universität innsbruck

Term Rewriting

Philipp Dablander and **Aart Middeldorp**

Outline

1. **Summary of Lecture 10**
2. **Optimal Strategies**
3. **Strategy Annotations**
4. **Simple Termination**
5. **Exercises**
6. **Further Reading**

Definitions

- one–step (many–step) rewrite strategy $\mathcal{S}$ is relation $\xrightarrow{\mathcal{S}} \subseteq \to_{\mathcal{R}}^{(+)}$ such that $\mathrm{NF}(\mathcal{S}) = \mathrm{NF}(\mathcal{R})$
- rewrite strategy $\mathcal{S}$ is deterministic if $\xleftarrow{\mathcal{S}} \cdot \xrightarrow{\mathcal{S}} \subseteq =$
- rewrite strategy $\mathcal{S}$ normalizes term t if all $\mathcal{S}$–rewrite sequences starting from t are finite
- rewrite strategy $\mathcal{S}$ is normalizing if it normalizes every normalizing term
- relative rewriting: $\to_{\mathcal{S}/\mathcal{R}} = \to_{\mathcal{R}}^* \cdot \to_{\mathcal{S}} \cdot \to_{\mathcal{R}}^*$
- rewrite strategy $\mathcal{S}$ is hyper–normalizing if every $\mathcal{R}$–normalizing term is $\mathcal{S}/\mathcal{R}$–terminating
- rewrite strategy $\mathcal{S}$ is perpetual if every maximal $\mathcal{S}$–rewrite sequence starting from any non–terminating term is infinite
- rewrite strategy $\mathcal{S}$ is cofinal for TRS $\mathcal{R}$ if for every $s \to_{\mathcal{R}}^* t$ and every maximal sequence $s = s_0 \xrightarrow{\mathcal{S}} s_1 \xrightarrow{\mathcal{S}} s_2 \xrightarrow{\mathcal{S}} \cdots$ there exists $k \geqslant 0$ such that $t \to_{\mathcal{R}}^* s_k$

Lemma

cofinal strategies are normalizing

Definitions

- leftmost outermost strategy contracts leftmost of outermost redexes
- leftmost innermost strategy contracts leftmost of innermost redexes
- maximal outermost strategy contracts all outermost redexes
- maximal innermost strategy contracts all innermost redexes
- maximal strategy performs maximal multi−step

Definition

TRS is left−normal if variables do not precede function symbols in left−hand sides

Theorem

- maximal strategy is cofinal for orthogonal TRSs
- maximal outermost strategy is hyper−normalizing for orthogonal TRSs
- leftmost outermost strategy is normalizing for orthogonal left−normal TRSs

Notation

$a \xleftrightarrow[\langle l, r\rangle]{*} b$ if conversion $a \leftrightarrow^* b$ has l left ($\leftarrow$) steps and r right ($\rightarrow$) steps

Definition

ARS $\mathcal{A} = \langle A, \rightarrow\rangle$ has random descent if $a \rightarrow^{r-l} b$ whenever $a \xleftrightarrow[\langle l, r\rangle]{*} b \in \mathrm{NF}(\mathcal{A})$

Theorem

if ARS $\mathcal{A}$ has random descent and $a \leftrightarrow^* b$ with normal form b then a is complete and all rewrite sequences from a to b have same length

Theorem

- innermost rewriting has random descent for orthogonal TRSs
- any innermost strategy is perpetual for orthogonal TRSs

Outline

Observation

maximal outermost and maximal strategies are not optimal

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Example

- TRS $\mathcal{R}$

$$0 + y \to y \qquad 0 \times y \to 0$$
$$\mathsf{s}(x) + y \to \mathsf{s}(x + y) \qquad \mathsf{s}(x) \times y \to (x \times y) + y$$

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- rewrite sequence $(0 \times \mathsf{s}(0)) \times (0 + \mathsf{s}(0)) \nrightarrow\!\!\!\!\rightarrow 0 \times \mathsf{s}(0)$

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maximal outermost and maximal strategies are not optimal

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- TRS $\mathcal{R}$
$$\begin{aligned} 0 + y &\to y & 0 \times y &\to 0 \\ \mathsf{s}(x) + y &\to \mathsf{s}(x + y) & \mathsf{s}(x) \times y &\to (x \times y) + y \end{aligned}$$
- rewrite sequence $(0 \times \mathsf{s}(0)) \times (0 + \mathsf{s}(0)) \mathrel{+\!\!\!\!\!\!\Vdash\!\!\!\to} 0 \times \mathsf{s}(0) \to 0$

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maximal outermost and maximal strategies are not optimal

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- TRS $\mathcal{R}$ $\quad 0 + y \to y \qquad 0 \times y \to 0$

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- redex $0 + \mathsf{s}(0)$ is not needed

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Definition (Needed Redex)

redex occurrence Δ in term $C[\Delta]$ with respect to TRS $\mathcal{R}$ is needed if $C[\bullet]$ has no normal form in $\mathcal{R} \cup \{\bullet \to \bullet\}$

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redex occurrence Δ in term $C[\Delta]$ with respect to TRS $\mathcal{R}$ is needed if $C[\bullet]$ has no normal form in $\mathcal{R} \cup \{\bullet \to \bullet\}$

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decidable approximations based on tree automata techniques exist

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decidable approximations based on tree automata techniques exist

Theorem

for left–normal orthogonal TRSs leftmost outermost redex is needed

Outline

Example

- rewrite rules

$$\mathsf{and}(x,\mathsf{T}) \xrightarrow{\alpha} x \qquad \mathsf{and}(x,\mathsf{F}) \xrightarrow{\beta} \mathsf{F} \qquad \mathsf{or}(\mathsf{T},x) \xrightarrow{\gamma} \mathsf{T} \qquad \mathsf{or}(\mathsf{F},x) \xrightarrow{\delta} x \qquad \infty \xrightarrow{\epsilon} \infty$$

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- strategy annotation

$$A(\mathsf{and}) = [2,\alpha,\beta,1] \qquad A(\mathsf{or}) = [1,\gamma,\delta,2] \qquad A(\infty) = [\epsilon]$$

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- strategy annotation for function symbol f is finite list $A(f)$ containing
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- rewrite rules

$$\mathsf{if}(\mathsf{T},x,y) \xrightarrow{\alpha} x \qquad \mathsf{if}(\mathsf{F},x,y) \xrightarrow{\beta} y \qquad \mathsf{if}(z,x,x) \xrightarrow{\gamma} x$$

- strategy annotations

$$A_1(\mathsf{if}) = [1,\alpha,\beta,2,3,\gamma]$$

$$A_2(\mathsf{if}) = [\alpha,\beta,1,2,3,\gamma]$$

$$A_3(\mathsf{if}) = [1,\alpha,\beta,\gamma,2,3]$$

$$A_4(\mathsf{if}) = [1,\alpha,\beta]$$

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- strategy annotation $A(f)$ for function symbol f is in–time if argument positions are listed in $A(f)$ before rewrite rules that need them
- rewrite rule $f(s_1, \ldots, s_n) \to t$ needs argument position i if
  - s_i is non–variable, or
  - s_i is variable that appears in $s_1, \ldots, s_{i-1}, s_{i+1}, \ldots, s_n$

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if argument position i is not needed for ℓ and $t \gtrdot \ell$ then $t[u]_i \gtrdot \ell$ for any term u

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Lemma

if argument position i is not needed for ℓ and $t \gtrdot \ell$ then $t[u]_i \gtrdot \ell$ for any term u

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$$\mathsf{redex}'_A(t, [\,]) = \bot \qquad \mathsf{redex}_A(t) = \mathsf{redex}'_A(t, A(\mathsf{root}(t)))$$

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$$\text{redex}'_A(t, [\ell \to r \mid L]) = \begin{cases} (\epsilon, \ell \to r) & \text{if } t \gtrdot \ell \\ \text{redex}'_A(t, L) & \text{otherwise} \end{cases}$$

Lemma

if argument position i is not needed for ℓ and $t \geqslant \ell$ then $t[u]_i \geqslant \ell$ for any term u

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$$\text{redex}'_A(t, [i \mid L]) = \begin{cases} (ip, \ell \to r) & \text{if } \text{redex}_A(t|_i) = (p, \ell \to r) \\ \text{redex}'_A(t, L) & \text{otherwise} \end{cases}$$

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- evaluation $\quad \mathsf{redex}_A(\mathsf{or}(\mathsf{and}(\infty,\mathsf{F}),\mathsf{or}(\mathsf{T},\infty)))$

Example

- rewrite rules

$$\mathsf{and}(x,\mathsf{T}) \xrightarrow{\alpha} x \qquad \mathsf{and}(x,\mathsf{F}) \xrightarrow{\beta} \mathsf{F} \qquad \mathsf{or}(\mathsf{T},x) \xrightarrow{\gamma} \mathsf{T} \qquad \mathsf{or}(\mathsf{F},x) \xrightarrow{\delta} x \qquad \infty \xrightarrow{\epsilon} \infty$$

- strategy annotation

$$A(\mathsf{and}) = [2,\alpha,\beta,1] \qquad A(\mathsf{or}) = [1,\gamma,\delta,2] \qquad A(\infty) = [\epsilon]$$

- evaluation

$$\begin{aligned} &\mathsf{redex}_A(\mathsf{or}(\mathsf{and}(\infty,\mathsf{F}),\mathsf{or}(\mathsf{T},\infty))) \\ &= \mathsf{redex}'_A(\mathsf{or}(\mathsf{and}(\infty,\mathsf{F}),\mathsf{or}(\mathsf{T},\infty)),[1,\gamma,\delta,2]) \end{aligned}$$

Example

- rewrite rules

$$\mathsf{and}(x,\mathsf{T}) \xrightarrow{\alpha} x \qquad \mathsf{and}(x,\mathsf{F}) \xrightarrow{\beta} \mathsf{F} \qquad \mathsf{or}(\mathsf{T},x) \xrightarrow{\gamma} \mathsf{T} \qquad \mathsf{or}(\mathsf{F},x) \xrightarrow{\delta} x \qquad \infty \xrightarrow{\epsilon} \infty$$

- strategy annotation

$$A(\mathsf{and}) = [2,\alpha,\beta,1] \qquad A(\mathsf{or}) = [1,\gamma,\delta,2] \qquad A(\infty) = [\epsilon]$$

- evaluation

$$\mathsf{redex}_A(\mathsf{or}(\mathsf{and}(\infty,\mathsf{F}),\mathsf{or}(\mathsf{T},\infty)))$$

$$= \mathsf{redex}'_A(\mathsf{or}(\mathsf{and}(\infty,\mathsf{F}),\mathsf{or}(\mathsf{T},\infty)), [1,\gamma,\delta,2])$$

$$\mathsf{redex}_A(\mathsf{and}(\infty,\mathsf{F}))$$

Example

- rewrite rules

$$\mathsf{and}(x,\mathsf{T}) \xrightarrow{\alpha} x \qquad \mathsf{and}(x,\mathsf{F}) \xrightarrow{\beta} \mathsf{F} \qquad \mathsf{or}(\mathsf{T},x) \xrightarrow{\gamma} \mathsf{T} \qquad \mathsf{or}(\mathsf{F},x) \xrightarrow{\delta} x \qquad \infty \xrightarrow{\epsilon} \infty$$

- strategy annotation

$$A(\mathsf{and}) = [2,\alpha,\beta,1] \qquad A(\mathsf{or}) = [1,\gamma,\delta,2] \qquad A(\infty) = [\epsilon]$$

- evaluation

$$\mathsf{redex}_A(\mathsf{or}(\mathsf{and}(\infty,\mathsf{F}),\mathsf{or}(\mathsf{T},\infty)))$$
$$= \mathsf{redex}'_A(\mathsf{or}(\mathsf{and}(\infty,\mathsf{F}),\mathsf{or}(\mathsf{T},\infty)),\ [1,\gamma,\delta,2])$$
$$\mathsf{redex}_A(\mathsf{and}(\infty,\mathsf{F}))$$
$$= \mathsf{redex}'_A(\mathsf{and}(\infty,\mathsf{F}),\ [2,\alpha,\beta,1])$$

Example

- rewrite rules

$$\mathsf{and}(x,\mathsf{T}) \xrightarrow{\alpha} x \qquad \mathsf{and}(x,\mathsf{F}) \xrightarrow{\beta} \mathsf{F} \qquad \mathsf{or}(\mathsf{T},x) \xrightarrow{\gamma} \mathsf{T} \qquad \mathsf{or}(\mathsf{F},x) \xrightarrow{\delta} x \qquad \infty \xrightarrow{\epsilon} \infty$$

- strategy annotation

$$A(\mathsf{and}) = [2,\alpha,\beta,1] \qquad A(\mathsf{or}) = [1,\gamma,\delta,2] \qquad A(\infty) = [\epsilon]$$

- evaluation

$$\mathsf{redex}_A(\mathsf{or}(\mathsf{and}(\infty,\mathsf{F}),\mathsf{or}(\mathsf{T},\infty)))$$
$$= \mathsf{redex}'_A(\mathsf{or}(\mathsf{and}(\infty,\mathsf{F}),\mathsf{or}(\mathsf{T},\infty)),\ [1,\gamma,\delta,2])$$
$$\mathsf{redex}_A(\mathsf{and}(\infty,\mathsf{F}))$$
$$= \mathsf{redex}'_A(\mathsf{and}(\infty,\mathsf{F}),\ [2,\alpha,\beta,1])$$
$$\mathsf{redex}_A(\mathsf{F})$$

Example

- rewrite rules

$$\mathsf{and}(x,\mathsf{T}) \xrightarrow{\alpha} x \qquad \mathsf{and}(x,\mathsf{F}) \xrightarrow{\beta} \mathsf{F} \qquad \mathsf{or}(\mathsf{T},x) \xrightarrow{\gamma} \mathsf{T} \qquad \mathsf{or}(\mathsf{F},x) \xrightarrow{\delta} x \qquad \infty \xrightarrow{\epsilon} \infty$$

- strategy annotation

$$A(\mathsf{and}) = [2,\alpha,\beta,1] \qquad A(\mathsf{or}) = [1,\gamma,\delta,2] \qquad A(\infty) = [\epsilon]$$

- evaluation

$$\mathsf{redex}_A(\mathsf{or}(\mathsf{and}(\infty,\mathsf{F}),\mathsf{or}(\mathsf{T},\infty)))$$
$$= \mathsf{redex}'_A(\mathsf{or}(\mathsf{and}(\infty,\mathsf{F}),\mathsf{or}(\mathsf{T},\infty)),[1,\gamma,\delta,2])$$
$$\mathsf{redex}_A(\mathsf{and}(\infty,\mathsf{F}))$$
$$= \mathsf{redex}'_A(\mathsf{and}(\infty,\mathsf{F}),[2,\alpha,\beta,1])$$

$$\mathsf{redex}_A(\mathsf{F}) = \bot$$

Example

- rewrite rules

$$\mathsf{and}(x,\mathsf{T}) \xrightarrow{\alpha} x \qquad \mathsf{and}(x,\mathsf{F}) \xrightarrow{\beta} \mathsf{F} \qquad \mathsf{or}(\mathsf{T},x) \xrightarrow{\gamma} \mathsf{T} \qquad \mathsf{or}(\mathsf{F},x) \xrightarrow{\delta} x \qquad \infty \xrightarrow{\epsilon} \infty$$

- strategy annotation

$$A(\mathsf{and}) = [2,\alpha,\beta,1] \qquad A(\mathsf{or}) = [1,\gamma,\delta,2] \qquad A(\infty) = [\epsilon]$$

- evaluation

$$\mathsf{redex}_A(\mathsf{or}(\mathsf{and}(\infty,\mathsf{F}),\mathsf{or}(\mathsf{T},\infty)))$$
$$= \mathsf{redex}'_A(\mathsf{or}(\mathsf{and}(\infty,\mathsf{F}),\mathsf{or}(\mathsf{T},\infty)),[1,\gamma,\delta,2])$$
$$\mathsf{redex}_A(\mathsf{and}(\infty,\mathsf{F}))$$
$$= \mathsf{redex}'_A(\mathsf{and}(\infty,\mathsf{F}),[2,\alpha,\beta,1])$$
$$= \mathsf{redex}'_A(\mathsf{and}(\infty,\mathsf{F}),[\alpha,\beta,1])$$

$$\mathsf{redex}_A(\mathsf{F}) = \bot$$

Example

- rewrite rules

$$\mathsf{and}(x,\mathsf{T}) \xrightarrow{\alpha} x \qquad \mathsf{and}(x,\mathsf{F}) \xrightarrow{\beta} \mathsf{F} \qquad \mathsf{or}(\mathsf{T},x) \xrightarrow{\gamma} \mathsf{T} \qquad \mathsf{or}(\mathsf{F},x) \xrightarrow{\delta} x \qquad \infty \xrightarrow{\epsilon} \infty$$

- strategy annotation

$$A(\mathsf{and}) = [2,\alpha,\beta,1] \qquad A(\mathsf{or}) = [1,\gamma,\delta,2] \qquad A(\infty) = [\epsilon]$$

- evaluation

$$\begin{aligned}
&\mathsf{redex}_A(\mathsf{or}(\mathsf{and}(\infty,\mathsf{F}),\mathsf{or}(\mathsf{T},\infty)))\\
&\quad = \mathsf{redex}'_A(\mathsf{or}(\mathsf{and}(\infty,\mathsf{F}),\mathsf{or}(\mathsf{T},\infty)),\ [1,\gamma,\delta,2])\\
&\mathsf{redex}_A(\mathsf{and}(\infty,\mathsf{F}))\\
&\quad = \mathsf{redex}'_A(\mathsf{and}(\infty,\mathsf{F}),\ [2,\alpha,\beta,1])\\
&\quad = \mathsf{redex}'_A(\mathsf{and}(\infty,\mathsf{F}),\ [\alpha,\beta,1])\\
&\quad = \mathsf{redex}'_A(\mathsf{and}(\infty,\mathsf{F}),\ [\beta,1])\\
&\mathsf{redex}_A(\mathsf{F}) = \bot
\end{aligned}$$

Example

- rewrite rules

$$\mathsf{and}(x,\mathsf{T}) \xrightarrow{\alpha} x \qquad \mathsf{and}(x,\mathsf{F}) \xrightarrow{\beta} \mathsf{F} \qquad \mathsf{or}(\mathsf{T},x) \xrightarrow{\gamma} \mathsf{T} \qquad \mathsf{or}(\mathsf{F},x) \xrightarrow{\delta} x \qquad \infty \xrightarrow{\epsilon} \infty$$

- strategy annotation

$$A(\mathsf{and}) = [2,\alpha,\beta,1] \qquad A(\mathsf{or}) = [1,\gamma,\delta,2] \qquad A(\infty) = [\epsilon]$$

- evaluation

$$\mathsf{redex}_A(\mathsf{or}(\mathsf{and}(\infty,\mathsf{F}),\mathsf{or}(\mathsf{T},\infty)))$$
$$= \mathsf{redex}'_A(\mathsf{or}(\mathsf{and}(\infty,\mathsf{F}),\mathsf{or}(\mathsf{T},\infty)),[1,\gamma,\delta,2])$$
$$\mathsf{redex}_A(\mathsf{and}(\infty,\mathsf{F}))$$
$$= \mathsf{redex}'_A(\mathsf{and}(\infty,\mathsf{F}),[2,\alpha,\beta,1])$$
$$= \mathsf{redex}'_A(\mathsf{and}(\infty,\mathsf{F}),[\alpha,\beta,1])$$
$$= \mathsf{redex}'_A(\mathsf{and}(\infty,\mathsf{F}),[\beta,1]) = (\epsilon,\beta)$$
$$\mathsf{redex}_A(\mathsf{F}) = \bot$$

Example

- rewrite rules

$$\mathsf{and}(x,\mathsf{T}) \xrightarrow{\alpha} x \qquad \mathsf{and}(x,\mathsf{F}) \xrightarrow{\beta} \mathsf{F} \qquad \mathsf{or}(\mathsf{T},x) \xrightarrow{\gamma} \mathsf{T} \qquad \mathsf{or}(\mathsf{F},x) \xrightarrow{\delta} x \qquad \infty \xrightarrow{\epsilon} \infty$$

- strategy annotation

$$A(\mathsf{and}) = [2,\alpha,\beta,1] \qquad A(\mathsf{or}) = [1,\gamma,\delta,2] \qquad A(\infty) = [\epsilon]$$

- evaluation

$$\begin{aligned}
&\mathsf{redex}_A(\mathsf{or}(\mathsf{and}(\infty,\mathsf{F}),\mathsf{or}(\mathsf{T},\infty)))\\
&\quad = \mathsf{redex}'_A(\mathsf{or}(\mathsf{and}(\infty,\mathsf{F}),\mathsf{or}(\mathsf{T},\infty)),[1,\gamma,\delta,2]) = (1,\beta)\\
&\mathsf{redex}_A(\mathsf{and}(\infty,\mathsf{F}))\\
&\quad = \mathsf{redex}'_A(\mathsf{and}(\infty,\mathsf{F}),[2,\alpha,\beta,1])\\
&\quad = \mathsf{redex}'_A(\mathsf{and}(\infty,\mathsf{F}),[\alpha,\beta,1])\\
&\quad = \mathsf{redex}'_A(\mathsf{and}(\infty,\mathsf{F}),[\beta,1]) = (\epsilon,\beta)\\
&\mathsf{redex}_A(\mathsf{F}) = \bot
\end{aligned}$$

Lemma

if argument position i is not needed for ℓ and $t \gtrsim \ell$ then $t[u]_i \gtrsim \ell$ for any term u

Definition

$$\text{redex}'_A(t, [\,]) = \bot \qquad \text{redex}_A(t) = \text{redex}'_A(t, A(\text{root}(t)))$$

$$\text{redex}'_A(t, [\ell \to r \mid L]) = \begin{cases} (\epsilon, \ell \to r) & \text{if } t \gtrsim \ell \\ \text{redex}'_A(t, L) & \text{otherwise} \end{cases}$$

$$\text{redex}'_A(t, [i \mid L]) = \begin{cases} (ip, \ell \to r) & \text{if } \text{redex}_A(t|_i) = (p, \ell \to r) \\ \text{redex}'_A(t, L) & \text{otherwise} \end{cases}$$

Theorem

$\forall$ full strategy annotation A $\forall$ term t $\quad \text{redex}_A(t) = \bot \iff t$ is normal form

Example

- rewrite rules

$$\mathsf{if}(\mathsf{T}, x, y) \xrightarrow{\alpha} x \qquad \mathsf{if}(\mathsf{F}, x, y) \xrightarrow{\beta} y \qquad \mathsf{if}(z, x, x) \xrightarrow{\gamma} x$$

- strategy annotation A with $A(\mathsf{if}) = [1, \alpha, 2, 3, \gamma]$ is not full

$$\mathsf{redex}_A(\mathsf{if}(\mathsf{F}, \mathsf{T}, \mathsf{F})) = \bot \qquad \mathsf{if}(\mathsf{F}, \mathsf{T}, \mathsf{F}) \to \mathsf{F}$$

Example

- rewrite rules

$$\mathsf{if}(\mathsf{T},x,y) \xrightarrow{\alpha} x \qquad \mathsf{if}(\mathsf{F},x,y) \xrightarrow{\beta} y \qquad \mathsf{if}(z,x,x) \xrightarrow{\gamma} x$$

- strategy annotation A with $A(\mathsf{if}) = [1,\alpha,2,3,\gamma]$ is not full

$$\mathsf{redex}_A(\mathsf{if}(\mathsf{F},\mathsf{T},\mathsf{F})) = \bot \qquad \mathsf{if}(\mathsf{F},\mathsf{T},\mathsf{F}) \to \mathsf{F}$$

Definition

$s \xrightarrow{\mathcal{S}_A} t$ if $\mathsf{redex}_A(s) = (p, \ell \to r)$ and $s \to_{p|\ell\to r} t$

Example

- rewrite rules

$$\mathsf{if}(\mathsf{T},x,y) \xrightarrow{\alpha} x \qquad \mathsf{if}(\mathsf{F},x,y) \xrightarrow{\beta} y \qquad \mathsf{if}(z,x,x) \xrightarrow{\gamma} x$$

- strategy annotation A with $A(\mathsf{if}) = [1,\alpha,2,3,\gamma]$ is not full

$$\mathsf{redex}_A(\mathsf{if}(\mathsf{F},\mathsf{T},\mathsf{F})) = \bot \qquad \mathsf{if}(\mathsf{F},\mathsf{T},\mathsf{F}) \to \mathsf{F}$$

Definition

$s \xrightarrow{\mathcal{S}_A} t$ if $\mathsf{redex}_A(s) = (p, \ell \to r)$ and $s \to_{p|\ell\to r} t$

Corollary

$\mathcal{S}_A$ is rewrite strategy for every full strategy annotation A

Definition

$$\text{normalize}_A(t) = \text{normalize}'_A(t, A(\text{root}(t)))$$

Definition

$$\textcolor{red}{\text{normalize}_A(t)} = \text{normalize}'_A(t, A(\text{root}(t)))$$

$$\text{normalize}'_A(t, [\,]) = t$$

Definition

$$\text{normalize}_A(t) = \text{normalize}'_A(t, A(\text{root}(t)))$$

$$\text{normalize}'_A(t, [\,]) = t$$

$$\text{normalize}'_A(t, [\ell \to r \mid L]) = \begin{cases} \text{normalize}_A(r\sigma) & \text{if } t = \ell\sigma \text{ for some substitution } \sigma \\ \text{normalize}'_A(t, L) & \text{otherwise} \end{cases}$$

Definition

$$\text{normalize}_A(t) = \text{normalize}'_A(t, A(\text{root}(t)))$$

$$\text{normalize}'_A(t, [\,]) = t$$

$$\text{normalize}'_A(t, [\ell \to r \mid L]) = \begin{cases} \text{normalize}_A(r\sigma) & \text{if } t = \ell\sigma \text{ for some substitution } \sigma \\ \text{normalize}'_A(t, L) & \text{otherwise} \end{cases}$$

$$\text{normalize}'_A(t, [i \mid L]) = \text{normalize}'_A(t[\text{normalize}_A(t|_i)]_i, L)$$

Example

- rewrite rules

$$\mathsf{and}(x,\mathsf{T}) \xrightarrow{\alpha} x \qquad \mathsf{and}(x,\mathsf{F}) \xrightarrow{\beta} \mathsf{F} \qquad \mathsf{or}(\mathsf{T},x) \xrightarrow{\gamma} \mathsf{T} \qquad \mathsf{or}(\mathsf{F},x) \xrightarrow{\delta} x \qquad \infty \xrightarrow{\epsilon} \infty$$

- strategy annotation

$$A(\mathsf{and}) = [2,\alpha,\beta,1] \qquad A(\mathsf{or}) = [1,\gamma,\delta,2] \qquad A(\infty) = [\epsilon]$$

- evaluation $\quad \mathsf{normalize}_A(\mathsf{or}(\mathsf{and}(\infty,\mathsf{F}),\mathsf{or}(\mathsf{T},\infty)))$

Example

- rewrite rules

$$\mathsf{and}(x,\mathsf{T}) \xrightarrow{\alpha} x \qquad \mathsf{and}(x,\mathsf{F}) \xrightarrow{\beta} \mathsf{F} \qquad \mathsf{or}(\mathsf{T},x) \xrightarrow{\gamma} \mathsf{T} \qquad \mathsf{or}(\mathsf{F},x) \xrightarrow{\delta} x \qquad \infty \xrightarrow{\epsilon} \infty$$

- strategy annotation

$$A(\mathsf{and}) = [2,\alpha,\beta,1] \qquad A(\mathsf{or}) = [1,\gamma,\delta,2] \qquad A(\infty) = [\epsilon]$$

- evaluation

$$\begin{aligned} &\mathsf{normalize}_A(\mathsf{or}(\mathsf{and}(\infty,\mathsf{F}),\mathsf{or}(\mathsf{T},\infty))) \\ &= \mathsf{normalize}'_A(\mathsf{or}(\mathsf{and}(\infty,\mathsf{F}),\mathsf{or}(\mathsf{T},\infty)), [1,\gamma,\delta,2]) \end{aligned}$$

Example

- rewrite rules

$$\mathsf{and}(x,\mathsf{T}) \xrightarrow{\alpha} x \qquad \mathsf{and}(x,\mathsf{F}) \xrightarrow{\beta} \mathsf{F} \qquad \mathsf{or}(\mathsf{T},x) \xrightarrow{\gamma} \mathsf{T} \qquad \mathsf{or}(\mathsf{F},x) \xrightarrow{\delta} x \qquad \infty \xrightarrow{\epsilon} \infty$$

- strategy annotation

$$A(\mathsf{and}) = [2,\alpha,\beta,1] \qquad A(\mathsf{or}) = [1,\gamma,\delta,2] \qquad A(\infty) = [\epsilon]$$

- evaluation

$$\begin{aligned} & \mathsf{normalize}_A(\mathsf{or}(\mathsf{and}(\infty,\mathsf{F}),\mathsf{or}(\mathsf{T},\infty))) \\ &= \mathsf{normalize}'_A(\mathsf{or}(\mathsf{and}(\infty,\mathsf{F}),\mathsf{or}(\mathsf{T},\infty)),\ [1,\gamma,\delta,2]) \\ &= \mathsf{normalize}'_A(\mathsf{or}(\mathsf{normalize}_A(\mathsf{and}(\infty,\mathsf{F})),\mathsf{or}(\mathsf{T},\infty)),\ [\gamma,\delta,2]) \end{aligned}$$

Example

- rewrite rules

$$\mathsf{and}(x,\mathsf{T}) \xrightarrow{\alpha} x \qquad \mathsf{and}(x,\mathsf{F}) \xrightarrow{\beta} \mathsf{F} \qquad \mathsf{or}(\mathsf{T},x) \xrightarrow{\gamma} \mathsf{T} \qquad \mathsf{or}(\mathsf{F},x) \xrightarrow{\delta} x \qquad \infty \xrightarrow{\epsilon} \infty$$

- strategy annotation

$$A(\mathsf{and}) = [2,\alpha,\beta,1] \qquad A(\mathsf{or}) = [1,\gamma,\delta,2] \qquad A(\infty) = [\epsilon]$$

- evaluation

$$\begin{aligned} &\mathsf{normalize}_A(\mathsf{or}(\mathsf{and}(\infty,\mathsf{F}),\mathsf{or}(\mathsf{T},\infty))) \\ &\quad = \mathsf{normalize}'_A(\mathsf{or}(\mathsf{and}(\infty,\mathsf{F}),\mathsf{or}(\mathsf{T},\infty)),\ [1,\gamma,\delta,2]) \\ &\quad = \mathsf{normalize}'_A(\mathsf{or}(\mathsf{normalize}_A(\mathsf{and}(\infty,\mathsf{F})),\mathsf{or}(\mathsf{T},\infty)),\ [\gamma,\delta,2]) \\ &\mathsf{normalize}_A(\mathsf{and}(\infty,\mathsf{F})) \\ &\quad = \mathsf{normalize}'_A(\mathsf{and}(\infty,\mathsf{F}),\ [2,\alpha,\beta,1]) \end{aligned}$$

Example

- rewrite rules

$$\mathsf{and}(x,\mathsf{T}) \xrightarrow{\alpha} x \qquad \mathsf{and}(x,\mathsf{F}) \xrightarrow{\beta} \mathsf{F} \qquad \mathsf{or}(\mathsf{T},x) \xrightarrow{\gamma} \mathsf{T} \qquad \mathsf{or}(\mathsf{F},x) \xrightarrow{\delta} x \qquad \infty \xrightarrow{\epsilon} \infty$$

- strategy annotation

$$A(\mathsf{and}) = [2,\alpha,\beta,1] \qquad A(\mathsf{or}) = [1,\gamma,\delta,2] \qquad A(\infty) = [\epsilon]$$

- evaluation

$$\begin{aligned}
&\mathsf{normalize}_A(\mathsf{or}(\mathsf{and}(\infty,\mathsf{F}),\mathsf{or}(\mathsf{T},\infty)))\\
&\quad = \mathsf{normalize}'_A(\mathsf{or}(\mathsf{and}(\infty,\mathsf{F}),\mathsf{or}(\mathsf{T},\infty)),\ [1,\gamma,\delta,2])\\
&\quad = \mathsf{normalize}'_A(\mathsf{or}(\mathsf{normalize}_A(\mathsf{and}(\infty,\mathsf{F})),\mathsf{or}(\mathsf{T},\infty)),\ [\gamma,\delta,2])\\
&\mathsf{normalize}_A(\mathsf{and}(\infty,\mathsf{F}))\\
&\quad = \mathsf{normalize}'_A(\mathsf{and}(\infty,\mathsf{F}),\ [2,\alpha,\beta,1])\\
&\quad = \mathsf{normalize}'_A(\mathsf{and}(\infty,\mathsf{normalize}_A(\mathsf{F})),\ [\alpha,\beta,1])
\end{aligned}$$

Example

- rewrite rules

$$\mathsf{and}(x,\mathsf{T}) \xrightarrow{\alpha} x \qquad \mathsf{and}(x,\mathsf{F}) \xrightarrow{\beta} \mathsf{F} \qquad \mathsf{or}(\mathsf{T},x) \xrightarrow{\gamma} \mathsf{T} \qquad \mathsf{or}(\mathsf{F},x) \xrightarrow{\delta} x \qquad \infty \xrightarrow{\epsilon} \infty$$

- strategy annotation

$$A(\mathsf{and}) = [2,\alpha,\beta,1] \qquad A(\mathsf{or}) = [1,\gamma,\delta,2] \qquad A(\infty) = [\epsilon]$$

- evaluation

$$\begin{aligned}
&\mathsf{normalize}_A(\mathsf{or}(\mathsf{and}(\infty,\mathsf{F}),\mathsf{or}(\mathsf{T},\infty)))\\
&\quad= \mathsf{normalize}'_A(\mathsf{or}(\mathsf{and}(\infty,\mathsf{F}),\mathsf{or}(\mathsf{T},\infty)),\ [1,\gamma,\delta,2])\\
&\quad= \mathsf{normalize}'_A(\mathsf{or}(\mathsf{normalize}_A(\mathsf{and}(\infty,\mathsf{F})),\mathsf{or}(\mathsf{T},\infty)),\ [\gamma,\delta,2])\\
&\mathsf{normalize}_A(\mathsf{and}(\infty,\mathsf{F}))\\
&\quad= \mathsf{normalize}'_A(\mathsf{and}(\infty,\mathsf{F}),\ [2,\alpha,\beta,1])\\
&\quad= \mathsf{normalize}'_A(\mathsf{and}(\infty,\mathsf{normalize}_A(\mathsf{F})),\ [\alpha,\beta,1])\\
&\quad= \mathsf{normalize}'_A(\mathsf{and}(\infty,\mathsf{F}),\ [\alpha,\beta,1])
\end{aligned}$$

Example

- rewrite rules

$$\mathsf{and}(x,\mathsf{T}) \xrightarrow{\alpha} x \qquad \mathsf{and}(x,\mathsf{F}) \xrightarrow{\beta} \mathsf{F} \qquad \mathsf{or}(\mathsf{T},x) \xrightarrow{\gamma} \mathsf{T} \qquad \mathsf{or}(\mathsf{F},x) \xrightarrow{\delta} x \qquad \infty \xrightarrow{\epsilon} \infty$$

- strategy annotation

$$A(\mathsf{and}) = [2,\alpha,\beta,1] \qquad A(\mathsf{or}) = [1,\gamma,\delta,2] \qquad A(\infty) = [\epsilon]$$

- evaluation

$$\begin{aligned}
&\mathsf{normalize}_A(\mathsf{or}(\mathsf{and}(\infty,\mathsf{F}),\mathsf{or}(\mathsf{T},\infty)))\\
&\quad = \mathsf{normalize}'_A(\mathsf{or}(\mathsf{and}(\infty,\mathsf{F}),\mathsf{or}(\mathsf{T},\infty)),\ [1,\gamma,\delta,2])\\
&\quad = \mathsf{normalize}'_A(\mathsf{or}(\mathsf{normalize}_A(\mathsf{and}(\infty,\mathsf{F})),\mathsf{or}(\mathsf{T},\infty)),\ [\gamma,\delta,2])
\end{aligned}$$

$$\begin{aligned}
&\mathsf{normalize}_A(\mathsf{and}(\infty,\mathsf{F}))\\
&\quad = \mathsf{normalize}'_A(\mathsf{and}(\infty,\mathsf{F}),\ [2,\alpha,\beta,1])\\
&\quad = \mathsf{normalize}'_A(\mathsf{and}(\infty,\mathsf{normalize}_A(\mathsf{F})),\ [\alpha,\beta,1])\\
&\quad = \mathsf{normalize}'_A(\mathsf{and}(\infty,\mathsf{F}),\ [\alpha,\beta,1])\\
&\quad = \mathsf{normalize}'_A(\mathsf{and}(\infty,\mathsf{F}),\ [\beta,1])
\end{aligned}$$

Example

- rewrite rules

$$\mathsf{and}(x,\mathsf{T}) \xrightarrow{\alpha} x \qquad \mathsf{and}(x,\mathsf{F}) \xrightarrow{\beta} \mathsf{F} \qquad \mathsf{or}(\mathsf{T},x) \xrightarrow{\gamma} \mathsf{T} \qquad \mathsf{or}(\mathsf{F},x) \xrightarrow{\delta} x \qquad \infty \xrightarrow{\epsilon} \infty$$

- strategy annotation

$$A(\mathsf{and}) = [2,\alpha,\beta,1] \qquad A(\mathsf{or}) = [1,\gamma,\delta,2] \qquad A(\infty) = [\epsilon]$$

- evaluation

$$\begin{aligned}
&\mathsf{normalize}_A(\mathsf{or}(\mathsf{and}(\infty,\mathsf{F}),\mathsf{or}(\mathsf{T},\infty)))\\
&\quad= \mathsf{normalize}'_A(\mathsf{or}(\mathsf{and}(\infty,\mathsf{F}),\mathsf{or}(\mathsf{T},\infty)),\ [1,\gamma,\delta,2])\\
&\quad= \mathsf{normalize}'_A(\mathsf{or}(\mathsf{normalize}_A(\mathsf{and}(\infty,\mathsf{F})),\mathsf{or}(\mathsf{T},\infty)),\ [\gamma,\delta,2])
\end{aligned}$$

$$\begin{aligned}
&\mathsf{normalize}_A(\mathsf{and}(\infty,\mathsf{F}))\\
&\quad= \mathsf{normalize}'_A(\mathsf{and}(\infty,\mathsf{F}),\ [2,\alpha,\beta,1])\\
&\quad= \mathsf{normalize}'_A(\mathsf{and}(\infty,\mathsf{normalize}_A(\mathsf{F})),\ [\alpha,\beta,1])\\
&\quad= \mathsf{normalize}'_A(\mathsf{and}(\infty,\mathsf{F}),\ [\alpha,\beta,1])\\
&\quad= \mathsf{normalize}'_A(\mathsf{and}(\infty,\mathsf{F}),\ [\beta,1]) = \mathsf{normalize}_A(\mathsf{F})
\end{aligned}$$

Example

- rewrite rules

$$\mathsf{and}(x,\mathsf{T}) \xrightarrow{\alpha} x \qquad \mathsf{and}(x,\mathsf{F}) \xrightarrow{\beta} \mathsf{F} \qquad \mathsf{or}(\mathsf{T},x) \xrightarrow{\gamma} \mathsf{T} \qquad \mathsf{or}(\mathsf{F},x) \xrightarrow{\delta} x \qquad \infty \xrightarrow{\epsilon} \infty$$

- strategy annotation

$$A(\mathsf{and}) = [2,\alpha,\beta,1] \qquad A(\mathsf{or}) = [1,\gamma,\delta,2] \qquad A(\infty) = [\epsilon]$$

- evaluation

$$\begin{aligned}
&\mathsf{normalize}_A(\mathsf{or}(\mathsf{and}(\infty,\mathsf{F}),\mathsf{or}(\mathsf{T},\infty)))\\
&\quad = \mathsf{normalize}'_A(\mathsf{or}(\mathsf{and}(\infty,\mathsf{F}),\mathsf{or}(\mathsf{T},\infty)),\ [1,\gamma,\delta,2])\\
&\quad = \mathsf{normalize}'_A(\mathsf{or}(\mathsf{normalize}_A(\mathsf{and}(\infty,\mathsf{F})),\mathsf{or}(\mathsf{T},\infty)),\ [\gamma,\delta,2])
\end{aligned}$$

$$\begin{aligned}
&\mathsf{normalize}_A(\mathsf{and}(\infty,\mathsf{F}))\\
&\quad = \mathsf{normalize}'_A(\mathsf{and}(\infty,\mathsf{F}),\ [2,\alpha,\beta,1])\\
&\quad = \mathsf{normalize}'_A(\mathsf{and}(\infty,\mathsf{normalize}_A(\mathsf{F})),\ [\alpha,\beta,1])\\
&\quad = \mathsf{normalize}'_A(\mathsf{and}(\infty,\mathsf{F}),\ [\alpha,\beta,1])\\
&\quad = \mathsf{normalize}'_A(\mathsf{and}(\infty,\mathsf{F}),\ [\beta,1]) = \mathsf{normalize}_A(\mathsf{F}) = \mathsf{F}
\end{aligned}$$

Example

- rewrite rules

$$\mathsf{and}(x, \mathsf{T}) \xrightarrow{\alpha} x \qquad \mathsf{and}(x, \mathsf{F}) \xrightarrow{\beta} \mathsf{F} \qquad \mathsf{or}(\mathsf{T}, x) \xrightarrow{\gamma} \mathsf{T} \qquad \mathsf{or}(\mathsf{F}, x) \xrightarrow{\delta} x \qquad \infty \xrightarrow{\epsilon} \infty$$

- strategy annotation

$$A(\mathsf{and}) = [2, \alpha, \beta, 1] \qquad A(\mathsf{or}) = [1, \gamma, \delta, 2] \qquad A(\infty) = [\epsilon]$$

- evaluation

$$\begin{aligned} & \mathsf{normalize}_A(\mathsf{or}(\mathsf{and}(\infty, \mathsf{F}), \mathsf{or}(\mathsf{T}, \infty))) \\ &= \mathsf{normalize}'_A(\mathsf{or}(\mathsf{and}(\infty, \mathsf{F}), \mathsf{or}(\mathsf{T}, \infty)), [1, \gamma, \delta, 2]) \\ &= \mathsf{normalize}'_A(\mathsf{or}(\mathsf{normalize}_A(\mathsf{and}(\infty, \mathsf{F})), \mathsf{or}(\mathsf{T}, \infty)), [\gamma, \delta, 2]) \\ &= \mathsf{normalize}'_A(\mathsf{or}(\mathsf{F}, \mathsf{or}(\mathsf{T}, \infty)), [\gamma, \delta, 2]) \end{aligned}$$

Example

- rewrite rules

$$\mathsf{and}(x, \mathsf{T}) \xrightarrow{\alpha} x \qquad \mathsf{and}(x, \mathsf{F}) \xrightarrow{\beta} \mathsf{F} \qquad \mathsf{or}(\mathsf{T}, x) \xrightarrow{\gamma} \mathsf{T} \qquad \mathsf{or}(\mathsf{F}, x) \xrightarrow{\delta} x \qquad \infty \xrightarrow{\epsilon} \infty$$

- strategy annotation

$$A(\mathsf{and}) = [2, \alpha, \beta, 1] \qquad A(\mathsf{or}) = [1, \gamma, \delta, 2] \qquad A(\infty) = [\epsilon]$$

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$$\begin{aligned} & \mathsf{normalize}_A(\mathsf{or}(\mathsf{and}(\infty, \mathsf{F}), \mathsf{or}(\mathsf{T}, \infty))) \\ &= \mathsf{normalize}'_A(\mathsf{or}(\mathsf{and}(\infty, \mathsf{F}), \mathsf{or}(\mathsf{T}, \infty)), [1, \gamma, \delta, 2]) \\ &= \mathsf{normalize}'_A(\mathsf{or}(\mathsf{normalize}_A(\mathsf{and}(\infty, \mathsf{F})), \mathsf{or}(\mathsf{T}, \infty)), [\gamma, \delta, 2]) \\ &= \mathsf{normalize}'_A(\mathsf{or}(\mathsf{F}, \mathsf{or}(\mathsf{T}, \infty)), [\gamma, \delta, 2]) \\ &= \mathsf{normalize}'_A(\mathsf{or}(\mathsf{F}, \mathsf{or}(\mathsf{T}, \infty)), [\delta, 2]) \end{aligned}$$

Example

- rewrite rules

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\end{aligned}$$

Definition

$$\mathrm{normalize}_A(t) = \mathrm{normalize}'_A(t, A(\mathrm{root}(t)))$$

$$\mathrm{normalize}'_A(t, [\,]) = t$$

$$\mathrm{normalize}'_A(t, [\ell \to r \mid L]) = \begin{cases} \mathrm{normalize}_A(r\sigma) & \text{if } t = \ell\sigma \text{ for some substitution } \sigma \\ \mathrm{normalize}'_A(t, L) & \text{otherwise} \end{cases}$$

$$\mathrm{normalize}'_A(t, [i \mid L]) = \mathrm{normalize}'_A(t[\mathrm{normalize}_A(t|_i)]_i, L)$$

Remark

normalize and normalize′ are evaluated in call−by−value manner

Theorem

$\forall$ full and in–time strategy annotation A

1. $\mathcal{S}_A$ normalizes term t $\iff$ normalize$_A(t)$ is defined

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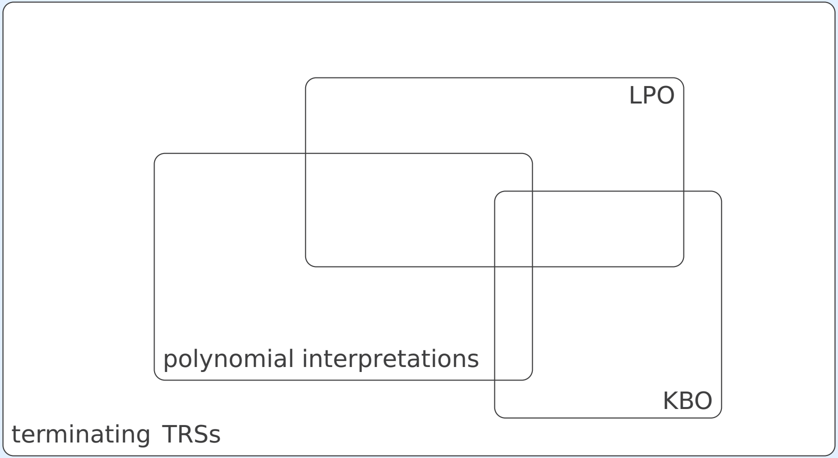
Theorem

$\forall$ full and in–time strategy annotation A $\forall$ term t

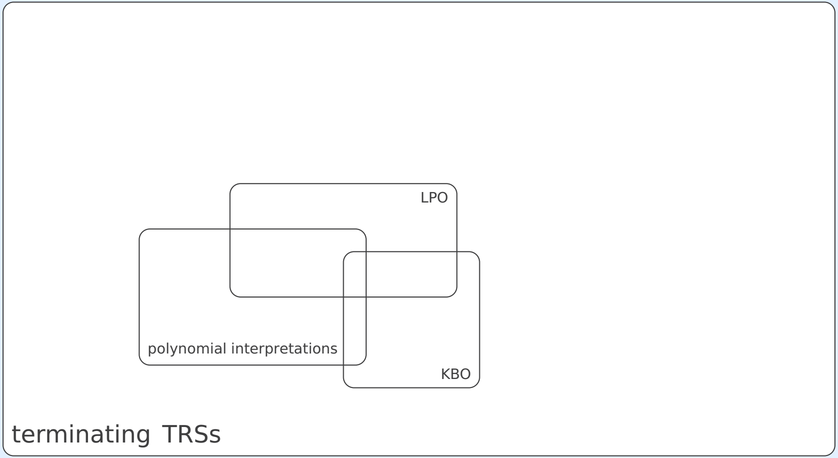
$$\text{leftmost innermost strategy normalizes } t \implies \mathcal{S}_A \text{ normalizes } t$$

Outline

Termination Methods



Termination Methods: Realistic View



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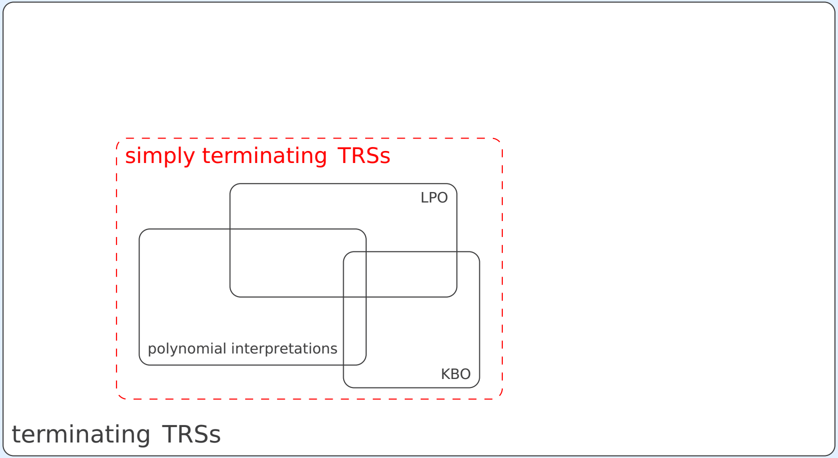

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Definitions (Subterm Property)

- term relation $>$ has subterm property if $C[t] > t$ for all non−empty contexts C and terms t

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term $\mathsf{f}(\mathsf{g}(\mathsf{a}),\mathsf{b})$ is (properly) embedded in $\mathsf{f}(\mathsf{h}(\mathsf{g}(\mathsf{a}),\mathsf{a}),\mathsf{g}(\mathsf{b}))$ since

$$\mathsf{f}(\mathsf{h}(\mathsf{g}(\mathsf{a}),\mathsf{a}),\mathsf{g}(\mathsf{b})) \to \mathsf{f}(\mathsf{g}(\mathsf{a}),\mathsf{g}(\mathsf{b})) \to \mathsf{f}(\mathsf{g}(\mathsf{a}),\mathsf{b})$$

in TRS

$$\mathcal{E}\text{mb}(\{\mathsf{a},\mathsf{b},\mathsf{f},\mathsf{g},\mathsf{h}\}) = \left\{\begin{array}{lll} \mathsf{f}(x,y) \to x & \mathsf{g}(x) \to x & \mathsf{h}(x,y) \to x \\ \mathsf{f}(x,y) \to y & & \mathsf{h}(x,y) \to y \end{array}\right\}$$

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assumption: signatures are finite

Outline

Kruskal's Tree Theorem

$\forall$ infinite sequence of ground terms $t_1, t_2, t_3, \ldots$ $\exists$ $i < j$ such that $t_i \trianglelefteq_{\text{emb}} t_j$

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Proof (by contradiction)

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simply terminating TRSs are terminating

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- TRS $\mathsf{f}(\mathsf{a}, \mathsf{b}, x) \to \mathsf{f}(x, x, x)$ is not simply terminating:

$$\begin{aligned} \mathsf{f}(\mathsf{a}, \mathsf{b}, \mathsf{f}(\mathsf{a}, \mathsf{b}, \mathsf{b})) &\to \mathsf{f}(\mathsf{f}(\mathsf{a}, \mathsf{b}, \mathsf{b}), \mathsf{f}(\mathsf{a}, \mathsf{b}, \mathsf{b}), \mathsf{f}(\mathsf{a}, \mathsf{b}, \mathsf{b})) \\ &\to \mathsf{f}(\mathsf{a}, \mathsf{f}(\mathsf{a}, \mathsf{b}, \mathsf{b}), \mathsf{f}(\mathsf{a}, \mathsf{b}, \mathsf{b})) \end{aligned}$$

Corollary

simply terminating TRSs are terminating

Lemma

TRS $\mathcal{R}$ is simply terminating if and only if $\mathcal{R} \cup \mathcal{E}\mathsf{mb}$ is terminating

Examples

- SRS $\mathsf{aa} \to \mathsf{aba}$ is not simply terminating:

$$\mathsf{aa} \to \mathsf{aba} \to \mathsf{aa}$$

- TRS $\mathsf{f}(\mathsf{a},\mathsf{b},x) \to \mathsf{f}(x,x,x)$ is not simply terminating:

$$\begin{aligned} \mathsf{f}(\mathsf{a},\mathsf{b},\mathsf{f}(\mathsf{a},\mathsf{b},\mathsf{b})) &\to \mathsf{f}(\mathsf{f}(\mathsf{a},\mathsf{b},\mathsf{b}),\mathsf{f}(\mathsf{a},\mathsf{b},\mathsf{b}),\mathsf{f}(\mathsf{a},\mathsf{b},\mathsf{b})) \\ &\to \mathsf{f}(\mathsf{a},\mathsf{f}(\mathsf{a},\mathsf{b},\mathsf{b}),\mathsf{f}(\mathsf{a},\mathsf{b},\mathsf{b})) \\ &\to \mathsf{f}(\mathsf{a},\mathsf{b},\mathsf{f}(\mathsf{a},\mathsf{b},\mathsf{b})) \end{aligned}$$

Outline

Definition (Simple Monotone Algebra)

simple monotone $\mathcal{F}$–algebra $(\mathcal{A}, >)$ consists of non–empty algebra $\mathcal{A} = (A, \{f_\mathcal{A}\}_{f \in \mathcal{F}})$ with well–founded order $>$ on A such that every $f_\mathcal{A}$ is simple and weakly monotone:

$$f_\mathcal{A}(a_1, \ldots, a_i, \ldots, a_n) \geqslant a_i$$

for all $a_1, \ldots, a_n \in A$ and $i \in \{1, \ldots, n\}$

$$f_\mathcal{A}(a_1, \ldots, a_i, \ldots, a_n) \geqslant f_\mathcal{A}(a_1, \ldots, b, \ldots, a_n)$$

for all $a_1, \ldots, a_n, b \in A$ and $i \in \{1, \ldots, n\}$ with $a_i > b$

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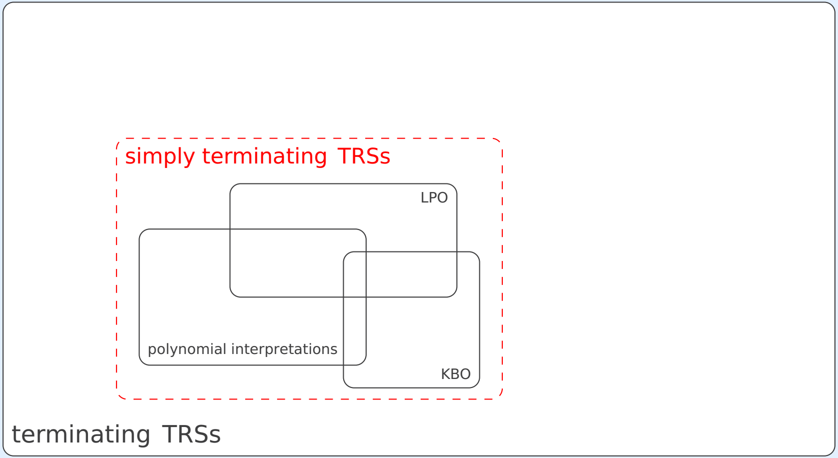
Theorem

TRS $\mathcal{R}$ is simply terminating $\iff$ $\mathcal{R}$ is compatible with simple monotone algebra

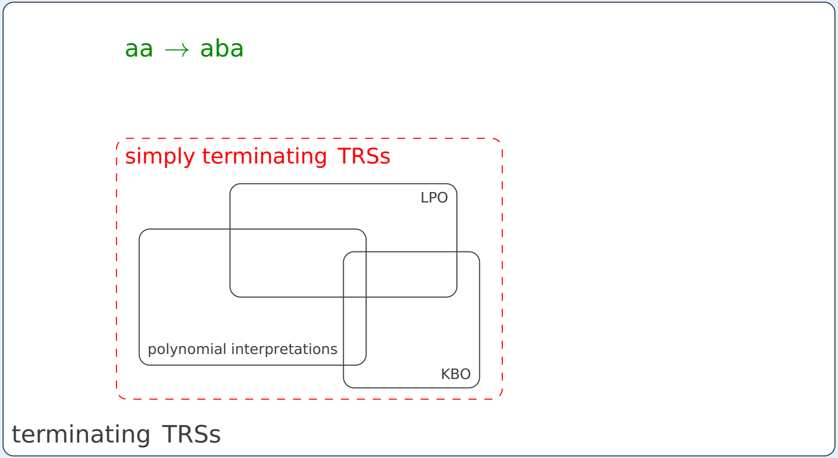
Corollary

polynomially terminating TRSs are simply terminating

Termination Methods: Realistic View



Termination Methods: Realistic View



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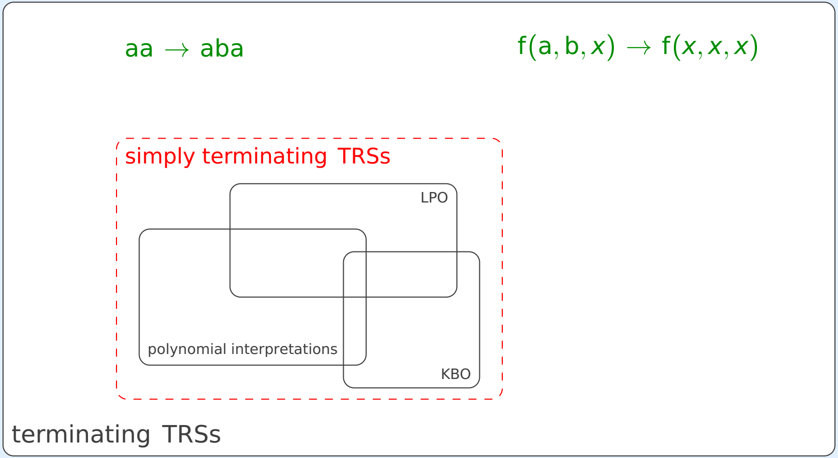

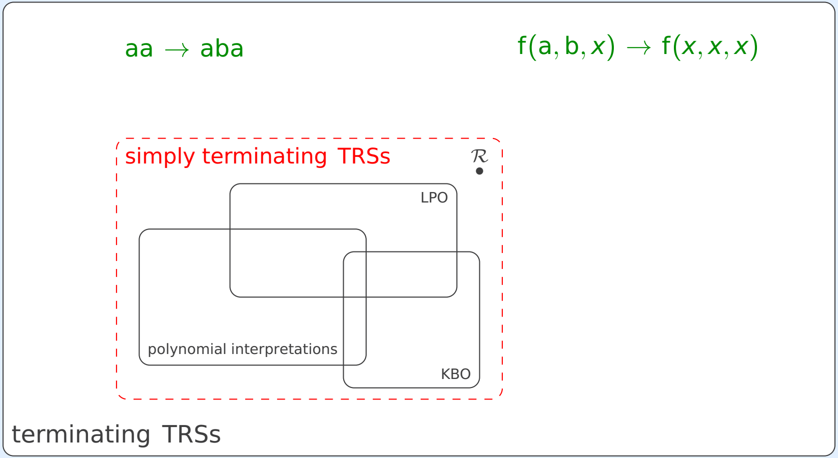

$$\mathcal{R}: \quad f(a) \to f(b) \qquad g(b) \to g(a)$$

Outline

Homework Exercises for June 8

1. Exercise 7.12. 1
2. Exercise 7.17. 2
3. Exercise 7.19. 2
4. Exercise 4.24. 2

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next lecture (June 8): online evaluation in presence

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Outline

Lecture Notes

- Section 7.3
- Section 7.4
- Section 4.2
- Appendix B.1
- Appendix B.2

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- Section 7.3
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Important Concepts

- embedding
- full strategy annotation
- in–time strategy annotation
- Kruskal's Tree Theorem
- needed
- simple monotone algebra
- simple termination
- simplification order
- subterm property
- strategy annotation