Term Rewriting — Lecture 2

26S — March 16, 2026

All Solutions

1 (a) In the ARS

$$\mathsf{a} \circlearrowleft$$

we have $\to^+ \;=\; \to^* \;=\; \{\,(\mathsf{a},\mathsf{a})\,\}$.

(b) In the ARS

$$\mathsf{a} \longrightarrow \mathsf{b} \longleftarrow \mathsf{c}$$

we have $\mathsf{a} \leftrightarrow^* \mathsf{c}$ but $\mathsf{a} \uparrow \mathsf{c}$ does not hold.

(c) In the ARS

$$\mathsf{a} \longrightarrow \mathsf{b} \longleftarrow \mathsf{c} \longleftarrow \mathsf{d}$$

we have $\mathsf{a} \downarrow \mathsf{d}$ but $\mathsf{a} \to^* \cdot \leftrightarrow \mathsf{d}$ does not hold. A simpler example is provided by the ARS

$$\mathsf{a} \qquad \mathsf{b} \circlearrowleft$$

which satisfies $\mathsf{a} \downarrow \mathsf{a}$ but not $\mathsf{a} \to^* \cdot \leftrightarrow \mathsf{a}$.

(d) In the ARS

$$\mathsf{a} \longrightarrow \mathsf{b} \longrightarrow \mathsf{c}$$

we have $\mathsf{a} \to^2 \mathsf{c}$ but $\mathsf{a} \to^3 \cdot \leftarrow \mathsf{c}$ does not hold.

(e) In the ARS

$$\mathsf{a} \longrightarrow \mathsf{b}$$

we have ${}^!\!\leftarrow \;=\; \{\,(\mathsf{b},\mathsf{a}),(\mathsf{b},\mathsf{b})\,\} \neq \{\,(\mathsf{b},\mathsf{a}),(\mathsf{a},\mathsf{a})\,\} = \leftarrow^!$. Note that

$$\leftarrow^! = \{\,(x,y) \mid x \leftarrow^* y \text{ and } y \in \mathsf{NF}(\leftarrow)\,\}$$

by definition, and a is the only element in $\mathsf{NF}(\leftarrow)$.

2 (a) There are 8 rewrite sequences starting at 12: 12, $12 \to 6$, $12 \to 4$, $12 \to 3$, $12 \to 2$, $12 \to 6 \to 3$, $12 \to 6 \to 2$, and $12 \to 4 \to 2$.

(b) The set of prime factors of a.

(c) All prime numbers.

(d) If $a, b \in A$ then $a \times b \in A$ and $a \leftarrow a \times b \rightarrow b$. Moreover, $a \downarrow b$ if and only if a and b have a common factor greater than one.

(e) There are 2^{n-1} different rewrite sequences starting at 2^n. This is easily proved by induction on n. If $n = 1$ then $2^n = 2$ is a normal form and hence admits only $1 = 2^{n-1}$ rewrite sequence. Suppose $n > 1$. In one step we can reach the elements 2^1, 2^2, …, 2^{n-1}. Hence, by the induction hypothesis, there are $2^0 + 2^1 + \cdots + 2^{n-2} = 2^{n-1} - 1$ different non-empty rewrite sequences starting at 2^n. Adding the empty rewrite sequence yields the desired number 2^{n-1}.

3 Let $\mathcal{A} = \langle \mathbb{N}, \rightarrow \rangle$ with $\rightarrow\, = \{(0,2),(1,3)\} \cup \{(x^2, x), (x+3, x) \mid x \in \mathbb{N}\}$. The ARS $\mathcal{A}$ is not terminating because $0 \rightarrow 0$ by using the rule $x^2 \rightarrow x$. By using the rule $x + 3 \rightarrow x$ repeatedly, every natural number rewrites to 0, 1, or 2. Since $0 \rightarrow 2$ and $1 \rightarrow 3 \rightarrow 0 \rightarrow 2$, every natural number rewrites to 2. Hence $\mathcal{A}$ is (locally) confluent. Because 2 is a normal form, $\mathcal{A}$ is normalizing. Because 2 is the only normal form, $\mathcal{A}$ has unique normal forms.

4 (a) $\mathsf{FORT\text{-}s}$ produces the ARS $\langle \{\mathsf{a}, \mathsf{b}, \mathsf{c}\}, \rightarrow \rangle$ with $\rightarrow\, = \{(\mathsf{a},\mathsf{a}), (\mathsf{b},\mathsf{a}), (\mathsf{b},\mathsf{c})\}$.

(b) $\mathsf{FORT\text{-}s}$ produces the ARS $\langle \{\mathsf{c}_0, \mathsf{c}_1\}, \rightarrow \rangle$ with $\rightarrow\, = \{(\mathsf{c}_0,\mathsf{c}_0), (\mathsf{c}_0,\mathsf{c}_1)\}$.

(c) $\mathsf{FORT\text{-}s}$ produces the ARS $\langle \{\mathsf{c}_0, \mathsf{c}_1, \mathsf{c}_2, \mathsf{c}_3\}, \rightarrow \rangle$ with $\rightarrow\, = \{(\mathsf{c}_0,\mathsf{c}_1), (\mathsf{c}_0,\mathsf{c}_2), (\mathsf{c}_1,\mathsf{c}_0), (\mathsf{c}_1,\mathsf{c}_3)\}$.

5 (a) From the inclusion $\rightarrow\, \subseteq\, \rightarrowtail^*$ we infer $\rightarrow^* \subseteq (\rightarrowtail^*)^* = \rightarrowtail^*$. Hence

$$
\begin{aligned}
{}^*\!\leftarrowtail \cdot \rightarrowtail^* &\subseteq\, \downarrow \cdot \downarrow && (\rightarrowtail^* \subseteq\, \downarrow)\\
&\subseteq\, \leftrightarrow^* \\
&\subseteq\, \downarrow && (\mathcal{A} \text{ is confluent})\\
&=\, \rightarrow^* \cdot {}^*\!\leftarrow \\
&\subseteq\, \rightarrowtail^* \cdot {}^*\!\leftarrowtail && (\rightarrow^* \subseteq\, \rightarrowtail^*)
\end{aligned}
$$

(b) We have

$$
\begin{aligned}
{}^*\!\leftarrow \cdot \rightarrow^* &\subseteq\, {}^*\!\leftarrowtail \cdot \rightarrowtail^* && (\rightarrow^* \subseteq\, \rightarrowtail^*)\\
&\subseteq\, \rightarrowtail^* \cdot {}^*\!\leftarrowtail && (\mathcal{B} \text{ is confluent})\\
&\subseteq\, \rightarrow^* \cdot {}^*\!\leftarrow \cdot {}^*\!\leftarrowtail && (\rightarrowtail^* \subseteq\, \downarrow)\\
&\subseteq\, \rightarrow^* \cdot {}^*\!\leftarrowtail \cdot {}^*\!\leftarrowtail && (\rightarrow^* \subseteq\, \rightarrowtail^*)\\
&=\, \rightarrow^* \cdot {}^*\!\leftarrowtail \\
&\subseteq\, \rightarrow^* \cdot \rightarrow^* \cdot {}^*\!\leftarrow && (\rightarrowtail^* \subseteq\, \downarrow)\\
&=\, \rightarrow^* \cdot {}^*\!\leftarrow
\end{aligned}
$$

(c) No. Consider for instance the ARSs

We clearly have $\rightarrow\, \subseteq\, \rightarrowtail$ and $\rightarrowtail^* \subseteq\, \leftrightarrow^*$. Note that $\mathcal{B}$ is confluent but $\mathcal{A}$ is not.