universität innsbruck

26S lecture 12

Term Rewriting

Philipp Dablander and **Aart Middeldorp**

Outline

Definitions

- strategy annotation for function symbol f is finite list $A(f)$ containing argument positions of f and (labels of) rewrite rules for f
- strategy annotation $A(f)$ for function symbol f is full if $A(f)$ contains all argument positions of f and all rewrite rules for f
- strategy annotation $A(f)$ for function symbol f is in–time if argument positions are listed in $A(f)$ before rewrite rules that need them
- rewrite rule $f(s_1, \dots, s_n) \to t$ needs argument position i if
  - s_i is non–variable, or
  - s_i is variable that appears in $s_1, \dots, s_{i-1}, s_{i+1}, \dots, s_n$

Definitions

$$\mathsf{redex}_A(t) = \mathsf{redex}'_A(t, A(\mathsf{root}(t)))$$

$$\mathsf{redex}'_A(t, [\,]) = \bot$$

$$\mathsf{redex}'_A(t, [\ell \to r \mid L]) = \begin{cases} (\epsilon, \ell \to r) & \text{if } t \geq \ell \\ \mathsf{redex}'_A(t, L) & \text{otherwise} \end{cases}$$

$$\mathsf{redex}'_A(t, [i \mid L]) = \begin{cases} (ip, \ell \to r) & \text{if } \mathsf{redex}_A(t|_i) = (p, \ell \to r) \\ \mathsf{redex}'_A(t, L) & \text{otherwise} \end{cases}$$

$$\mathsf{normalize}_A(t) = \mathsf{normalize}'_A(t, A(\mathsf{root}(t)))$$

$$\mathsf{normalize}'_A(t, [\,]) = t$$

$$\mathsf{normalize}'_A(t, [\ell \to r \mid L]) = \begin{cases} \mathsf{normalize}_A(r\sigma) & \text{if } t = \ell\sigma \text{ for some substitution } \sigma \\ \mathsf{normalize}'_A(t, L) & \text{otherwise} \end{cases}$$

$$\mathsf{normalize}'_A(t, [i \mid L]) = \mathsf{normalize}'_A(t[t|_i]_i, L)$$

Definition

$s \xrightarrow{\mathcal{S}_A} t$ if $\text{redex}_A(s) = (p, \ell \to r)$ and $s \to_{p|\ell \to r} t$

Lemma

$\mathcal{S}_A$ is rewrite strategy for every full strategy annotation A

Theorem

$\forall$ full and in–time strategy annotation A $\forall$ term t

- $\mathcal{S}_A$ normalizes term t $\iff$ $\text{normalize}_A(t)$ is defined
- $t{\downarrow}_{\mathcal{S}_A} = \text{normalize}_A(t)$ for all normalizing terms t
- leftmost innermost strategy normalizes t $\implies$ $\mathcal{S}_A$ normalizes t

Definitions

- term relation $>$ has subterm property if $C[t] > t$ for all non–empty contexts C and terms t
- simplification order is rewrite order with subterm property
- TRS $\mathcal{R}$ is simply terminating if $\mathcal{R}$ is compatible with simplification order

Theorem

$>_{\text{lpo}}$ and $>_{\text{kbo}}$ are simplification orders

Theorem

for finite signatures

- simplification orders are well–founded
- simply terminating TRSs are terminating

Definitions

- TRS $\mathcal{E}\text{mb} = \{f(x_1, \ldots, x_n) \to x_i \mid f$ is n-ary function symbol, $1 \leqslant i \leqslant n\}$
- simple monotone $\mathcal{F}$–algebra $(\mathcal{A}, >)$ consists of non–empty algebra $\mathcal{A} = (A, \{f_{\mathcal{A}}\}_{f \in \mathcal{F}})$ with well–founded order $>$ on A such that every $f_{\mathcal{A}}$ is simple and weakly monotone:

$$f_{\mathcal{A}}(a_1, \ldots, a_i, \ldots, a_n) \geqslant a_i$$

for all $a_1, \ldots, a_n \in A$ and $i \in \{1, \ldots, n\}$

$$f_{\mathcal{A}}(a_1, \ldots, a_i, \ldots, a_n) \geqslant f_{\mathcal{A}}(a_1, \ldots, b, \ldots, a_n)$$

for all $a_1, \ldots, a_n, b \in A$ and $i \in \{1, \ldots, n\}$ with $a_i > b$

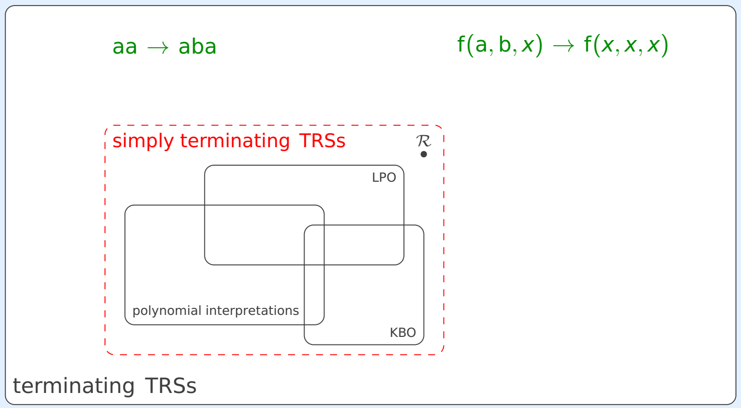
Lemma

TRS $\mathcal{R}$ is simply terminating

$\iff$ $\mathcal{R} \cup \mathcal{E}\text{mb}$ is terminating

$\iff$ $\mathcal{R}$ is compatible with simple monotone algebra

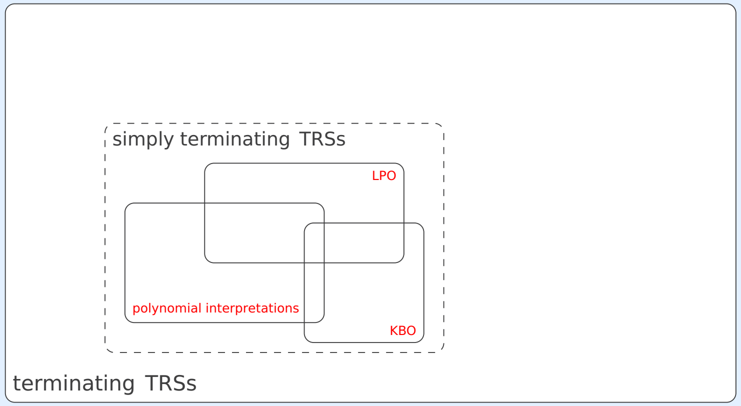
Termination Methods: Realistic View



$\mathcal{R}$: $\mathsf{f(a)} \to \mathsf{f(b)}$ $\quad \mathsf{g(b)} \to \mathsf{g(a)}$

Outline

Termination Methods: Realistic View



dependency pairs make direct termination methods much more powerful

Definitions (Dependency Pair)

TRS $\mathcal{R}$ over signature $\mathcal{F}$

- $\mathcal{F}^\sharp = \mathcal{F} \cup \{f^\sharp \mid f$ is defined symbol of $\mathcal{R}\}$
- if $t = f(t_1, \ldots, t_n)$ with f defined then $t^\sharp = f^\sharp(t_1, \ldots, t_n)$
- dependency pair $\ell^\sharp \to u^\sharp$ of rewrite rule $\ell \to r$ satisfies
  - $u \trianglelefteq r$ and $u \ntriangleleft \ell$
  - $\mathrm{root}(u)$ is defined symbol
- $\mathsf{DP}(\mathcal{R})$ is set of all dependency pairs of $\mathcal{R}$

Example

- rewrite rules

$$0 - y \to 0 \qquad 0 \div \mathsf{s}(y) \to 0$$
$$x - 0 \to x \qquad \mathsf{s}(x) \div \mathsf{s}(y) \to \mathsf{s}((x - y) \div \mathsf{s}(y))$$
$$\mathsf{s}(x) - \mathsf{s}(y) \to x - y$$

- dependency pairs

$$\mathsf{s}(x) -^\sharp \mathsf{s}(y) \to x -^\sharp y$$
$$\mathsf{s}(x) \div^\sharp \mathsf{s}(y) \to (x - y) \div^\sharp \mathsf{s}(y)$$
$$\mathsf{s}(x) \div^\sharp \mathsf{s}(y) \to x -^\sharp y$$

- polynomial interpretation

$$0_\mathbb{N} = 0 \qquad \mathsf{s}_\mathbb{N}(x) = x + 1 \qquad -_\mathbb{N}(x, y) = \div_\mathbb{N}(x, y) = -^\sharp_\mathbb{N}(x, y) = \div^\sharp_\mathbb{N}(x, y) = x \qquad ?$$

Theorem

$\forall$ non–terminating TRS $\mathcal{R}$ $\exists$ infinite rewrite sequence

$$t_1 \xrightarrow[\mathcal{R}]{*} t_2 \xrightarrow[\mathsf{DP}(\mathcal{R})]{\epsilon} t_3 \xrightarrow[\mathcal{R}]{*} t_4 \xrightarrow[\mathsf{DP}(\mathcal{R})]{\epsilon} \cdots$$

such that t_1 is terminating with respect to $\mathcal{R}$

Definition (Reduction Pair)

reduction pair $(>, \gtrsim)$ consists of well–founded order $>$ and preorder $\gtrsim$ such that

① $>$ is closed under substitutions

② $\gtrsim$ is closed under contexts and substitutions

③ $> \cdot \gtrsim \; \subseteq \; >$ or $\gtrsim \cdot > \; \subseteq \; >$

Theorem

TRS $\mathcal{R}$ is terminating $\iff$ $\mathsf{DP}(\mathcal{R}) \subseteq >$ and $\mathcal{R} \subseteq \gtrsim$ for some reduction pair $(>, \gtrsim)$

Definitions (Well–Founded Weakly Monotone Algebra)

- well–founded weakly monotone $\mathcal{F}$–algebra (WFWMA) $(\mathcal{A}, >, \gtrsim)$ consists of non–empty algebra $\mathcal{A} = (A, \{f_{\mathcal{A}}\}_{f \in \mathcal{F}})$ with well–founded order $>$ and preorder $\gtrsim$ on A such that
  ① every $f_{\mathcal{A}}$ is monotone with respect to $\gtrsim$ in all coordinates
  ② $> \cdot \gtrsim \; \subseteq \; >$ or $\gtrsim \cdot > \; \subseteq \; >$
- relation $>_{\mathcal{A}}$ on terms: $s >_{\mathcal{A}} t$ if $[\alpha]_{\mathcal{A}}(s) > [\alpha]_{\mathcal{A}}(t)$ for all assignments α
- relation $\gtrsim_{\mathcal{A}}$ on terms: $s \gtrsim_{\mathcal{A}} t$ if $[\alpha]_{\mathcal{A}}(s) \gtrsim [\alpha]_{\mathcal{A}}(t)$ for all assignments α

Lemmata

- $(>_{\mathcal{A}}, \gtrsim_{\mathcal{A}})$ is reduction pair for every WFWMA $(\mathcal{A}, >, \gtrsim)$
- $(\mathcal{A}, >, \geqslant)$ is WFWMA for every well–founded monotone algebra $(\mathcal{A}, >)$

Example ❶

- rewrite rules

$$\begin{aligned} \mathsf{append}(\mathsf{nil}, z) &\to z & \mathsf{append}(x : y, z) &\to x : \mathsf{append}(y, z) \\ \mathsf{reverse}(\mathsf{nil}) &\to \mathsf{nil} & \mathsf{reverse}(x : y) &\to \mathsf{append}(\mathsf{reverse}(y), x : \mathsf{nil}) \\ \mathsf{shuffle}(\mathsf{nil}) &\to \mathsf{nil} & \mathsf{shuffle}(x : y) &\to x : \mathsf{shuffle}(\mathsf{reverse}(y)) \end{aligned}$$

- dependency pairs

$$\begin{aligned} &\mathsf{append}^\sharp(x : y, z) \to \mathsf{append}^\sharp(y, z) \\ &\mathsf{reverse}^\sharp(x : y) \to \mathsf{reverse}^\sharp(y) & &\mathsf{reverse}^\sharp(x : y) \to \mathsf{append}^\sharp(\mathsf{reverse}(y), x : \mathsf{nil}) \\ &\mathsf{shuffle}^\sharp(x : y) \to \mathsf{reverse}^\sharp(y) & &\mathsf{shuffle}^\sharp(x : y) \to \mathsf{shuffle}^\sharp(\mathsf{reverse}(y)) \end{aligned}$$

- polynomial interpretation

$$\mathsf{reverse}_{\mathbb{N}}(x) = \mathsf{shuffle}_{\mathbb{N}}(x) = \mathsf{append}^\sharp_{\mathbb{N}}(x, y) = x \qquad \mathsf{append}_{\mathbb{N}}(x, y) = x + y$$

$$\mathsf{nil}_{\mathbb{N}} = 0 \qquad \mathsf{reverse}^\sharp_{\mathbb{N}}(x) = \mathsf{shuffle}^\sharp_{\mathbb{N}}(x) = x + 1 \qquad :_{\mathbb{N}}(x, y) = x + y + 1$$

Example ❷

- rewrite rules

$$\begin{aligned} \mathsf{primes} &\to \mathsf{sieve}(\mathsf{from}(\mathsf{s}(\mathsf{s}(0)))) & \mathsf{sieve}(0 : y) &\to \mathsf{sieve}(y) \\ & & \mathsf{sieve}(\mathsf{s}(x) : y) &\to \mathsf{s}(x) : \mathsf{sieve}(\mathsf{filter}(y, x, x)) \\ \mathsf{head}(x : y) &\to x & \mathsf{filter}(y : z, 0, w) &\to 0 : \mathsf{filter}(z, w, w) \\ \mathsf{tail}(x : y) &\to y & \mathsf{filter}(y : z, \mathsf{s}(x), w) &\to y : \mathsf{filter}(z, x, w) \end{aligned}$$

- dependency pairs

$$\begin{aligned} \mathsf{primes}^\sharp &\to \mathsf{sieve}^\sharp(\mathsf{from}(\mathsf{s}(\mathsf{s}(0)))) & \mathsf{sieve}^\sharp(0 : y) &\to \mathsf{sieve}^\sharp(y) \\ \mathsf{sieve}^\sharp(\mathsf{s}(x) : y) &\to \mathsf{sieve}^\sharp(\mathsf{filter}(y, x, x)) & \mathsf{filter}^\sharp(y : z, 0, w) &\to \mathsf{filter}^\sharp(z, w, w) \\ \mathsf{sieve}^\sharp(\mathsf{s}(x) : y) &\to \mathsf{filter}^\sharp(y, x, x) & \mathsf{filter}^\sharp(y : z, \mathsf{s}(x), w) &\to \mathsf{filter}^\sharp(z, x, w) \end{aligned}$$

- polynomial interpretation

$$0_{\mathbb{N}} = 0 \quad \mathsf{s}_{\mathbb{N}}(x) = x + 1 \quad \mathsf{from}_{\mathbb{N}}(x) = x \quad :_{\mathbb{N}}(x, y) = x + y + 1 \quad \mathsf{primes}_{\mathbb{N}} = 2 \quad \mathsf{primes}^\sharp_{\mathbb{N}} = 3$$

$$\mathsf{head}_{\mathbb{N}}(x) = \mathsf{tail}_{\mathbb{N}}(x) = x \quad \mathsf{filter}_{\mathbb{N}}(x, y, z) = \mathsf{filter}^\sharp_{\mathbb{N}}(x, y, z) = \mathsf{sieve}_{\mathbb{N}}(x) = \mathsf{sieve}^\sharp_{\mathbb{N}}(x) = x$$

Outline

Online Evaluation in Presence

https://lv-analyse.uibk.ac.at/evasys/public/online/index

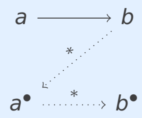
Outline

Definition

ARS $\mathcal{A} = \langle A, \rightarrow \rangle$ has Z property if

$$a \rightarrow b \quad \Longrightarrow \quad b \rightarrow^* \bullet(a) \rightarrow^* \bullet(b)$$

for some function $\bullet$ on A

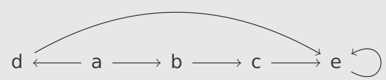


Notation

$a^\bullet$ for $\bullet(a)$

Example

ARS



- define $a^\bullet = b^\bullet = c^\bullet = d^\bullet = e^\bullet = e$
- every element rewrites to e $\Longrightarrow$ Z property is trivially satisfied

Lemma (Monotonicity)

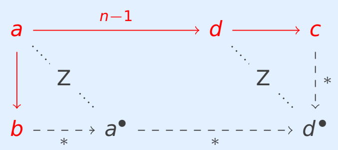
$a \to^* b \implies a^\bullet \to^* b^\bullet$ for every ARS $\langle A, \to \rangle$ with Z property for $\bullet$

Theorem

every ARS with Z property is confluent

Proof

- $b \leftarrow a \to^n c \implies b \downarrow c$ by induction on n:
  - $n = 0 \implies c = a \to b$
  - $n > 0 \implies a \to^{n-1} d \to c$ for some element d



- $\leftarrow \cdot \to^* \subseteq \downarrow$ (semi–confluence) $\implies$ ${}^*\!\leftarrow \cdot \to^* \subseteq \downarrow$ (confluence)

Application: Confluence of Combinatory Logic

how to find suitable bullet function $\bullet$ for CL ?

Definition

functions $\diamond$ and $\star$ on CL–terms:

$$t^\diamond = \begin{cases} u^\diamond \star v^\diamond & \text{if } t = u\,v \\ t & \text{otherwise} \end{cases} \qquad s \star t = \begin{cases} t & \text{if } s = \mathsf{I} \\ u & \text{if } s = \mathsf{K}\,u \\ u\,t\,(v\,t) & \text{if } s = \mathsf{S}\,u\,v \\ s\,t & \text{otherwise} \end{cases}$$

Example

$$\begin{aligned}(\mathsf{SK}(\mathsf{IK})(\mathsf{IIS}))^{\diamond\diamond} &= ((\mathsf{SK}(\mathsf{IK}))^\diamond \star (\mathsf{IIS})^\diamond)^\diamond = (((\mathsf{SK})^\diamond \star (\mathsf{IK})^\diamond) \star ((\mathsf{II})^\diamond \star \mathsf{S}^\diamond))^\diamond \\ &= (((\mathsf{S}^\diamond \star \mathsf{K}^\diamond) \star (\mathsf{I}^\diamond \star \mathsf{K}^\diamond)) \star ((\mathsf{I}^\diamond \star \mathsf{I}^\diamond) \star \mathsf{S}))^\diamond = (((\mathsf{S} \star \mathsf{K}) \star (\mathsf{I} \star \mathsf{K})) \star ((\mathsf{I} \star \mathsf{I}) \star \mathsf{S}))^\diamond \\ &= ((\mathsf{SK} \star \mathsf{K}) \star (\mathsf{I} \star \mathsf{S}))^\diamond = (\mathsf{SKK} \star \mathsf{S})^\diamond = (\mathsf{KS}(\mathsf{KS}))^\diamond = \mathsf{KS} \star \mathsf{KS} = \mathsf{S}\end{aligned}$$

Example (cont'd)

$(\mathsf{SK}(\mathsf{IK})(\mathsf{IIS}))^{\diamond\diamond}$ is common reduct of IIS and $\mathsf{SKK}(\mathsf{IS})$

$$\mathsf{IIS} \leftarrow \mathsf{K}(\mathsf{IIS})(\mathsf{IK}(\mathsf{IIS})) \leftarrow \mathsf{SK}(\mathsf{IK})(\mathsf{IIS}) \to \mathsf{SKK}(\mathsf{IIS}) \to \mathsf{SKK}(\mathsf{IS})$$

Theorem

CL has Z property for $\diamond$

Proof (sketch)

for all CL–terms s, t, u, v

① $s\,t \to^= s \star t$

② $t \to^* t^\diamond$

③ $s \to^* t$ and $u \to^* v \implies s \star u \to^* t \star v$

④ $s \to^= t \implies t \to^* s^\diamond \to^* t^\diamond$

Remark

method extends to all orthogonal TRSs

Definition

strategy $\mathcal{S}_\bullet$ for ARS $\mathcal{A}$ with Z property for $\bullet$: $a \twoheadrightarrow b$ if $a \notin \mathrm{NF}(\mathcal{A})$ and $b = a^\bullet$

Theorem

$\mathcal{S}_\bullet$ is normalizing strategy for every ARS with Z property for $\bullet$

Proof

① $a \to^n b$ and $n > 0 \implies b \to^* \bullet^n(a)$ by induction on n:

$a \to c \to^{n-1} b \implies c \to^* \bullet(a)$ (Z property)

- $n = 1 \implies b = c$
- $n > 1 \implies b \to^* \bullet^{n-1}(c)$ (induction hypothesis)

$\bullet^{n-1}(c) \to^* \bullet^n(a) \implies b \to^* \bullet^n(a)$ ($n-1$ applications of monotonicity)

Theorem

$\mathcal{S}_\bullet$ is normalizing strategy for every ARS with Z property for $\bullet$

Proof (cont'd)

① $a \to^n b$ and $n > 0 \implies b \to^* \bullet^n(a)$

② $a \twoheadrightarrow^{\leqslant n} \bullet^n(a)$ for all $n \geqslant 0$ by induction on n

- $n = 0 \implies a = \bullet^n(a)$
- $n > 0 \implies a \twoheadrightarrow^{\leqslant n-1} \bullet^{n-1}(a) \twoheadrightarrow^{=} \bullet^{n-1}(a)^\bullet = \bullet^n(a)$ (induction hypothesis)

③ $a \to^n b$ with $n > 0$ and $b \in \mathrm{NF}(\to)$

$a \twoheadrightarrow^{\leqslant n} \bullet^n(a) \; {}^*\!\!\leftarrow b \implies a \twoheadrightarrow^{\leqslant n} b \implies \mathcal{S}_\bullet$ is normalizing

Theorem

$\mathcal{S}_\bullet$ is hyper–normalizing strategy for every ARS with Z property for $\bullet$

Proof (sketch)

① $\to^* \cdot \twoheadrightarrow \subseteq \twoheadrightarrow \cdot \to^*$ ($\twoheadrightarrow$ commutes over $\to^*$)

- suppose $a \to^* b \twoheadrightarrow c$
- $b \notin \mathrm{NF} \implies a \notin \mathrm{NF} \implies a \twoheadrightarrow \bullet(a) \to^* \bullet(b) = c$ (monotonicity)

② $\mathcal{S}_\bullet$ is normalizing strategy $\implies \mathcal{S}_\bullet$ is hyper–normalizing strategy

Outline

Homework Exercises for June 15

① Exercise 4.49(a,c,d). ③

② Exercise 3.34. ②

③ Exercise 6.5. ②

④ Exercise 6.24. ☆☆☆

Outline

Lecture Notes

- Section 4.5 (until Example 4.5.15)
- Section 1.2
- Section 1.5
- Section 3.4

Important Concepts

- $\mathcal{S}_\bullet$
- $s \star t$
- $t^\diamond$
- dependency pair
- reduction pair
- well–founded weakly monotone algebra
- Z property

Outline

Grading

$$\text{score} = \min\left(\max\left(\tfrac{50}{69}(E+P) + \tfrac{1}{3}T + B,\; T + B\right),\; 100\right)$$

E: points for solved exercises (at most 84)

P: points for presentation of solutions (at most 8)

T: points for test (at most 100)

B: points for bonus exercises (at most 20)

$$\text{grade} = \text{score} \in \; (-50) \to 5 \quad [50-63) \to 4 \quad [63-75) \to 3 \quad [75-88) \to 2 \quad [88-) \to 1$$

- (optional) test on June 22
- closed book, 15:30 – 18:00, HS 10
- online registration is required until 23:59 on June 15
- earlier tests: 20W 22W 23W