots, A_n)) = \text{rhs}(\alpha)(\text{tgt}(A_1), \dots, \text{tgt}(A_n))_\alpha$$

- ▶ proof terms A and B are **co-initial** if $\text{src}(A) = \text{src}(B)$

Remark

proof term A **witnesses multi-step** $\text{src}(A) \twoheadrightarrow \text{tgt}(A)$

Example (cont'd)

$$\begin{aligned} h(f(x, g(y))) &\xrightarrow{\alpha} h(f(x, g(x))) & g(a) &\xrightarrow{\gamma} g(b) & s &= h(f(h(f(g(a), g(a))), g(a))) \\ f(g(x), y) &\xrightarrow{\beta} f(g(x), g(x)) & b &\xrightarrow{\delta} a & A &= h(f(\alpha(\gamma, a), \gamma)) & B &= \alpha(h(\beta(a, \gamma)), a) \end{aligned}$$

- ▶ $\text{src}(A) = h(\text{src}(f(\alpha(\gamma, a), \gamma))) = h(f(\text{src}(\alpha(\gamma, a)), \text{src}(\gamma)))$
 $= h(f(h(f(\text{src}(\gamma), g(\text{src}(a))), g(a)))) = h(f(h(f(g(a), g(a))), g(a)))$
- ▶ $\text{tgt}(B) = h(f(\text{tgt}(h(\beta(a, \gamma))), g(\text{tgt}(h(\beta(a, \gamma))))) = h(f(h(\text{tgt}(\beta(a, \gamma))), g(h(\text{tgt}(\beta(a, \gamma)))))$
 $= h(f(h(f(g(\text{tgt}(a)), g(\text{tgt}(a))), g(h(f(g(\text{tgt}(a)), g(\text{tgt}(a))))))) = h(f(h(f(g(a), g(a))), g(h(f(g(a), g(a))))))$

Lemma

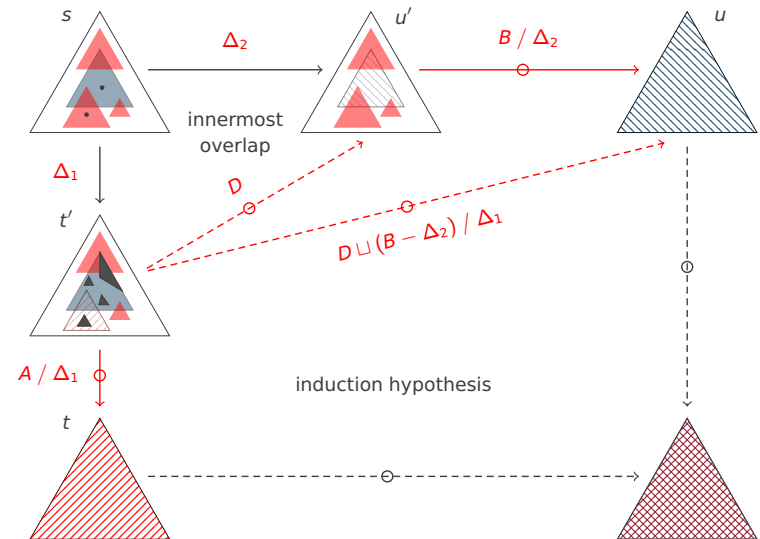
$s \twoheadrightarrow t \iff \text{src}(A) = s$ and $\text{tgt}(A) = t$ for some proof term A

Theorem

left-linear development closed TRSs are confluent

Remarks

- ▶ formalized proof employs proof terms
- ▶ result follows from **diamond property of \twoheadrightarrow**
- ▶ proof employs **induction on amount of overlap between two multi-steps**



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Definitions (Strategies, cont'd)

- ▶ rewrite strategy \mathcal{S} is **perpetual** if every maximal \mathcal{S} -rewrite sequence starting from any non-terminating term is infinite:

$$\forall t (WN_{\mathcal{S}}(t) \implies SN_{\mathcal{R}}(t))$$

Remarks

- ▶ (hyper-)normalizing strategy avoids non-terminating computations, if possible
- ▶ perpetual strategy avoids terminating computations, if possible

Lemma

for terminating TRSs every strategy is **hyper-normalizing** and **perpetual**

Definitions (Strategies)

- ▶ **one-step rewrite strategy** \mathcal{S} for TRS \mathcal{R} is relation $\xrightarrow{\mathcal{S}} \subseteq \rightarrow_{\mathcal{R}}$ such that $NF(\mathcal{S}) = NF(\mathcal{R})$
- ▶ **many-step** rewrite strategy \mathcal{S} is relation $\xrightarrow{\mathcal{S}} \subseteq \rightarrow_{\mathcal{R}}^+$ such that $NF(\mathcal{S}) = NF(\mathcal{R})$
- ▶ rewrite strategy \mathcal{S} is **deterministic** if $\xleftarrow{\mathcal{S}} \cdot \xrightarrow{\mathcal{S}} \subseteq =$
- ▶ rewrite strategy \mathcal{S} **normalizes** term t if all \mathcal{S} -rewrite sequences starting from t are finite
- ▶ rewrite strategy \mathcal{S} is **normalizing** if it normalizes every normalizing term:

$$\forall t (WN_{\mathcal{R}}(t) \implies SN_{\mathcal{S}}(t))$$

- ▶ **relative rewriting**: $\rightarrow_{\mathcal{S}/\mathcal{R}} = \rightarrow_{\mathcal{R}}^* \cdot \rightarrow_{\mathcal{S}} \cdot \rightarrow_{\mathcal{R}}^*$
- ▶ rewrite strategy \mathcal{S} is **hyper-normalizing** if every \mathcal{R} -normalizing term is \mathcal{S}/\mathcal{R} -terminating

Example

- ▶ rewrite rules

$0 + 0 \rightarrow 0$	$1 + 0 \rightarrow 1$	\dots	$9 + 0 \rightarrow 9$	$0 : x \rightarrow x$
$0 + 1 \rightarrow 1$	$1 + 1 \rightarrow 2$	\dots	$9 + 1 \rightarrow 1 : 0$	$x + (y : z) \rightarrow y : (x + z)$
$0 + 2 \rightarrow 2$	$1 + 2 \rightarrow 3$	\dots	$9 + 2 \rightarrow 1 : 1$	$(x : y) + z \rightarrow x : (y + z)$
$0 + 3 \rightarrow 3$	$1 + 3 \rightarrow 4$	\dots	$9 + 3 \rightarrow 1 : 2$	$x : (y : z) \rightarrow (x + y) : z$
$0 + 4 \rightarrow 4$	$1 + 4 \rightarrow 5$	\dots	$9 + 4 \rightarrow 1 : 3$	
$0 + 5 \rightarrow 5$	$1 + 5 \rightarrow 6$	\dots	$9 + 5 \rightarrow 1 : 4$	
$0 + 6 \rightarrow 6$	$1 + 6 \rightarrow 7$	\dots	$9 + 6 \rightarrow 1 : 5$	
$0 + 7 \rightarrow 7$	$1 + 7 \rightarrow 8$	\dots	$9 + 7 \rightarrow 1 : 6$	
$0 + 8 \rightarrow 8$	$1 + 8 \rightarrow 9$	\dots	$9 + 8 \rightarrow 1 : 7$	
$0 + 9 \rightarrow 9$	$1 + 9 \rightarrow 1 : 0$	\dots	$9 + 9 \rightarrow 1 : 8$	

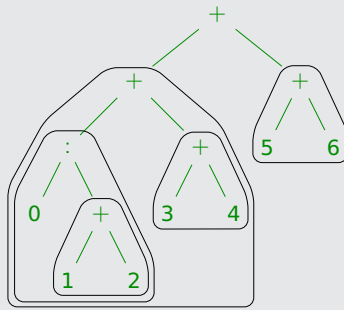
- ▶ term $((0 : (1 + 2)) + (3 + 4)) + (5 + 6)$

Example (cont'd)

▶ term

$$\boxed{0 : \boxed{1 + 2}} + \boxed{3 + 4} + \boxed{5 + 6}$$

▶ tree representation



maximal/leftmost outermost/innermost strategies

Example (cont'd)

▶ **leftmost outermost** strategy (12 redexes)

$$\begin{aligned} & ((0 : (1 + 2)) + (3 + 4)) + (5 + 6) \\ & \rightarrow (0 : ((1 + 2) + (3 + 4))) + (5 + 6) \rightarrow 0 : (((1 + 2) + (3 + 4)) + (5 + 6)) \\ & \rightarrow ((1 + 2) + (3 + 4)) + (5 + 6) \rightarrow (3 + (3 + 4)) + (5 + 6) \rightarrow (3 + 7) + (5 + 6) \\ & \rightarrow (1 : 0) + (5 + 6) \rightarrow 1 : (0 + (5 + 6)) \rightarrow 1 : (0 + (1 : 1)) \rightarrow 1 : (1 : (0 + 1)) \\ & \rightarrow (1 + 1) : (0 + 1) \rightarrow 2 : (0 + 1) \rightarrow 2 : 1 \end{aligned}$$

▶ **leftmost innermost** strategy (10 redexes)

$$\begin{aligned} & ((0 : (1 + 2)) + (3 + 4)) + (5 + 6) \\ & \rightarrow ((0 : 3) + (3 + 4)) + (5 + 6) \rightarrow (3 + (3 + 4)) + (5 + 6) \rightarrow (3 + 7) + (5 + 6) \\ & \rightarrow (1 : 0) + (5 + 6) \rightarrow (1 : 0) + (1 : 1) \rightarrow 1 : ((1 : 0) + 1) \rightarrow 1 : (1 : (0 + 1)) \\ & \rightarrow 1 : (1 : 1) \rightarrow (1 + 1) : 1 \rightarrow 2 : 1 \end{aligned}$$

Example (cont'd)

▶ **maximal outermost** strategy (12 redexes in 9 steps)

$$\begin{aligned} & ((0 : (1 + 2)) + (3 + 4)) + (5 + 6) \\ & \mapsto (0 : ((1 + 2) + (3 + 4))) + (1 : 1) \mapsto 0 : (((1 + 2) + (3 + 4)) + (1 : 1)) \\ & \mapsto ((1 + 2) + (3 + 4)) + (1 : 1) \mapsto 1 : (((1 + 2) + (3 + 4)) + 1) \\ & \mapsto 1 : ((3 + 7) + 1) \mapsto 1 : ((1 : 0) + 1) \mapsto 1 : (1 : (0 + 1)) \\ & \mapsto (1 + 1) : (0 + 1) \mapsto 2 : 1 \end{aligned}$$

▶ **maximal innermost** strategy (10 redexes in 8 steps)

$$\begin{aligned} & ((0 : (1 + 2)) + (3 + 4)) + (5 + 6) \\ & \mapsto ((0 : 3) + 7) + (1 : 1) \mapsto (3 + 7) + (1 : 1) \mapsto (1 : 0) + (1 : 1) \\ & \mapsto 1 : (0 + (1 : 1)) \mapsto 1 : (1 : (0 + 1)) \mapsto 1 : (1 : 1) \mapsto (1 + 1) : 1 \mapsto 2 : 1 \end{aligned}$$

Definition (Maximal Multi-Step Rewriting)

maximal multi-step relation \twoheadrightarrow is inductively defined as follows:

- ① $x \twoheadrightarrow x$ for all variables x
- ② $f(s_1, \dots, s_n) \twoheadrightarrow f(t_1, \dots, t_n)$ if $s_i \twoheadrightarrow t_i$ for all $1 \leq i \leq n$ and $f(s_1, \dots, s_n)$ is no redex
- ③ $l\sigma \twoheadrightarrow r\tau$ if $l \rightarrow r \in \mathcal{R}$ and $\sigma \twoheadrightarrow \tau$

Definition (Maximal Strategy)

maximal strategy performs maximal multi-step for every reducible term

Example (cont'd)

► maximal strategy (11 redexes in 5 steps)

$$t = ((0 : (1 + 2)) + (3 + 4)) + (5 + 6)$$
$$\rightarrow (0 : (3 + 7)) + (1 : 1) \rightarrow 1 : ((1 : 0) + 1) \rightarrow 1 : (1 : (0 + 1))$$
$$\rightarrow (1 + 1) : 1 \rightarrow 2 : 1$$

► maximal strategy (12 redexes in 6 steps)

$$t \rightarrow (0 : (3 + 7)) + (1 : 1) \rightarrow 0 : ((1 : 0) + (1 : 1)) \rightarrow 1 : (0 + (1 : 1))$$
$$\rightarrow 1 : (1 : (0 + 1)) \rightarrow (1 + 1) : 1 \rightarrow 2 : 1$$

► maximal strategy (12 redexes in 6 steps)

$$t \rightarrow (0 : (3 + 7)) + (1 : 1) \rightarrow 0 : ((1 : 0) + (1 : 1)) \rightarrow 1 : ((1 : 0) + 1)$$
$$\rightarrow 1 : (1 : (0 + 1)) \rightarrow (1 + 1) : 1 \rightarrow 2 : 1$$

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1. Summary of Lecture 9

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Maximal Strategy

Innermost Strategies

Leftmost Outermost Strategy

5. Exercises

6. Further Reading

Remark

all strategies defined so-far are **deterministic** for orthogonal TRSs

Theorem

for orthogonal TRSs

- maximal and maximal outermost strategies are **hyper-normalizing**
- innermost strategies are **perpetual**

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Definition (Cofinality)

rewrite strategy \mathcal{S} is **cofinal** for TRS \mathcal{R} if for every maximal sequence

$$s = s_0 \xrightarrow{\mathcal{S}} s_1 \xrightarrow{\mathcal{S}} s_2 \xrightarrow{\mathcal{S}} \dots$$

and every $s \rightarrow_{\mathcal{R}}^* t$ there exists $k \geq 0$ such that $t \rightarrow_{\mathcal{R}}^* s_k$

Lemma

cofinal strategies are normalizing

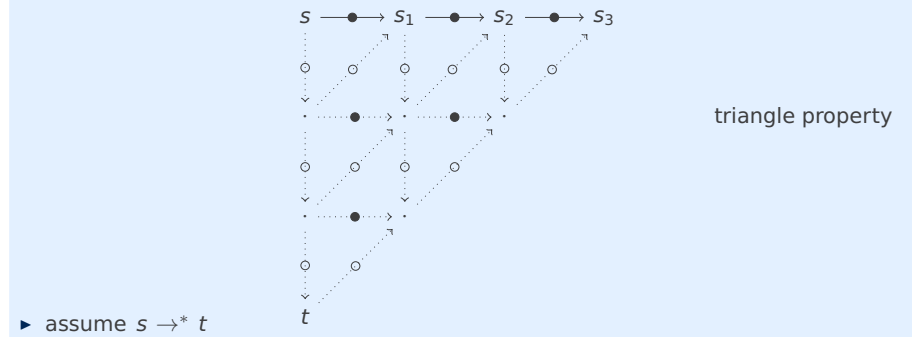
Proof

- ▶ let \mathcal{S} be cofinal strategy for TRS \mathcal{R}
- ▶ consider maximal sequence $s = s_0 \xrightarrow{\mathcal{S}} s_1 \xrightarrow{\mathcal{S}} s_2 \xrightarrow{\mathcal{S}} \dots$ and let $s \rightarrow_{\mathcal{R}}^! t$
- ▶ cofinality $\implies t \rightarrow_{\mathcal{R}}^* s_k$ for some $k \geq 0$
- ▶ $t = s_k$

Theorem

maximal strategy is cofinal for orthogonal TRSs

Proof



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Theorem

every innermost strategy is **perpetual** for orthogonal TRSs

Proof Outline

- ① innermost rewriting has random descent
- ② innermost normalizing terms are innermost terminating
- ③ innermost terminating terms are terminating

Notation

$a \xrightarrow[\langle l, r \rangle]{*} b$ if conversion $a \leftrightarrow^* b$ has l left (\leftarrow) steps and r right (\rightarrow) steps

Definition (Random Descent)

ARS $\mathcal{A} = \langle A, \rightarrow \rangle$ has **random descent** if $a \rightarrow^{r-l} b$ whenever $a \xrightarrow[\langle l, r \rangle]{*} b$ with $b \in \text{NF}(\mathcal{A})$

Theorem

if ARS \mathcal{A} has random descent and $a \leftrightarrow^* b$ with $b \in \text{NF}(\mathcal{A})$ then

- 1 a is **complete** (confluent and terminating)
- 2 all rewrite sequences from a to b have **same length**

▶ skip proofs

Theorem

innermost rewriting has random descent for orthogonal TRSs

▶ skip proof

Proof

$\leftarrow^i \cdot \xrightarrow{i} \subseteq \xrightarrow{i} \cdot \leftarrow^i \cup = :$

- ▶ let $s \xrightarrow{i} t_1$ by contracting innermost redex Δ_1 at position p_1
let $s \xrightarrow{i} t_2$ by contracting innermost redex Δ_2 at position p_2
- ▶ if p_1 and p_2 are parallel then $t_1 \xrightarrow{i} \cdot \leftarrow^i t_2$
- ▶ $p_1 < p_2$ and $p_2 < p_1$ are impossible because p_1 and p_2 are positions of innermost redexes
- ▶ if $p_1 = p_2$ then $\Delta_1 = \Delta_2$ and thus $t_1 = t_2$ by orthogonality

$\leftarrow^i \cdot \xrightarrow{i} \subseteq \xrightarrow{i} \cdot \leftarrow^i \cup = \implies \xrightarrow{i}$ has random descent

Proof Outline

- 1 innermost rewriting has random descent
- 2 innermost normalizing terms are innermost terminating
- 3 **innermost terminating terms are terminating**

Proof

$s \xrightarrow{i}^! t \implies s$ is complete with respect to $\xrightarrow{i} \implies s$ is innermost terminating

Theorem

innermost termination and termination coincide for (terms in) orthogonal TRSs

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Remark

leftmost outermost strategy is **not** normalizing for all orthogonal TRSs

Example

$$a \rightarrow b$$

$$c \rightarrow c$$

$$f(x, b) \rightarrow b$$

- ▶ leftmost outermost $f(c, a) \rightarrow f(c, a) \rightarrow f(c, a) \rightarrow \dots$
- ▶ leftmost innermost $f(c, a) \rightarrow f(c, a) \rightarrow f(c, a) \rightarrow \dots$
- ▶ maximal outermost $f(c, a) \twoheadrightarrow f(c, b) \twoheadrightarrow b$
- ▶ maximal innermost $f(c, a) \twoheadrightarrow f(c, b) \twoheadrightarrow f(c, b) \twoheadrightarrow \dots$
- ▶ maximal $f(c, a) \dashrightarrow f(c, b) \dashrightarrow b$

Definitions (Left-Normality)

- ▶ term t is **left-normal** if variables do not precede (in prefix notation) function symbols
- ▶ TRS \mathcal{R} is left-normal if all left-hand sides of rules in \mathcal{R} are left-normal

Examples

- ▶ $x + (y : z) \rightarrow y : (x + z)$ **not** left-normal $+ x : y z$
- ▶ $(x : y) + z \rightarrow x : (y + z)$ left-normal $+ : x y z$

Theorem

leftmost outermost strategy is normalizing for orthogonal left-normal TRSs

Remark

important result: **Combinatory Logic** is left-normal

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Homework Exercises for June 1

- ① Exercise 6.7. ①
- ② Exercise 6.8(a,b). ②
- ③ Exercise 7.1. ②
- ④ Exercise 7.8. ①
- ⑤ Exercise 7.9. ①
- ⑥ Exercise 7.6. ☆☆☆

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Lecture Notes

- ▶ Section 1.5
- ▶ Section 6.2
- ▶ Section 7.1
- ▶ Section 7.2

Important Concepts

- ▶ cofinal
- ▶ hyper-normalization
- ▶ innermost normalization
- ▶ innermost rewriting
- ▶ innermost termination
- ▶ left-normality
- ▶ leftmost innermost strategy
- ▶ leftmost outermost strategy
- ▶ maximal innermost strategy
- ▶ maximal outermost strategy
- ▶ maximal strategy
- ▶ normalization
- ▶ outermost rewriting
- ▶ perpetual
- ▶ proof term
- ▶ random descent
- ▶ relative rewriting
- ▶ rewrite strategy
- ▶ $\text{src}(A)$
- ▶ $\text{tgt}(A)$