



Term Rewriting

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Outline

- 1. Summary of Lecture 11**
- 2. Dependency Pairs**
- 3. Evaluation**
- 4. Z Property**
- 5. Exercises**
- 6. Further Reading**
- 7. Test**

Definitions

- ▶ **strategy annotation** for function symbol f is finite list $A(f)$ containing argument positions of f and (labels of) rewrite rules for f
- ▶ strategy annotation $A(f)$ for function symbol f is **full** if $A(f)$ contains all argument positions of f and all rewrite rules for f
- ▶ strategy annotation $A(f)$ for function symbol f is **in-time** if argument positions are listed in $A(f)$ before rewrite rules that **need** them
- ▶ rewrite rule $f(s_1, \dots, s_n) \rightarrow t$ **needs** argument position i if
 - ▶ s_i is non-variable, or
 - ▶ s_i is variable that appears in $s_1, \dots, s_{i-1}, s_{i+1}, \dots, s_n$

$$\text{redex}_A(t) = \text{redex}'_A(t, A(\text{root}(t)))$$

$$\text{redex}'_A(t, []) = \perp$$

$$\text{redex}'_A(t, [\ell \rightarrow r \mid L]) = \begin{cases} (\epsilon, \ell \rightarrow r) & \text{if } t \geq \ell \\ \text{redex}'_A(t, L) & \text{otherwise} \end{cases}$$

$$\text{redex}'_A(t, [i \mid L]) = \begin{cases} (ip, \ell \rightarrow r) & \text{if } \text{redex}_A(t|_i) = (p, \ell \rightarrow r) \\ \text{redex}'_A(t, L) & \text{otherwise} \end{cases}$$

$$\text{normalize}_A(t) = \text{normalize}'_A(t, A(\text{root}(t)))$$

$$\text{normalize}'_A(t, []) = t$$

$$\text{normalize}'_A(t, [\ell \rightarrow r \mid L]) = \begin{cases} \text{normalize}_A(r\sigma) & \text{if } t = \ell\sigma \text{ for some substitution } \sigma \\ \text{normalize}'_A(t, L) & \text{otherwise} \end{cases}$$

$$\text{normalize}'_A(t, [i \mid L]) = \text{normalize}'_A(t[t|_i]_i, L)$$

Definition

$s \xrightarrow{\mathcal{S}_A} t$ if $\text{redex}_A(s) = (p, \ell \rightarrow r)$ and $s \rightarrow_{p|\ell \rightarrow r} t$

Lemma

\mathcal{S}_A is rewrite strategy for every full strategy annotation A

Theorem

\forall full and in-time strategy annotation $A \quad \forall$ term t

- ▶ \mathcal{S}_A normalizes term $t \iff \text{normalize}_A(t)$ is defined
- ▶ $t \downarrow_{\mathcal{S}_A} = \text{normalize}_A(t)$ for all normalizing terms t
- ▶ leftmost innermost strategy normalizes $t \implies \mathcal{S}_A$ normalizes t

Definitions

- ▶ term relation $>$ has **subterm property** if $C[t] > t$ for all non-empty contexts C and terms t
- ▶ **simplification order** is rewrite order with subterm property
- ▶ TRS \mathcal{R} is **simply terminating** if \mathcal{R} is compatible with simplification order

Theorem

$>_{\text{lpo}}$ and $>_{\text{kbo}}$ are simplification orders

Theorem

for finite signatures

- ▶ simplification orders are **well-founded**
- ▶ simply terminating TRSs are **terminating**

Definitions

- ▶ TRS $\mathcal{E}mb = \{f(x_1, \dots, x_n) \rightarrow x_i \mid f \text{ is } n\text{-ary function symbol, } 1 \leq i \leq n\}$
- ▶ **simple monotone** \mathcal{F} -algebra $(\mathcal{A}, >)$ consists of non-empty algebra $\mathcal{A} = (A, \{f_{\mathcal{A}}\}_{f \in \mathcal{F}})$ with well-founded order $>$ on A such that every $f_{\mathcal{A}}$ is simple and weakly monotone:

$$f_{\mathcal{A}}(a_1, \dots, a_i, \dots, a_n) \geq a_i$$

for all $a_1, \dots, a_n \in A$ and $i \in \{1, \dots, n\}$

$$f_{\mathcal{A}}(a_1, \dots, a_i, \dots, a_n) \geq f_{\mathcal{A}}(a_1, \dots, b, \dots, a_n)$$

for all $a_1, \dots, a_n, b \in A$ and $i \in \{1, \dots, n\}$ with $a_i > b$

Lemma

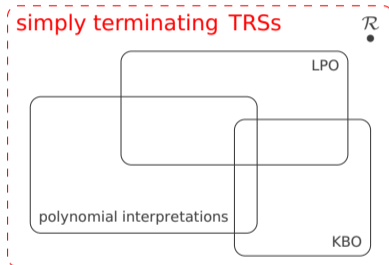
TRS \mathcal{R} is simply terminating

$\iff \mathcal{R} \cup \mathcal{E}mb$ is terminating

$\iff \mathcal{R}$ is compatible with simple monotone algebra

$aa \rightarrow aba$

$f(a, b, x) \rightarrow f(x, x, x)$



$\mathcal{R}: \quad f(a) \rightarrow f(b) \quad g(b) \rightarrow g(a)$

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1. Summary of Lecture 11

2. Dependency Pairs

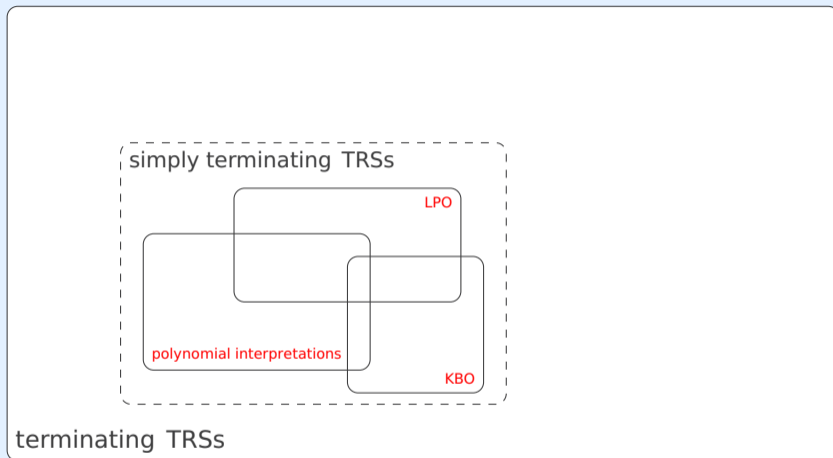
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dependency pairs make direct termination methods much more powerful

Definitions (Dependency Pair)

TRS \mathcal{R} over signature \mathcal{F}

- ▶ $\mathcal{F}^\# = \mathcal{F} \cup \{f^\# \mid f \text{ is defined symbol of } \mathcal{R}\}$
- ▶ if $t = f(t_1, \dots, t_n)$ with f defined then $t^\# = f^\#(t_1, \dots, t_n)$
- ▶ **dependency pair** $l^\# \rightarrow u^\#$ of rewrite rule $l \rightarrow r$ satisfies
 - ▶ $u \sqsubseteq r$ and $u \not\sqsubseteq l$
 - ▶ $\text{root}(u)$ is defined symbol
- ▶ **DP(\mathcal{R})** is set of all dependency pairs of \mathcal{R}

Example

► rewrite rules

$$0 - y \rightarrow 0$$

$$x - 0 \rightarrow x$$

$$s(x) - s(y) \rightarrow x - y$$

$$0 \div s(y) \rightarrow 0$$

$$s(x) \div s(y) \rightarrow s((x - y) \div s(y))$$

► dependency pairs

$$s(x) -^{\#} s(y) \rightarrow x -^{\#} y$$

$$s(x) \div^{\#} s(y) \rightarrow (x - y) \div^{\#} s(y)$$

$$s(x) \div^{\#} s(y) \rightarrow x -^{\#} y$$

► polynomial interpretation

$$0_{\mathbb{N}} = 0 \quad s_{\mathbb{N}}(x) = x + 1 \quad -_{\mathbb{N}}(x, y) = \div_{\mathbb{N}}(x, y) = -^{\#}_{\mathbb{N}}(x, y) = \div^{\#}_{\mathbb{N}}(x, y) = x \quad ?$$

Theorem

\forall non-terminating TRS \mathcal{R} \exists infinite rewrite sequence

$$t_1 \xrightarrow{\mathcal{R}^*} t_2 \xrightarrow{\text{DP}(\mathcal{R})^\epsilon} t_3 \xrightarrow{\mathcal{R}^*} t_4 \xrightarrow{\text{DP}(\mathcal{R})^\epsilon} \dots$$

such that t_1 is terminating with respect to \mathcal{R}

Definition (Reduction Pair)

reduction pair $(>, \succsim)$ consists of well-founded order $>$ and preorder \succsim such that

- ① $>$ is closed under substitutions
- ② \succsim is closed under contexts and substitutions
- ③ $> \cdot \succsim \subseteq >$ or $\succsim \cdot > \subseteq >$

Theorem

TRS \mathcal{R} is terminating \iff $\text{DP}(\mathcal{R}) \subseteq >$ and $\mathcal{R} \subseteq \succsim$ for some reduction pair $(>, \succsim)$

Definitions (Well-Founded Weakly Monotone Algebra)

- ▶ **well-founded weakly monotone \mathcal{F} -algebra (WFWMA)** $(\mathcal{A}, >, \succeq)$ consists of non-empty algebra $\mathcal{A} = (A, \{f_{\mathcal{A}}\}_{f \in \mathcal{F}})$ with well-founded order $>$ and preorder \succeq on A such that
 - ① every $f_{\mathcal{A}}$ is monotone with respect to \succeq in all coordinates
 - ② $> \cdot \succeq \subseteq >$ or $\succeq \cdot > \subseteq >$
- ▶ relation $>_{\mathcal{A}}$ on terms: $s >_{\mathcal{A}} t$ if $[\alpha]_{\mathcal{A}}(s) > [\alpha]_{\mathcal{A}}(t)$ for all assignments α
- ▶ relation $\succeq_{\mathcal{A}}$ on terms: $s \succeq_{\mathcal{A}} t$ if $[\alpha]_{\mathcal{A}}(s) \succeq [\alpha]_{\mathcal{A}}(t)$ for all assignments α

Lemmata

- ▶ $(>_{\mathcal{A}}, \succeq_{\mathcal{A}})$ is reduction pair for every WFWMA $(\mathcal{A}, >, \succeq)$
- ▶ $(\mathcal{A}, >, \succeq)$ is WFWMA for every well-founded monotone algebra $(\mathcal{A}, >)$

Example 1

► rewrite rules

$$\text{append}(\text{nil}, z) \rightarrow z$$
$$\text{reverse}(\text{nil}) \rightarrow \text{nil}$$
$$\text{shuffle}(\text{nil}) \rightarrow \text{nil}$$
$$\text{append}(x : y, z) \rightarrow x : \text{append}(y, z)$$
$$\text{reverse}(x : y) \rightarrow \text{append}(\text{reverse}(y), x : \text{nil})$$
$$\text{shuffle}(x : y) \rightarrow x : \text{shuffle}(\text{reverse}(y))$$

► dependency pairs

$$\text{append}^\#(x : y, z) \rightarrow \text{append}^\#(y, z)$$
$$\text{reverse}^\#(x : y) \rightarrow \text{reverse}^\#(y)$$
$$\text{shuffle}^\#(x : y) \rightarrow \text{reverse}^\#(y)$$
$$\text{reverse}^\#(x : y) \rightarrow \text{append}^\#(\text{reverse}(y), x : \text{nil})$$
$$\text{shuffle}^\#(x : y) \rightarrow \text{shuffle}^\#(\text{reverse}(y))$$

► polynomial interpretation

$$\text{reverse}_{\mathbb{N}}(x) = \text{shuffle}_{\mathbb{N}}(x) = \text{append}_{\mathbb{N}}^\#(x, y) = x \quad \text{append}_{\mathbb{N}}(x, y) = x + y$$
$$\text{nil}_{\mathbb{N}} = 0 \quad \text{reverse}_{\mathbb{N}}^\#(x) = \text{shuffle}_{\mathbb{N}}^\#(x) = x + 1 \quad :_{\mathbb{N}}(x, y) = x + y + 1$$

Example 2

► rewrite rules

$$\text{primes} \rightarrow \text{sieve}(\text{from}(\text{s}(\text{s}(0))))$$
$$\text{head}(x : y) \rightarrow x$$
$$\text{tail}(x : y) \rightarrow y$$
$$\text{sieve}(0 : y) \rightarrow \text{sieve}(y)$$
$$\text{sieve}(s(x) : y) \rightarrow s(x) : \text{sieve}(\text{filter}(y, x, x))$$
$$\text{filter}(y : z, 0, w) \rightarrow 0 : \text{filter}(z, w, w)$$
$$\text{filter}(y : z, s(x), w) \rightarrow y : \text{filter}(z, x, w)$$

► dependency pairs

$$\text{primes}^\# \rightarrow \text{sieve}^\#(\text{from}(\text{s}(\text{s}(0))))$$
$$\text{sieve}^\#(s(x) : y) \rightarrow \text{sieve}^\#(\text{filter}(y, x, x))$$
$$\text{sieve}^\#(s(x) : y) \rightarrow \text{filter}^\#(y, x, x)$$
$$\text{sieve}^\#(0 : y) \rightarrow \text{sieve}^\#(y)$$
$$\text{filter}^\#(y : z, 0, w) \rightarrow \text{filter}^\#(z, w, w)$$
$$\text{filter}^\#(y : z, s(x), w) \rightarrow \text{filter}^\#(z, x, w)$$

► polynomial interpretation

$$0_{\mathbb{N}} = 0 \quad s_{\mathbb{N}}(x) = x + 1 \quad \text{from}_{\mathbb{N}}(x) = x \quad :_{\mathbb{N}}(x, y) = x + y + 1 \quad \text{primes}_{\mathbb{N}} = 2 \quad \text{primes}_{\mathbb{N}}^\# = 3$$
$$\text{head}_{\mathbb{N}}(x) = \text{tail}_{\mathbb{N}}(x) = x \quad \text{filter}_{\mathbb{N}}(x, y, z) = \text{filter}_{\mathbb{N}}^\#(x, y, z) = \text{sieve}_{\mathbb{N}}(x) = \text{sieve}_{\mathbb{N}}^\#(x) = x$$

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Online Evaluation in Presence

<https://lv-analyse.uibk.ac.at/evasys/public/online/index>



Outline

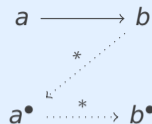
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Definition

ARS $\mathcal{A} = \langle A, \rightarrow \rangle$ has **Z property** if

$$a \rightarrow b \implies b \rightarrow^* \bullet(a) \rightarrow^* \bullet(b)$$

for some function \bullet on A

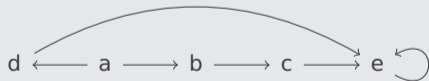


Notation

a^\bullet for $\bullet(a)$

Example

ARS



- ▶ define $a^\bullet = b^\bullet = c^\bullet = d^\bullet = e^\bullet = e$
- ▶ every element rewrites to $e \implies$ Z property is trivially satisfied

Lemma (Monotonicity)

$a \rightarrow^* b \implies a^\bullet \rightarrow^* b^\bullet$ for every ARS $\langle A, \rightarrow \rangle$ with Z property for \bullet

Theorem

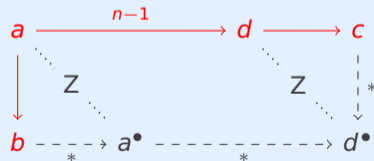
every ARS with Z property is confluent

Proof

▶ $b \leftarrow a \rightarrow^n c \implies b \downarrow c$ by induction on n :

▶ $n = 0 \implies c = a \rightarrow b$

▶ $n > 0 \implies a \rightarrow^{n-1} d \rightarrow c$ for some element d



▶ $\leftarrow \cdot \rightarrow^* \subseteq \downarrow$ (semi-confluence) $\implies \bullet \leftarrow \cdot \rightarrow^* \subseteq \downarrow$ (confluence)

Application: Confluence of Combinatory Logic

how to find suitable bullet function \bullet for CL ?

Definition

functions \diamond and \star on CL-terms:

$$t^\diamond = \begin{cases} u^\diamond \star v^\diamond & \text{if } t = uv \\ t & \text{otherwise} \end{cases} \quad s \star t = \begin{cases} t & \text{if } s = I \\ u & \text{if } s = Ku \\ ut(vt) & \text{if } s = Suv \\ st & \text{otherwise} \end{cases}$$

Example

$$\begin{aligned} (\mathbf{SK}(\mathbf{IK})(\mathbf{IIS}))^{\diamond\diamond} &= ((\mathbf{SK}(\mathbf{IK}))^\diamond \star (\mathbf{IIS})^\diamond)^\diamond = (((\mathbf{SK})^\diamond \star (\mathbf{IK})^\diamond) \star ((\mathbf{II})^\diamond \star \mathbf{S}^\diamond))^\diamond \\ &= (((\mathbf{S}^\diamond \star \mathbf{K}^\diamond) \star (\mathbf{I}^\diamond \star \mathbf{K}^\diamond)) \star ((\mathbf{I}^\diamond \star \mathbf{I}^\diamond) \star \mathbf{S}^\diamond))^\diamond = (((\mathbf{S} \star \mathbf{K}) \star (\mathbf{I} \star \mathbf{K})) \star ((\mathbf{I} \star \mathbf{I}) \star \mathbf{S}))^\diamond \\ &= ((\mathbf{SK} \star \mathbf{K}) \star (\mathbf{I} \star \mathbf{S}))^\diamond = (\mathbf{SKK} \star \mathbf{S})^\diamond = (\mathbf{KS}(\mathbf{KS}))^\diamond = \mathbf{KS} \star \mathbf{KS} = \mathbf{S} \end{aligned}$$

Example (cont'd)

$(SK(IK)(IIS))^\diamond$ is common reduct of IIS and $SKK(IS)$

$$IIS \leftarrow K(IIS)(IK(IIS)) \leftarrow SK(IK)(IIS) \rightarrow SKK(IIS) \rightarrow SKK(IS)$$

Theorem

CL has Z property for \diamond

Proof (sketch)

for all CL-terms s, t, u, v

① $st \rightarrow^= s \star t$

③ $s \rightarrow^* t$ and $u \rightarrow^* v \implies s \star u \rightarrow^* t \star v$

② $t \rightarrow^* t^\diamond$

④ $s \rightarrow^= t \implies t \rightarrow^* s^\diamond \rightarrow^* t^\diamond$

Remark

method extends to all orthogonal TRSs

Definition

strategy \mathcal{S}_\bullet for ARS \mathcal{A} with Z property for \bullet : $a \dashrightarrow b$ if $a \notin \text{NF}(\mathcal{A})$ and $b = a^\bullet$

Theorem

\mathcal{S}_\bullet is normalizing strategy for every ARS with Z property for \bullet

Proof

① $a \rightarrow^n b$ and $n > 0 \implies b \rightarrow^* \bullet^n(a)$ by induction on n :

$a \rightarrow c \rightarrow^{n-1} b \implies c \rightarrow^* \bullet(a)$ (Z property)

▶ $n = 1 \implies b = c$

▶ $n > 1 \implies b \rightarrow^* \bullet^{n-1}(c)$ (induction hypothesis)

$\bullet^{n-1}(c) \rightarrow^* \bullet^n(a) \implies b \rightarrow^* \bullet^n(a)$ ($n - 1$ applications of monotonicity)

Theorem

\mathcal{S}_\bullet is normalizing strategy for every ARS with Z property for \bullet

Proof (cont'd)

① $a \rightarrow^n b$ and $n > 0 \implies b \rightarrow^* \bullet^n(a)$

② $a \rightarrow^{\leq n} \bullet^n(a)$ for all $n \geq 0$ by induction on n

▶ $n = 0 \implies a = \bullet^0(a)$

▶ $n > 0 \implies a \rightarrow^{\leq n-1} \bullet^{n-1}(a) \rightarrow^{\bullet} \bullet^{n-1}(a)^\bullet = \bullet^n(a)$ (induction hypothesis)

③ $a \rightarrow^n b$ with $n > 0$ and $b \in \text{NF}(\rightarrow)$

$a \rightarrow^{\leq n} \bullet^n(a) \bullet^* \leftarrow b \implies a \rightarrow^{\leq n} b \implies \mathcal{S}_\bullet$ is normalizing

Theorem

\mathcal{S}_\bullet is hyper-normalizing strategy for every ARS with Z property for \bullet

Proof (sketch)

① $\rightarrow^* \cdot \rightarrow \subseteq \rightarrow \cdot \rightarrow^*$ (\rightarrow commutes over \rightarrow^*)

▶ suppose $a \rightarrow^* b \rightarrow c$

▶ $b \notin \text{NF} \implies a \notin \text{NF} \implies a \rightarrow \bullet(a) \rightarrow^* \bullet(b) = c$ (monotonicity)

② \mathcal{S}_\bullet is normalizing strategy $\implies \mathcal{S}_\bullet$ is hyper-normalizing strategy

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Homework Exercises for June 15

① Exercise 4.49(a,c,d).

3

② Exercise 3.34.

2

③ Exercise 6.5.

2

④ Exercise 6.24.



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Lecture Notes

- ▶ Section 4.5 (until Example 4.5.15)
- ▶ Section 1.2
- ▶ Section 1.5
- ▶ Section 3.4

Important Concepts

- ▶ \mathcal{S}_\bullet
- ▶ $s \star t$
- ▶ t^\diamond
- ▶ dependency pair
- ▶ reduction pair
- ▶ well-founded weakly monotone algebra
- ▶ Z property

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$$\text{score} = \min\left(\max\left(\frac{50}{69}(E + P) + \frac{1}{3}T + B, T + B\right), 100\right)$$

E : points for solved exercises (at most 84)

P : points for presentation of solutions (at most 8)

T : points for **test** (at most 100)

B : points for bonus exercises (at most 20)

$$\text{grade} = \text{score} \in (-50) \rightarrow 5 \quad [50 - 63) \rightarrow 4 \quad [63 - 75) \rightarrow 3 \quad [75 - 88) \rightarrow 2 \quad [88 -) \rightarrow 1$$

- ▶ (optional) test on June 22
- ▶ **closed book**, 15:30 – 18:00, HS 10
- ▶ online registration is required until **23:59 on June 15**
- ▶ earlier tests: 20W 22W 23W