Term Rewriting

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Outline

1. **Summary of Lecture 10**
2. **Optimal Strategies**
3. **Strategy Annotations**
4. **Simple Termination**
5. **Exercises**
6. **Further Reading**

Definitions

- one–step (many–step) rewrite strategy $\mathcal{S}$ is relation $\xrightarrow{\mathcal{S}} \subseteq \to_{\mathcal{R}}^{(+)}$ such that $\mathrm{NF}(\mathcal{S}) = \mathrm{NF}(\mathcal{R})$
- rewrite strategy $\mathcal{S}$ is deterministic if $\xleftarrow{\mathcal{S}} \cdot \xrightarrow{\mathcal{S}} \subseteq =$
- rewrite strategy $\mathcal{S}$ normalizes term t if all $\mathcal{S}$–rewrite sequences starting from t are finite
- rewrite strategy $\mathcal{S}$ is normalizing if it normalizes every normalizing term
- relative rewriting: $\to_{\mathcal{S}/\mathcal{R}} = \to_{\mathcal{R}}^{*} \cdot \to_{\mathcal{S}} \cdot \to_{\mathcal{R}}^{*}$
- rewrite strategy $\mathcal{S}$ is hyper–normalizing if every $\mathcal{R}$–normalizing term is $\mathcal{S}/\mathcal{R}$–terminating
- rewrite strategy $\mathcal{S}$ is perpetual if every maximal $\mathcal{S}$–rewrite sequence starting from any non–terminating term is infinite
- rewrite strategy $\mathcal{S}$ is cofinal for TRS $\mathcal{R}$ if for every $s \to_{\mathcal{R}}^{*} t$ and every maximal sequence $s = s_0 \xrightarrow{\mathcal{S}} s_1 \xrightarrow{\mathcal{S}} s_2 \xrightarrow{\mathcal{S}} \cdots$ there exists $k \geqslant 0$ such that $t \to_{\mathcal{R}}^{*} s_k$

Lemma

cofinal strategies are normalizing

Definitions

- leftmost outermost strategy contracts leftmost of outermost redexes
- leftmost innermost strategy contracts leftmost of innermost redexes
- maximal outermost strategy contracts all outermost redexes
- maximal innermost strategy contracts all innermost redexes
- maximal strategy performs maximal multi−step

Definition

TRS is left−normal if variables do not precede function symbols in left−hand sides

Theorem

- maximal strategy is cofinal for orthogonal TRSs
- maximal outermost strategy is hyper−normalizing for orthogonal TRSs
- leftmost outermost strategy is normalizing for orthogonal left−normal TRSs

Notation

$a \xleftrightarrow[\langle l,r \rangle]{*} b$ if conversion $a \leftrightarrow^* b$ has l left $(\leftarrow)$ steps and r right $(\rightarrow)$ steps

Definition

ARS $\mathcal{A} = \langle A, \rightarrow \rangle$ has random descent if $a \rightarrow^{r-l} b$ whenever $a \xleftrightarrow[\langle l,r \rangle]{*} b \in \mathrm{NF}(\mathcal{A})$

Theorem

if ARS $\mathcal{A}$ has random descent and $a \leftrightarrow^* b$ with normal form b then a is complete and all rewrite sequences from a to b have same length

Theorem

- innermost rewriting has random descent for orthogonal TRSs
- any innermost strategy is perpetual for orthogonal TRSs

Outline

Observation

maximal outermost and maximal strategies are not optimal

Example

- TRS $\mathcal{R}$
$$0 + y \to y \qquad 0 \times y \to 0$$
$$\mathsf{s}(x) + y \to \mathsf{s}(x + y) \qquad \mathsf{s}(x) \times y \to (x \times y) + y$$
- rewrite sequence $(0 \times \mathsf{s}(0)) \times (0 + \mathsf{s}(0)) \not\twoheadrightarrow 0 \times \mathsf{s}(0) \to 0$
- redex $0 + \mathsf{s}(0)$ is not needed: $(0 \times \mathsf{s}(0)) \times \bullet \to 0 \times \bullet \to 0$

Definition (Needed Redex)

redex occurrence Δ in term $C[\Delta]$ with respect to TRS $\mathcal{R}$ is needed if $C[\bullet]$ has no normal form in $\mathcal{R} \cup \{\bullet \to \bullet\}$

Theorem

for orthogonal TRSs

- every reducible term has needed redex
- needed reduction is hyper−normalizing

Remark

for orthogonal TRSs it is undecidable whether redex is needed

Remark

decidable approximations based on tree automata techniques exist

Theorem

for left−normal orthogonal TRSs leftmost outermost redex is needed

Outline

Example

- rewrite rules

$$\mathsf{and}(x,\mathsf{T}) \xrightarrow{\alpha} x \qquad \mathsf{and}(x,\mathsf{F}) \xrightarrow{\beta} \mathsf{F} \qquad \mathsf{or}(\mathsf{T},x) \xrightarrow{\gamma} \mathsf{T} \qquad \mathsf{or}(\mathsf{F},x) \xrightarrow{\delta} x \qquad \infty \xrightarrow{\epsilon} \infty$$

- strategy annotation

$$A(\mathsf{and}) = [2,\alpha,\beta,1] \qquad A(\mathsf{or}) = [1,\gamma,\delta,2] \qquad A(\infty) = [\epsilon]$$

- evaluation

$$\mathsf{or}(\mathsf{and}(\infty,\mathsf{F}),\mathsf{or}(\mathsf{T},\infty)) \to \mathsf{or}(\mathsf{F},\mathsf{or}(\mathsf{T},\infty)) \to \mathsf{or}(\mathsf{T},\infty) \to \mathsf{T}$$

Definitions (Strategy Annotation)

- strategy annotation for function symbol f is finite list $A(f)$ containing
  - argument positions of f
  - (labels of) rewrite rules for f
- strategy annotation $A(f)$ for function symbol f is full if $A(f)$ contains all argument positions of f and all rewrite rules for f
- strategy annotation $A(f)$ for function symbol f is in–time if argument positions are listed in $A(f)$ before rewrite rules that need them
- rewrite rule $f(s_1, \ldots, s_n) \to t$ needs argument position i if
  - s_i is non–variable, or
  - s_i is variable that appears in $s_1, \ldots, s_{i-1}, s_{i+1}, \ldots, s_n$

Example

- rewrite rules

$$\mathsf{if}(\mathsf{T}, x, y) \xrightarrow{\alpha} x \qquad \mathsf{if}(\mathsf{F}, x, y) \xrightarrow{\beta} y \qquad \mathsf{if}(z, x, x) \xrightarrow{\gamma} x$$

- strategy annotations

	full	in–time
$A_1(\mathsf{if}) = [1, \alpha, \beta, 2, 3, \gamma]$	🙂	🙂
$A_2(\mathsf{if}) = [\alpha, \beta, 1, 2, 3, \gamma]$	🙂	🙁
$A_3(\mathsf{if}) = [1, \alpha, \beta, \gamma, 2, 3]$	🙂	🙁
$A_4(\mathsf{if}) = [1, \alpha, \beta]$	🙁	🙂

Lemma

if argument position i is not needed for ℓ and $t \gtrsim \ell$ then $t[u]_i \gtrsim \ell$ for any term u

Definition

$$\mathrm{redex}'_A(t, [\,]) = \bot \qquad \mathrm{redex}_A(t) = \mathrm{redex}'_A(t, A(\mathrm{root}(t)))$$

$$\mathrm{redex}'_A(t, [\ell \to r \mid L]) = \begin{cases} (\epsilon, \ell \to r) & \text{if } t \gtrsim \ell \\ \mathrm{redex}'_A(t, L) & \text{otherwise} \end{cases}$$

$$\mathrm{redex}'_A(t, [i \mid L]) = \begin{cases} (ip, \ell \to r) & \text{if } \mathrm{redex}_A(t|_i) = (p, \ell \to r) \\ \mathrm{redex}'_A(t, L) & \text{otherwise} \end{cases}$$

Theorem

$\forall$ full strategy annotation A $\forall$ term t $\quad \mathrm{redex}_A(t) = \bot \iff t$ is normal form

Example

- rewrite rules

$$\mathsf{and}(x,\mathsf{T}) \xrightarrow{\alpha} x \qquad \mathsf{and}(x,\mathsf{F}) \xrightarrow{\beta} \mathsf{F} \qquad \mathsf{or}(\mathsf{T},x) \xrightarrow{\gamma} \mathsf{T} \qquad \mathsf{or}(\mathsf{F},x) \xrightarrow{\delta} x \qquad \infty \xrightarrow{\epsilon} \infty$$

- strategy annotation

$$A(\mathsf{and}) = [2,\alpha,\beta,1] \qquad A(\mathsf{or}) = [1,\gamma,\delta,2] \qquad A(\infty) = [\epsilon]$$

- evaluation

$$\begin{aligned}
&\mathsf{redex}_A(\mathsf{or}(\mathsf{and}(\infty,\mathsf{F}),\mathsf{or}(\mathsf{T},\infty)))\\
&\quad= \mathsf{redex}'_A(\mathsf{or}(\mathsf{and}(\infty,\mathsf{F}),\mathsf{or}(\mathsf{T},\infty)),[1,\gamma,\delta,2]) = (1,\beta)\\
&\mathsf{redex}_A(\mathsf{and}(\infty,\mathsf{F}))\\
&\quad= \mathsf{redex}'_A(\mathsf{and}(\infty,\mathsf{F}),[2,\alpha,\beta,1])\\
&\quad= \mathsf{redex}'_A(\mathsf{and}(\infty,\mathsf{F}),[\alpha,\beta,1])\\
&\quad= \mathsf{redex}'_A(\mathsf{and}(\infty,\mathsf{F}),[\beta,1]) = (\epsilon,\beta)\\
&\mathsf{redex}_A(\mathsf{F}) = \bot
\end{aligned}$$

Example

- rewrite rules

$$\mathsf{if}(\mathsf{T},x,y) \xrightarrow{\alpha} x \qquad \mathsf{if}(\mathsf{F},x,y) \xrightarrow{\beta} y \qquad \mathsf{if}(z,x,x) \xrightarrow{\gamma} x$$

- strategy annotation A with $A(\mathsf{if}) = [1,\alpha,2,3,\gamma]$ is not full

$$\mathsf{redex}_A(\mathsf{if}(\mathsf{F},\mathsf{T},\mathsf{F})) = \bot \qquad \mathsf{if}(\mathsf{F},\mathsf{T},\mathsf{F}) \to \mathsf{F}$$

Definition

$s \xrightarrow{\mathcal{S}_A} t$ if $\mathsf{redex}_A(s) = (p, \ell \to r)$ and $s \to_{p|\ell\to r} t$

Corollary

$\mathcal{S}_A$ is rewrite strategy for every full strategy annotation A

Definition

$$\text{normalize}_A(t) = \text{normalize}'_A(t, A(\text{root}(t)))$$

$$\text{normalize}'_A(t, [\,]) = t$$

$$\text{normalize}'_A(t, [\ell \to r \mid L]) = \begin{cases} \text{normalize}_A(r\sigma) & \text{if } t = \ell\sigma \text{ for some substitution } \sigma \\ \text{normalize}'_A(t, L) & \text{otherwise} \end{cases}$$

$$\text{normalize}'_A(t, [i \mid L]) = \text{normalize}'_A(t[\text{normalize}_A(t|_i)]_i, L)$$

Remark

normalize and normalize$'$ are evaluated in call−by−value manner

Example

- rewrite rules

$$\mathsf{and}(x,\mathsf{T}) \xrightarrow{\alpha} x \qquad \mathsf{and}(x,\mathsf{F}) \xrightarrow{\beta} \mathsf{F} \qquad \mathsf{or}(\mathsf{T},x) \xrightarrow{\gamma} \mathsf{T} \qquad \mathsf{or}(\mathsf{F},x) \xrightarrow{\delta} x \qquad \infty \xrightarrow{\epsilon} \infty$$

- strategy annotation

$$A(\mathsf{and}) = [2,\alpha,\beta,1] \qquad A(\mathsf{or}) = [1,\gamma,\delta,2] \qquad A(\infty) = [\epsilon]$$

- evaluation

$$\begin{aligned}
&\mathsf{normalize}_A(\mathsf{or}(\mathsf{and}(\infty,\mathsf{F}),\mathsf{or}(\mathsf{T},\infty)))\\
&= \mathsf{normalize}'_A(\mathsf{or}(\mathsf{and}(\infty,\mathsf{F}),\mathsf{or}(\mathsf{T},\infty)),\ [1,\gamma,\delta,2])\\
&= \mathsf{normalize}'_A(\mathsf{or}(\mathsf{normalize}_A(\mathsf{and}(\infty,\mathsf{F})),\mathsf{or}(\mathsf{T},\infty)),\ [\gamma,\delta,2])\\
&= \ \cdots\ = \mathsf{normalize}'_A(\mathsf{or}(\mathsf{F},\mathsf{or}(\mathsf{T},\infty)),\ [\gamma,\delta,2])\\
&= \mathsf{normalize}'_A(\mathsf{or}(\mathsf{F},\mathsf{or}(\mathsf{T},\infty)),\ [\delta,2])\\
&= \mathsf{normalize}_A(\mathsf{or}(\mathsf{T},\infty)) = \mathsf{normalize}'_A(\mathsf{or}(\mathsf{T},\infty),\ [1,\gamma,\delta,2])\\
&= \mathsf{normalize}'_A(\mathsf{or}(\mathsf{normalize}_A(\mathsf{T}),\infty),\ [\gamma,\delta,2])\\
&= \mathsf{normalize}'_A(\mathsf{or}(\mathsf{T},\infty),\ [\gamma,\delta,2]) = \mathsf{normalize}_A(\mathsf{T}) = \mathsf{T}
\end{aligned}$$

Theorem

$\forall$ full and in–time strategy annotation A

1. $\mathcal{S}_A$ normalizes term $t \iff \text{normalize}_A(t)$ is defined
2. $t{\downarrow_{\mathcal{S}_A}} = \text{normalize}_A(t)$ for all normalizing terms t

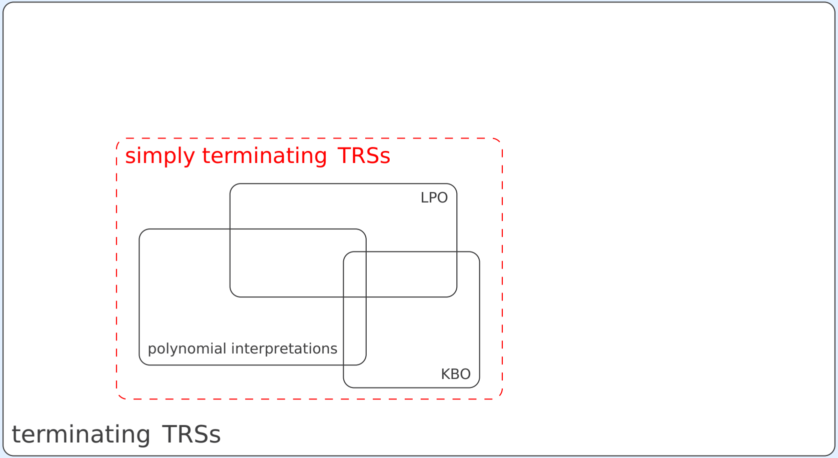
Theorem

$\forall$ full and in–time strategy annotation A $\forall$ term t

$$\text{leftmost innermost strategy normalizes } t \implies \mathcal{S}_A \text{ normalizes } t$$

Outline

Termination Methods: Realistic View



Outline

Definitions (Subterm Property)

- term relation $>$ has **subterm property** if $C[t] > t$ for all non−empty contexts C and terms t
- TRS $\mathcal{E}\mathrm{mb} = \{ f(x_1, \ldots, x_n) \to x_i \mid f$ is n-ary function symbol, $1 \leqslant i \leqslant n \}$
- $\trianglelefteq_{\mathrm{emb}} = \leftarrow^*_{\mathcal{E}\mathrm{mb}}$ (**embedding**)
- $\triangleright_{\mathrm{emb}} = \to^+_{\mathcal{E}\mathrm{mb}}$

Example

term $\mathsf{f}(\mathsf{g}(\mathsf{a}),\mathsf{b})$ is (properly) embedded in $\mathsf{f}(\mathsf{h}(\mathsf{g}(\mathsf{a}),\mathsf{a}),\mathsf{g}(\mathsf{b}))$ since

$$\mathsf{f}(\mathsf{h}(\mathsf{g}(\mathsf{a}),\mathsf{a}),\mathsf{g}(\mathsf{b})) \to \mathsf{f}(\mathsf{g}(\mathsf{a}),\mathsf{g}(\mathsf{b})) \to \mathsf{f}(\mathsf{g}(\mathsf{a}),\mathsf{b})$$

in TRS

$$\mathcal{E}\mathrm{mb}(\{\mathsf{a},\mathsf{b},\mathsf{f},\mathsf{g},\mathsf{h}\}) = \left\{ \begin{matrix} \mathsf{f}(x,y) \to x & \mathsf{g}(x) \to x & \mathsf{h}(x,y) \to x \\ \mathsf{f}(x,y) \to y & & \mathsf{h}(x,y) \to y \end{matrix} \right\}$$

Lemma

$\rhd_{\text{emb}}$ is smallest rewrite order with subterm property

Definitions (Simple Termination)

- reduction order is well–founded rewrite order
- simplification order is rewrite order with subterm property
- TRS is simply terminating if it is compatible with simplification order

Theorem

- $>_{\text{lpo}}$ is simplification order
- $>_{\text{kbo}}$ is simplification order

assumption: signatures are finite

Outline

Kruskal's Tree Theorem

$\forall$ infinite sequence of ground terms $t_1, t_2, t_3, \ldots$ $\exists$ $i < j$ such that $t_i \trianglelefteq_{\mathsf{emb}} t_j$

Lemma

simplification orders are well–founded

Proof (by contradiction)

- consider simplification order $>$ $\implies$ $\rhd_{\mathsf{emb}} \subseteq >$
- $\mathcal{V}\mathsf{ar}(t) \subseteq \mathcal{V}\mathsf{ar}(s)$ whenever $s > t$
- consider infinite sequence $t_1 > t_2 > t_3 > \cdots$
- define substitution $\tau = \{x \mapsto \mathsf{c} \mid x \in \mathcal{V}\mathsf{ar}(t_1)\}$ where c is (fresh) constant
- infinite sequence $t_1\tau > t_2\tau > t_3\tau > \cdots$ of ground terms
- $\exists$ $i < j$ such that $t_i\tau \trianglelefteq_{\mathsf{emb}} t_j\tau$ $\implies$ $t_i\tau \leqslant t_j\tau$ ↯

Corollary

simply terminating TRSs are terminating

Lemma

TRS $\mathcal{R}$ is simply terminating if and only if $\mathcal{R} \cup \mathcal{E}\mathsf{mb}$ is terminating

Examples

- SRS $\mathsf{aa} \to \mathsf{aba}$ is not simply terminating:

$$\mathsf{aa} \to \mathsf{aba} \to \mathsf{aa}$$

- TRS $\mathsf{f}(\mathsf{a},\mathsf{b},x) \to \mathsf{f}(x,x,x)$ is not simply terminating:

$$\begin{aligned} \mathsf{f}(\mathsf{a},\mathsf{b},\mathsf{f}(\mathsf{a},\mathsf{b},\mathsf{b})) &\to \mathsf{f}(\mathsf{f}(\mathsf{a},\mathsf{b},\mathsf{b}),\mathsf{f}(\mathsf{a},\mathsf{b},\mathsf{b}),\mathsf{f}(\mathsf{a},\mathsf{b},\mathsf{b})) \\ &\to \mathsf{f}(\mathsf{a},\mathsf{f}(\mathsf{a},\mathsf{b},\mathsf{b}),\mathsf{f}(\mathsf{a},\mathsf{b},\mathsf{b})) \\ &\to \mathsf{f}(\mathsf{a},\mathsf{b},\mathsf{f}(\mathsf{a},\mathsf{b},\mathsf{b})) \end{aligned}$$

Outline

Definition (Simple Monotone Algebra)

simple monotone $\mathcal{F}$–algebra $(\mathcal{A}, >)$ consists of non–empty algebra $\mathcal{A} = (A, \{f_\mathcal{A}\}_{f \in \mathcal{F}})$ with well–founded order $>$ on A such that every $f_\mathcal{A}$ is simple and weakly monotone:

$$f_\mathcal{A}(a_1, \ldots, a_i, \ldots, a_n) \geqslant a_i$$

for all $a_1, \ldots, a_n \in A$ and $i \in \{1, \ldots, n\}$

$$f_\mathcal{A}(a_1, \ldots, a_i, \ldots, a_n) \geqslant f_\mathcal{A}(a_1, \ldots, b, \ldots, a_n)$$

for all $a_1, \ldots, a_n, b \in A$ and $i \in \{1, \ldots, n\}$ with $a_i > b$

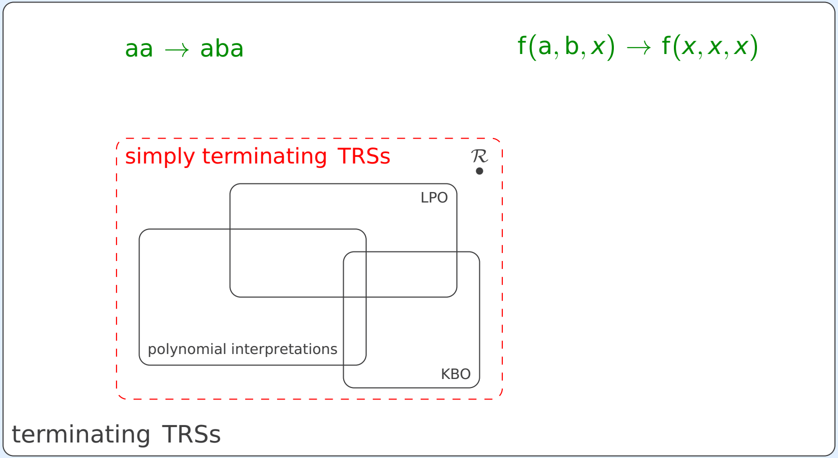
Theorem

TRS $\mathcal{R}$ is simply terminating $\iff$ $\mathcal{R}$ is compatible with simple monotone algebra

Corollary

polynomially terminating TRSs are simply terminating

Termination Methods: Realistic View



$$\mathcal{R}: \qquad \mathsf{f(a)} \to \mathsf{f(b)} \qquad \mathsf{g(b)} \to \mathsf{g(a)}$$

Outline

Homework Exercises for June 8

1. Exercise 7.12. 1
2. Exercise 7.17. 2
3. Exercise 7.19. 2
4. Exercise 4.24. 2

next lecture (June 8): online evaluation in presence $\Longrightarrow$ bring device

Outline

Lecture Notes

- Section 7.3
- Section 7.4
- Section 4.2
- Appendix B.1
- Appendix B.2

Important Concepts

- embedding
- full strategy annotation
- in−time strategy annotation
- Kruskal's Tree Theorem
- needed
- simple monotone algebra
- simple termination
- simplification order
- subterm property
- strategy annotation