universität innsbruck

26S lecture 12

Term Rewriting

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Outline

1. **Summary of Lecture 11**
2. **Dependency Pairs**
3. **Evaluation**
4. **Z Property**
5. **Exercises**
6. **Further Reading**
7. **Test**

Definitions

- strategy annotation for function symbol f is finite list $A(f)$ containing argument positions of f and (labels of) rewrite rules for f
- strategy annotation $A(f)$ for function symbol f is full if $A(f)$ contains all argument positions of f and all rewrite rules for f
- strategy annotation $A(f)$ for function symbol f is in–time if argument positions are listed in $A(f)$ before rewrite rules that need them
- rewrite rule $f(s_1, \ldots, s_n) \to t$ needs argument position i if
  - s_i is non–variable, or
  - s_i is variable that appears in $s_1, \ldots, s_{i-1}, s_{i+1}, \ldots, s_n$

Definitions

$$\textcolor{red}{\text{redex}_A(t)} = \text{redex}'_A(t, A(\text{root}(t)))$$

$$\text{redex}'_A(t, [\,]) = \bot$$

$$\text{redex}'_A(t, [\ell \to r \mid L]) = \begin{cases} (\epsilon, \ell \to r) & \text{if } t \geq \ell \\ \text{redex}'_A(t, L) & \text{otherwise} \end{cases}$$

$$\text{redex}'_A(t, [i \mid L]) = \begin{cases} (ip, \ell \to r) & \text{if } \text{redex}_A(t|_i) = (p, \ell \to r) \\ \text{redex}'_A(t, L) & \text{otherwise} \end{cases}$$

$$\textcolor{red}{\text{normalize}_A(t)} = \text{normalize}'_A(t, A(\text{root}(t)))$$

$$\text{normalize}'_A(t, [\,]) = t$$

$$\text{normalize}'_A(t, [\ell \to r \mid L]) = \begin{cases} \text{normalize}_A(r\sigma) & \text{if } t = \ell\sigma \text{ for some substitution } \sigma \\ \text{normalize}'_A(t, L) & \text{otherwise} \end{cases}$$

$$\text{normalize}'_A(t, [i \mid L]) = \text{normalize}'_A(t[t|_i]_i, L)$$

Definition

$s \xrightarrow{\mathcal{S}_A} t$ if $\text{redex}_A(s) = (p, \ell \to r)$ and $s \to_{p \mid \ell \to r} t$

Lemma

$\mathcal{S}_A$ is rewrite strategy for every full strategy annotation A

Theorem

$\forall$ full and in–time strategy annotation A $\forall$ term t

- $\mathcal{S}_A$ normalizes term $t \iff \text{normalize}_A(t)$ is defined
- $t{\downarrow_{\mathcal{S}_A}} = \text{normalize}_A(t)$ for all normalizing terms t
- leftmost innermost strategy normalizes $t \implies \mathcal{S}_A$ normalizes t

Definitions

- term relation $>$ has subterm property if $C[t] > t$ for all non–empty contexts C and terms t
- simplification order is rewrite order with subterm property
- TRS $\mathcal{R}$ is simply terminating if $\mathcal{R}$ is compatible with simplification order

Theorem

$>_{\mathsf{lpo}}$ and $>_{\mathsf{kbo}}$ are simplification orders

Theorem

for finite signatures

- simplification orders are well–founded
- simply terminating TRSs are terminating

Definitions

- TRS $\mathcal{E}\text{mb} = \{f(x_1, \ldots, x_n) \to x_i \mid f$ is n-ary function symbol, $1 \leqslant i \leqslant n\}$
- simple monotone $\mathcal{F}$–algebra $(\mathcal{A}, >)$ consists of non–empty algebra $\mathcal{A} = (A, \{f_\mathcal{A}\}_{f \in \mathcal{F}})$ with well–founded order $>$ on A such that every $f_\mathcal{A}$ is simple and weakly monotone:

$$f_\mathcal{A}(a_1, \ldots, a_i, \ldots, a_n) \geqslant a_i$$

for all $a_1, \ldots, a_n \in A$ and $i \in \{1, \ldots, n\}$

$$f_\mathcal{A}(a_1, \ldots, a_i, \ldots, a_n) \geqslant f_\mathcal{A}(a_1, \ldots, b, \ldots, a_n)$$

for all $a_1, \ldots, a_n, b \in A$ and $i \in \{1, \ldots, n\}$ with $a_i > b$

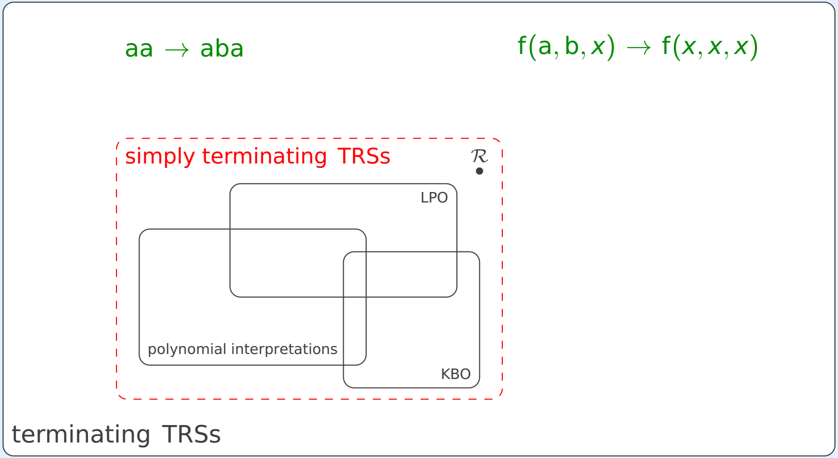
Lemma

TRS $\mathcal{R}$ is simply terminating

$\iff$ $\mathcal{R} \cup \mathcal{E}\text{mb}$ is terminating

$\iff$ $\mathcal{R}$ is compatible with simple monotone algebra

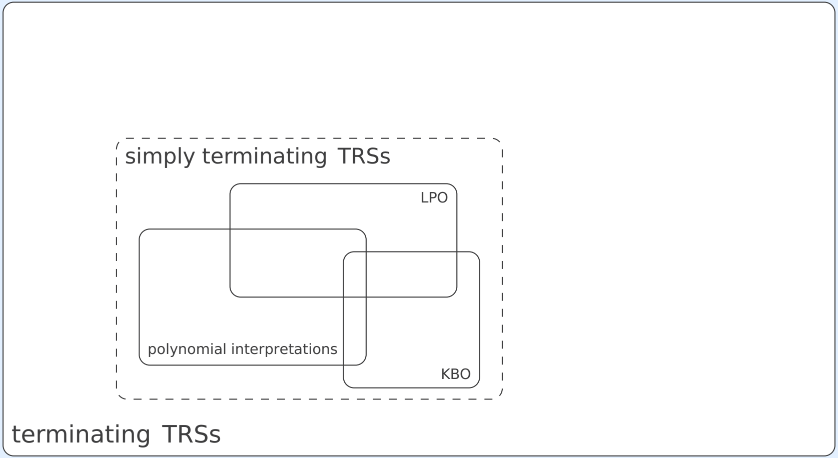
Termination Methods: Realistic View



$\mathcal{R}$: $\quad \mathsf{f(a)} \to \mathsf{f(b)} \qquad \mathsf{g(b)} \to \mathsf{g(a)}$

Outline

Termination Methods: Realistic View



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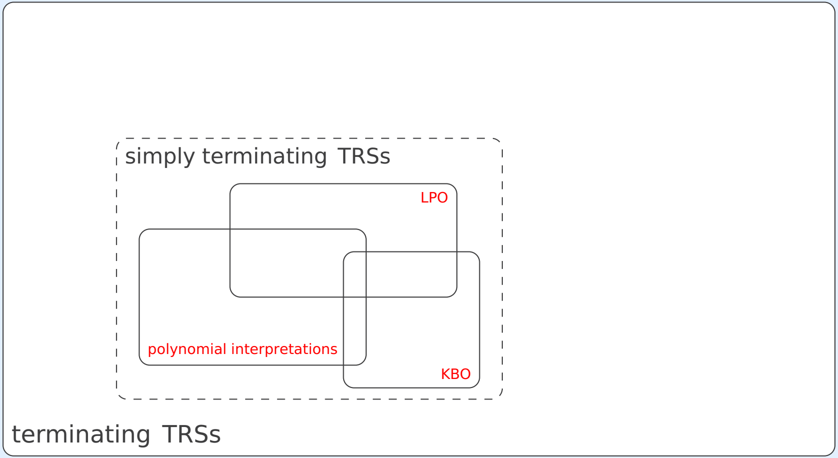

dependency pairs make direct termination methods much more powerful

Definitions (Dependency Pair)

TRS $\mathcal{R}$ over signature $\mathcal{F}$

- $\mathcal{F}^\sharp = \mathcal{F} \cup \{ f^\sharp \mid f$ is defined symbol of $\mathcal{R}\}$

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  - $u \trianglelefteq r$ and $u \ntrianglelefteq \ell$

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  - $u \trianglelefteq r$ and $u \ntrianglelefteq \ell$
  - $\mathrm{root}(u)$ is defined symbol
- DP($\mathcal{R}$) is set of all dependency pairs of $\mathcal{R}$

Example

- rewrite rules

$$
\begin{aligned}
0 - y &\to 0 \\
x - 0 &\to x \\
\mathsf{s}(x) - \mathsf{s}(y) &\to x - y
\end{aligned}
\qquad
\begin{aligned}
0 \div \mathsf{s}(y) &\to 0 \\
\mathsf{s}(x) \div \mathsf{s}(y) &\to \mathsf{s}((x - y) \div \mathsf{s}(y))
\end{aligned}
$$

Example

- rewrite rules

$$0 - y \to 0 \qquad 0 \div s(y) \to 0$$

$$x - 0 \to x \qquad s(x) \div s(y) \to s((x - y) \div s(y))$$

$$s(x) - s(y) \to x - y$$

- dependency pairs

$$s(x) -^{\sharp} s(y) \to x -^{\sharp} y$$

Example

- rewrite rules

$$0 - y \to 0 \qquad 0 \div \mathsf{s}(y) \to 0$$
$$x - 0 \to x \qquad \mathsf{s}(x) \div \mathsf{s}(y) \to \mathsf{s}((x - y) \div \mathsf{s}(y))$$
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- dependency pairs

$$\mathsf{s}(x) -^{\sharp} \mathsf{s}(y) \to x -^{\sharp} y$$
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$$\mathsf{s}(x) \div^\sharp \mathsf{s}(y) \to x -^\sharp y$$

Theorem

$\forall$ non−terminating TRS $\mathcal{R}$ $\exists$ infinite rewrite sequence

$$t_1 \xrightarrow[\mathcal{R}]{*} t_2 \xrightarrow[\mathsf{DP}(\mathcal{R})]{\epsilon} t_3 \xrightarrow[\mathcal{R}]{*} t_4 \xrightarrow[\mathsf{DP}(\mathcal{R})]{\epsilon} \cdots$$

such that t_1 is terminating with respect to $\mathcal{R}$

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reduction pair $(>, \gtrsim)$ consists of well−founded order $>$ and preorder $\gtrsim$ such that

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TRS $\mathcal{R}$ is terminating $\iff$ $\mathsf{DP}(\mathcal{R}) \subseteq >$ and $\mathcal{R} \subseteq \gtrsim$ for some reduction pair $(>, \gtrsim)$

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$$t_1 \gtrsim t_2 > t_3 \gtrsim t_4 > \cdots$$

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Example

- rewrite rules

$$0 - y \to 0 \qquad 0 \div \mathsf{s}(y) \to 0$$
$$x - 0 \to x \qquad \mathsf{s}(x) \div \mathsf{s}(y) \to \mathsf{s}((x - y) \div \mathsf{s}(y))$$
$$\mathsf{s}(x) - \mathsf{s}(y) \to x - y$$

- dependency pairs

$$\mathsf{s}(x) -^\sharp \mathsf{s}(y) \to x -^\sharp y$$
$$\mathsf{s}(x) \div^\sharp \mathsf{s}(y) \to (x - y) \div^\sharp \mathsf{s}(y)$$
$$\mathsf{s}(x) \div^\sharp \mathsf{s}(y) \to x -^\sharp y$$

- polynomial interpretation

$$0_{\mathbb{N}} = 0 \qquad \mathsf{s}_{\mathbb{N}}(x) = x + 1 \qquad -_{\mathbb{N}}(x, y) = \div_{\mathbb{N}}(x, y) = -^\sharp_{\mathbb{N}}(x, y) = \div^\sharp_{\mathbb{N}}(x, y) = x$$

Example

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Definitions (Well–Founded Weakly Monotone Algebra)

- well–founded weakly monotone $\mathcal{F}$–algebra (WFWMA) $(\mathcal{A}, >, \gtrsim)$ consists of non–empty algebra $\mathcal{A} = (A, \{f_\mathcal{A}\}_{f \in \mathcal{F}})$ with well–founded order $>$ and preorder $\gtrsim$ on A

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  2. $> \cdot \gtrsim \; \subseteq \; >$ or $\gtrsim \cdot > \; \subseteq \; >$
- relation $>_{\mathcal{A}}$ on terms: $s >_{\mathcal{A}} t$ if $[\alpha]_{\mathcal{A}}(s) > [\alpha]_{\mathcal{A}}(t)$ for all assignments α

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  2. $> \cdot \gtrsim \, \subseteq \, >$ or $\gtrsim \cdot > \, \subseteq \, >$
- relation $>_\mathcal{A}$ on terms: $s >_\mathcal{A} t$ if $[\alpha]_\mathcal{A}(s) > [\alpha]_\mathcal{A}(t)$ for all assignments α
- relation $\gtrsim_\mathcal{A}$ on terms: $s \gtrsim_\mathcal{A} t$ if $[\alpha]_\mathcal{A}(s) \gtrsim [\alpha]_\mathcal{A}(t)$ for all assignments α

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Lemmata

- $(>_{\mathcal{A}}, \gtrsim_{\mathcal{A}})$ is reduction pair for every WFWMA $(\mathcal{A}, >, \gtrsim)$

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Lemmata

- $(>_{\mathcal{A}}, \gtrsim_{\mathcal{A}})$ is reduction pair for every WFWMA $(\mathcal{A}, >, \gtrsim)$
- $(\mathcal{A}, >, \geqslant)$ is WFWMA for every well–founded monotone algebra $(\mathcal{A}, >)$

Example ①

- rewrite rules

$$\mathsf{append}(\mathsf{nil}, z) \to z \qquad \mathsf{append}(x : y, z) \to x : \mathsf{append}(y, z)$$

$$\mathsf{reverse}(\mathsf{nil}) \to \mathsf{nil} \qquad \mathsf{reverse}(x : y) \to \mathsf{append}(\mathsf{reverse}(y), x : \mathsf{nil})$$

$$\mathsf{shuffle}(\mathsf{nil}) \to \mathsf{nil} \qquad \mathsf{shuffle}(x : y) \to x : \mathsf{shuffle}(\mathsf{reverse}(y))$$

Example ①

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- dependency pairs

$$\begin{aligned} \mathsf{append}^\sharp(x : y, z) &\to \mathsf{append}^\sharp(y, z) & & \\ \mathsf{reverse}^\sharp(x : y) &\to \mathsf{reverse}^\sharp(y) & \mathsf{reverse}^\sharp(x : y) &\to \mathsf{append}^\sharp(\mathsf{reverse}(y), x : \mathsf{nil}) \\ \mathsf{shuffle}^\sharp(x : y) &\to \mathsf{reverse}^\sharp(y) & \mathsf{shuffle}^\sharp(x : y) &\to \mathsf{shuffle}^\sharp(\mathsf{reverse}(y)) \end{aligned}$$

Example 1

- rewrite rules

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- polynomial interpretation

$$\mathsf{reverse}_{\mathbb{N}}(x) = \mathsf{shuffle}_{\mathbb{N}}(x) = \mathsf{append}^\sharp_{\mathbb{N}}(x, y) = x \qquad \mathsf{append}_{\mathbb{N}}(x, y) = x + y$$

$$\mathsf{nil}_{\mathbb{N}} = 0 \qquad \mathsf{reverse}^\sharp_{\mathbb{N}}(x) = \mathsf{shuffle}^\sharp_{\mathbb{N}}(x) = x + 1 \qquad :_{\mathbb{N}}(x, y) = x + y + 1$$

Example ❷

- rewrite rules

$$
\begin{aligned}
\mathsf{primes} &\to \mathsf{sieve}(\mathsf{from}(\mathsf{s}(\mathsf{s}(0)))) & \mathsf{sieve}(0 : y) &\to \mathsf{sieve}(y) \\
\mathsf{from}(x) &\to x : \mathsf{from}(\mathsf{s}(x)) & \mathsf{sieve}(\mathsf{s}(x) : y) &\to \mathsf{s}(x) : \mathsf{sieve}(\mathsf{filter}(y, x, x)) \\
\mathsf{head}(x : y) &\to x & \mathsf{filter}(y : z, 0, w) &\to 0 : \mathsf{filter}(z, w, w) \\
\mathsf{tail}(x : y) &\to y & \mathsf{filter}(y : z, \mathsf{s}(x), w) &\to y : \mathsf{filter}(z, x, w)
\end{aligned}
$$

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Example ❷

- rewrite rules

$$\mathsf{primes} \to \mathsf{sieve}(\mathsf{from}(\mathsf{s}(\mathsf{s}(0))))$$
$$\mathsf{head}(x : y) \to x$$
$$\mathsf{tail}(x : y) \to y$$
$$\mathsf{sieve}(0 : y) \to \mathsf{sieve}(y)$$
$$\mathsf{sieve}(\mathsf{s}(x) : y) \to \mathsf{s}(x) : \mathsf{sieve}(\mathsf{filter}(y, x, x))$$
$$\mathsf{filter}(y : z, 0, w) \to 0 : \mathsf{filter}(z, w, w)$$
$$\mathsf{filter}(y : z, \mathsf{s}(x), w) \to y : \mathsf{filter}(z, x, w)$$

- dependency pairs

$$\mathsf{primes}^\sharp \to \mathsf{sieve}^\sharp(\mathsf{from}(\mathsf{s}(\mathsf{s}(0))))$$
$$\mathsf{sieve}^\sharp(\mathsf{s}(x) : y) \to \mathsf{sieve}^\sharp(\mathsf{filter}(y, x, x))$$
$$\mathsf{sieve}^\sharp(\mathsf{s}(x) : y) \to \mathsf{filter}^\sharp(y, x, x)$$
$$\mathsf{sieve}^\sharp(0 : y) \to \mathsf{sieve}^\sharp(y)$$
$$\mathsf{filter}^\sharp(y : z, 0, w) \to \mathsf{filter}^\sharp(z, w, w)$$
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$$\mathsf{head}(x : y) \to x \qquad \mathsf{filter}(y : z, 0, w) \to 0 : \mathsf{filter}(z, w, w)$$
$$\mathsf{tail}(x : y) \to y \qquad \mathsf{filter}(y : z, \mathsf{s}(x), w) \to y : \mathsf{filter}(z, x, w)$$

- dependency pairs

$$\mathsf{primes}^\sharp \to \mathsf{sieve}^\sharp(\mathsf{from}(\mathsf{s}(\mathsf{s}(0)))) \qquad \mathsf{sieve}^\sharp(0 : y) \to \mathsf{sieve}^\sharp(y)$$
$$\mathsf{sieve}^\sharp(\mathsf{s}(x) : y) \to \mathsf{sieve}^\sharp(\mathsf{filter}(y, x, x)) \qquad \mathsf{filter}^\sharp(y : z, 0, w) \to \mathsf{filter}^\sharp(z, w, w)$$
$$\mathsf{sieve}^\sharp(\mathsf{s}(x) : y) \to \mathsf{filter}^\sharp(y, x, x) \qquad \mathsf{filter}^\sharp(y : z, \mathsf{s}(x), w) \to \mathsf{filter}^\sharp(z, x, w)$$

- polynomial interpretation

$$0_{\mathbb{N}} = 0 \quad \mathsf{s}_{\mathbb{N}}(x) = x + 1 \quad \mathsf{from}_{\mathbb{N}}(x) = x \quad :_{\mathbb{N}}(x, y) = x + y + 1 \quad \mathsf{primes}_{\mathbb{N}} = 2 \quad \mathsf{primes}^\sharp_{\mathbb{N}} = 3$$
$$\mathsf{head}_{\mathbb{N}}(x) = \mathsf{tail}_{\mathbb{N}}(x) = x \quad \mathsf{filter}_{\mathbb{N}}(x, y, z) = \mathsf{filter}^\sharp_{\mathbb{N}}(x, y, z) = \mathsf{sieve}_{\mathbb{N}}(x) = \mathsf{sieve}^\sharp_{\mathbb{N}}(x) = x$$

Outline

Online Evaluation in Presence

`https://lv-analyse.uibk.ac.at/evasys/public/online/index`

Outline

Definition

ARS $\mathcal{A} = \langle A, \to \rangle$ has Z property if

$$a \to b \quad \Longrightarrow \quad b \to^* \bullet(a) \to^* \bullet(b)$$

for some function $\bullet$ on A

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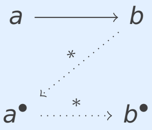

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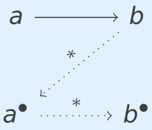

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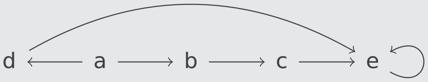

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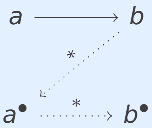

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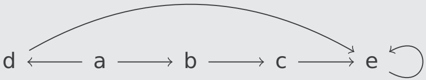

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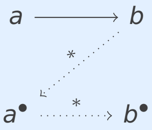

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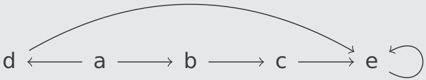

- define $a^\bullet = b^\bullet = c^\bullet = d^\bullet = e^\bullet = e$
- every element rewrites to $e \implies$ Z property is trivially satisfied

Lemma (Monotonicity)

$a \to^* b \implies a^\bullet \to^* b^\bullet$ for every ARS $\langle A, \to \rangle$ with Z property for $\bullet$

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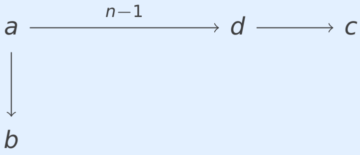

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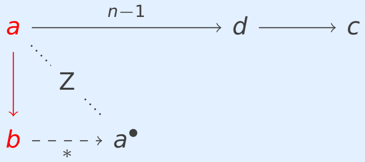

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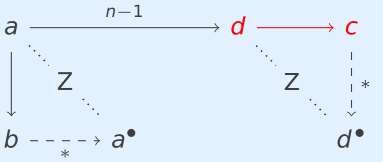

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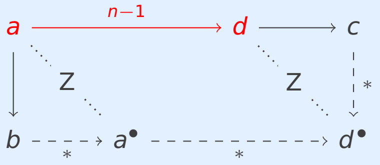

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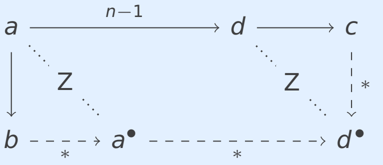

- $\leftarrow \cdot \to^* \subseteq \downarrow$ (semi–confluence)

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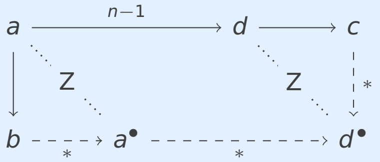

- $\leftarrow \cdot \to^* \subseteq \downarrow$ (semi–confluence) $\implies$ ${}^*\!\leftarrow \cdot \to^* \subseteq \downarrow$ (confluence)

Application: Confluence of Combinatory Logic

how to find suitable bullet function $\bullet$ for CL ?

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Definition

functions $\diamond$ and $\star$ on CL–terms:

$$t^\diamond = \begin{cases} u^\diamond \star v^\diamond & \text{if } t = u\,v \\ t & \text{otherwise} \end{cases}$$

$$s \star t = \begin{cases} t & \text{if } s = \mathsf{I} \\ u & \text{if } s = \mathsf{K}\,u \\ u\,t\,(v\,t) & \text{if } s = \mathsf{S}\,u\,v \\ s\,t & \text{otherwise} \end{cases}$$

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Example (cont'd)

$(\mathsf{SK}(\mathsf{IK})(\mathsf{IIS}))^{\diamond\diamond}$ is common reduct of IIS and $\mathsf{SKK}(\mathsf{IS})$

$$\mathsf{IIS} \leftarrow \mathsf{K}(\mathsf{IIS})(\mathsf{IK}(\mathsf{IIS})) \leftarrow \mathsf{SK}(\mathsf{IK})(\mathsf{IIS}) \rightarrow \mathsf{SKK}(\mathsf{IIS}) \rightarrow \mathsf{SKK}(\mathsf{IS})$$

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Theorem

CL has Z property for $\diamond$

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Proof (sketch)

for all CL–terms s, t, u, v

① $s\,t \rightarrow^{=} s \star t$

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① $s\,t \rightarrow^{=} s \star t$

② $t \rightarrow^{*} t^{\diamond}$

Example (cont'd)

$(\mathsf{SK}(\mathsf{IK})(\mathsf{IIS}))^{\diamond\diamond}$ is common reduct of IIS and $\mathsf{SKK}(\mathsf{IS})$

$$\mathsf{IIS} \leftarrow \mathsf{K}(\mathsf{IIS})(\mathsf{IK}(\mathsf{IIS})) \leftarrow \mathsf{SK}(\mathsf{IK})(\mathsf{IIS}) \to \mathsf{SKK}(\mathsf{IIS}) \to \mathsf{SKK}(\mathsf{IS})$$

Theorem

CL has Z property for $\diamond$

Proof (sketch)

for all CL–terms s, t, u, v

① $s\,t \to^{=} s \star t$

③ $s \to^* t$ and $u \to^* v \implies s \star u \to^* t \star v$

② $t \to^* t^\diamond$

Example (cont'd)

$(\mathsf{SK}(\mathsf{IK})(\mathsf{IIS}))^{\diamond\diamond}$ is common reduct of IIS and $\mathsf{SKK}(\mathsf{IS})$

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③ $s \to^{*} t$ and $u \to^{*} v \implies s \star u \to^{*} t \star v$

④ $s \to^{=} t \implies t \to^{*} s^{\diamond} \to^{*} t^{\diamond}$

Example (cont'd)

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CL has Z property for $\diamond$

Proof (sketch)

for all CL–terms s, t, u, v

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④ $s \rightarrow^{=} t \implies t \rightarrow^{*} s^{\diamond} \rightarrow^{*} t^{\diamond}$

Remark

method extends to all orthogonal TRSs

Definition

strategy $\mathcal{S}_\bullet$ for ARS $\mathcal{A}$ with Z property for $\bullet$: $a \xrightarrow{\bullet} b$ if $a \notin \mathrm{NF}(\mathcal{A})$ and $b = a^\bullet$

Definition

strategy $\mathcal{S}_\bullet$ for ARS $\mathcal{A}$ with Z property for $\bullet$: $a \twoheadrightarrow_\bullet b$ if $a \notin \mathsf{NF}(\mathcal{A})$ and $b = a^\bullet$

Theorem

$\mathcal{S}_\bullet$ is normalizing strategy for every ARS with Z property for $\bullet$

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strategy $\mathcal{S}_\bullet$ for ARS $\mathcal{A}$ with Z property for $\bullet$: $a \mathrel{\rightarrow\!\!\!\!\bullet\!\!\rightarrow} b$ if $a \notin \mathrm{NF}(\mathcal{A})$ and $b = a^\bullet$

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$\mathcal{S}_\bullet$ is normalizing strategy for every ARS with Z property for $\bullet$

Proof

① $a \to^n b$ and $n > 0 \implies b \to^* \bullet^n(a)$

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Proof

① $a \to^n b$ and $n > 0 \implies b \to^* \bullet^n(a)$ by induction on n:

$a \to c \to^{n-1} b$

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strategy $\mathcal{S}_\bullet$ for ARS $\mathcal{A}$ with Z property for $\bullet$: $a \twoheadrightarrow_\bullet b$ if $a \notin \mathsf{NF}(\mathcal{A})$ and $b = a^\bullet$

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- $n = 1 \implies b = c$

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strategy $\mathcal{S}_\bullet$ for ARS $\mathcal{A}$ with Z property for $\bullet$: $a \to_\bullet b$ if $a \notin \mathrm{NF}(\mathcal{A})$ and $b = a^\bullet$

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$a \to c \to^{n-1} b \implies c \to^* \bullet(a)$ (Z property)

- $n = 1 \implies b = c$
- $n > 1 \implies b \to^* \bullet^{n-1}(c)$ (induction hypothesis)

  $\bullet^{n-1}(c) \to^* \bullet^n(a)$ ($n-1$ applications of monotonicity)

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strategy $\mathcal{S}_\bullet$ for ARS $\mathcal{A}$ with Z property for $\bullet$: $a \xrightarrow{\bullet} b$ if $a \notin \mathrm{NF}(\mathcal{A})$ and $b = a^\bullet$

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Proof

① $a \to^n b$ and $n > 0 \implies b \to^* \bullet^n(a)$ by induction on n:

$$a \to c \to^{n-1} b \implies c \to^* \bullet(a) \qquad \text{(Z property)}$$

- $n = 1 \implies b = c$
- $n > 1 \implies b \to^* \bullet^{n-1}(c)$ (induction hypothesis)

$$\bullet^{n-1}(c) \to^* \bullet^n(a) \implies b \to^* \bullet^n(a) \qquad (n-1 \text{ applications of monotonicity})$$

Theorem

$\mathcal{S}_\bullet$ is normalizing strategy for every ARS with Z property for $\bullet$

Proof (cont'd)

① $a \to^n b$ and $n > 0 \implies b \to^* \bullet^n(a)$

② $a \xrightarrow{\bullet}^{\leqslant n} \bullet^n(a)$ for all $n \geqslant 0$

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Proof (cont'd)

① $a \to^n b$ and $n > 0 \implies b \to^* \bullet^n(a)$

② $a \xrightarrow{\bullet}^{\leqslant n} \bullet^n(a)$ for all $n \geqslant 0$ by induction on n

- $n = 0 \implies a = \bullet^n(a)$

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$\mathcal{S}_\bullet$ is normalizing strategy for every ARS with Z property for $\bullet$

Proof (cont'd)

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② $a \xrightarrow{\bullet}^{\leqslant n} \bullet^n(a)$ for all $n \geqslant 0$ by induction on n

- $n = 0 \implies a = \bullet^n(a)$
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Proof (cont'd)

① $a \to^n b$ and $n > 0 \implies b \to^* \bullet^n(a)$

② $a \twoheadrightarrow_\bullet^{\leqslant n} \bullet^n(a)$ for all $n \geqslant 0$ by induction on n

- $n = 0 \implies a = \bullet^n(a)$
- $n > 0 \implies a \twoheadrightarrow_\bullet^{\leqslant n-1} \bullet^{n-1}(a) \twoheadrightarrow_\bullet^{=} \bullet^{n-1}(a)^\bullet$ (induction hypothesis)

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③ $a \to^n b$ with $n > 0$ and $b \in \mathrm{NF}(\to)$

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Proof (cont'd)

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③ $a \to^n b$ with $n > 0$ and $b \in \mathrm{NF}(\to)$

$a \xrightarrow{\bullet}^{\leqslant n} \bullet^n(a) \;{}^*\!\!\leftarrow b$

Theorem

$\mathcal{S}_\bullet$ is normalizing strategy for every ARS with Z property for $\bullet$

Proof (cont'd)

① $a \to^n b$ and $n > 0 \implies b \to^* \bullet^n(a)$

② $a \mathrel{\multimap\!\!\to}^{\leqslant n} \bullet^n(a)$ for all $n \geqslant 0$ by induction on n

- $n = 0 \implies a = \bullet^n(a)$
- $n > 0 \implies a \mathrel{\multimap\!\!\to}^{\leqslant n-1} \bullet^{n-1}(a) \mathrel{\multimap\!\!\to}^{=} \bullet^{n-1}(a)^\bullet = \bullet^n(a)$ (induction hypothesis)

③ $a \to^n b$ with $n > 0$ and $b \in \mathsf{NF}(\to)$

$a \mathrel{\multimap\!\!\to}^{\leqslant n} \bullet^n(a) \; {}^*\!\!\leftarrow b \implies a \mathrel{\multimap\!\!\to}^{\leqslant n} b$

Theorem

$\mathcal{S}_\bullet$ is normalizing strategy for every ARS with Z property for $\bullet$

Proof (cont'd)

① $a \to^n b$ and $n > 0 \implies b \to^* \bullet^n(a)$

② $a \mathrel{\to\!\!\!\!\bullet\!\!\!\to}^{\leqslant n} \bullet^n(a)$ for all $n \geqslant 0$ by induction on n

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③ $a \to^n b$ with $n > 0$ and $b \in \mathrm{NF}(\to)$

$a \mathrel{\to\!\!\!\!\bullet\!\!\!\to}^{\leqslant n} \bullet^n(a) \; {}^*\!\!\leftarrow b \implies a \mathrel{\to\!\!\!\!\bullet\!\!\!\to}^{\leqslant n} b \implies \mathcal{S}_\bullet$ is normalizing

Theorem

$\mathcal{S}_\bullet$ is hyper−normalizing strategy for every ARS with Z property for $\bullet$

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$\mathcal{S}_\bullet$ is hyper−normalizing strategy for every ARS with Z property for $\bullet$

Proof (sketch)

① $\to^* \cdot \xrightarrow{\bullet} \subseteq \xrightarrow{\bullet} \cdot \to^*$ ($\xrightarrow{\bullet}$ commutes over $\to^*$)

Theorem

$\mathcal{S}_\bullet$ is hyper−normalizing strategy for every ARS with Z property for $\bullet$

Proof (sketch)

① $\to^* \cdot \overset{\bullet}{\to} \subseteq \overset{\bullet}{\to} \cdot \to^*$ ($\overset{\bullet}{\to}$ commutes over $\to^*$)

- suppose $a \to^* b \overset{\bullet}{\to} c$

Theorem

$\mathcal{S}_\bullet$ is hyper−normalizing strategy for every ARS with Z property for $\bullet$

Proof (sketch)

① $\to^* \cdot \xrightarrow{\bullet} \subseteq \xrightarrow{\bullet} \cdot \to^*$ ($\xrightarrow{\bullet}$ commutes over $\to^*$)

- suppose $a \to^* b \xrightarrow{\bullet} c$
- $b \notin \mathrm{NF}$

Theorem

$\mathcal{S}_\bullet$ is hyper-normalizing strategy for every ARS with Z property for $\bullet$

Proof (sketch)

① $\to^* \cdot \twoheadrightarrow_\bullet \subseteq \twoheadrightarrow_\bullet \cdot \to^*$ ($\twoheadrightarrow_\bullet$ commutes over $\to^*$)

- suppose $a \to^* b \twoheadrightarrow_\bullet c$
- $b \notin \mathrm{NF} \implies a \notin \mathrm{NF}$

Theorem

$\mathcal{S}_\bullet$ is hyper–normalizing strategy for every ARS with Z property for $\bullet$

Proof (sketch)

① $\to^* \cdot \xrightarrow{\bullet} \subseteq \xrightarrow{\bullet} \cdot \to^*$ ($\xrightarrow{\bullet}$ commutes over $\to^*$)

- suppose $a \to^* b \xrightarrow{\bullet} c$
- $b \notin \mathrm{NF} \implies a \notin \mathrm{NF} \implies a \xrightarrow{\bullet} \bullet(a)$

Theorem

$\mathcal{S}_\bullet$ is hyper–normalizing strategy for every ARS with Z property for $\bullet$

Proof (sketch)

① $\to^* \cdot \xrightarrow{\bullet} \subseteq \xrightarrow{\bullet} \cdot \to^*$ ($\xrightarrow{\bullet}$ commutes over $\to^*$)

- suppose $a \to^* b \xrightarrow{\bullet} c$
- $b \notin \mathrm{NF} \implies a \notin \mathrm{NF} \implies a \xrightarrow{\bullet} \bullet(a) \to^* \bullet(b)$ (monotonicity)

Theorem

$\mathcal{S}_\bullet$ is hyper–normalizing strategy for every ARS with Z property for $\bullet$

Proof (sketch)

① $\to^* \cdot \xrightarrow{\bullet} \subseteq \xrightarrow{\bullet} \cdot \to^*$ ($\xrightarrow{\bullet}$ commutes over $\to^*$)

- suppose $a \to^* b \xrightarrow{\bullet} c$
- $b \notin \mathsf{NF} \implies a \notin \mathsf{NF} \implies a \xrightarrow{\bullet} \bullet(a) \to^* \bullet(b) = c$ (monotonicity)

② $\mathcal{S}_\bullet$ is normalizing strategy

Theorem

$\mathcal{S}_\bullet$ is hyper–normalizing strategy for every ARS with Z property for $\bullet$

Proof (sketch)

① $\to^* \cdot \xrightarrow{\bullet} \subseteq \xrightarrow{\bullet} \cdot \to^*$ ($\xrightarrow{\bullet}$ commutes over $\to^*$)

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- $b \notin \mathsf{NF} \implies a \notin \mathsf{NF} \implies a \xrightarrow{\bullet} \bullet(a) \to^* \bullet(b) = c$ (monotonicity)

② $\mathcal{S}_\bullet$ is normalizing strategy $\implies$ $\mathcal{S}_\bullet$ is hyper–normalizing strategy

Outline

Homework Exercises for June 15

1. Exercise 4.49(a,c,d). ③
2. Exercise 3.34. ②
3. Exercise 6.5. ②
4. Exercise 6.24. ☆☆☆

Outline

Lecture Notes

- Section 4.5 (until Example 4.5.15)
- Section 1.2
- Section 1.5
- Section 3.4

Lecture Notes

- Section 4.5 (until Example 4.5.15)
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Important Concepts

- $\mathcal{S}_\bullet$
- $s \star t$
- $t^\diamond$
- dependency pair
- reduction pair
- well–founded weakly monotone algebra
- Z property

Outline

Grading

$$\text{score} = \min\left(\max\left(\tfrac{50}{69}(E+P) + \tfrac{1}{3}T + B,\ T+B\right),\ 100\right)$$

- E: points for solved exercises (at most 84)
- P: points for presentation of solutions (at most 8)
- T: points for test (at most 100)
- B: points for bonus exercises (at most 20)

$$\text{grade} = \text{score} \in \ (-50) \to 5 \quad [50-63) \to 4 \quad [63-75) \to 3 \quad [75-88) \to 2 \quad [88-) \to 1$$

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- (optional) test on June 22

Grading

$$\text{score} = \min\left(\max\left(\tfrac{50}{69}(E+P) + \tfrac{1}{3}T + B,\ T + B\right),\ 100\right)$$

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- (optional) test on June 22
- closed book, 15:30 – 18:00, HS 10

Grading

$$\text{score} \;=\; \min\left(\max\left(\tfrac{50}{69}(E+P) + \tfrac{1}{3}T + B,\; T+B\right),\; 100\right)$$

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- earlier tests: 20W 22W 23W